Quantalic Topological Theories

Walter Tholen[1]

Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada, M3J 1P3

Dedicated to Bill Lawvere and Peter Freyd, in admiration of their works

Abstract

The paper proposes the notions of topological platform and quantalic topological theory for the presentation and investigation of categories of interest beyond the realm of algebra. These notions are nevertheless grounded in algebra, through the notions of monad and distributive law. The paper shows how they entail previously proposed concepts with similar goals.

Keywords: monad, quantale, lax monad extension, (lax) distributive law, (lax) λ-algebra, $(\mathbb{T}, \mathsf{V})$-category, topological platform, quantalic topological theory, natural topological theory.
2010 MSC: 18 C15, 18C20, 18D99.

1. Introduction

Just before a decade of research on Abelian categories had come to a first great conclusion with the publication of Freyd's book [13], Lawvere's presentation of the notion of *algebraic theory* in his 1963 thesis [25] initiated an intense period in the development of categorical algebra. Through Linton's extension of that notion to infinitary theories as presented in [27] at the 1965 La Jolla conference, and with the publication of the seminal papers by Kleisli [20] and by Eilenberg and Moore [12] that same year, it quickly became clear that the generalized Lawvere-Linton algebraic theories are equivalently described by (what was later called) *monads* over **Set**.

It came as a surprise when Manes in his 1967 thesis [30] gave the first "topological" example of a monadic category over **Set**. Barr's [3] relational extension of the Manes result from compact Hausdorff spaces to all topological spaces showed that monads have a role to play beyond algebra or algebraic topology, specifically in general topology. Shortly afterwards, Lawvere's 1973 milestone paper [26] paved the way for the *enriched category theory* of Eilenberg and Kelly [11] to aid the the investigation of metric and analytic structures.

When in 2000 Bill Lawvere mentioned to me that Lowen's *approach spaces* [28] should be considered as some kind of V-multicategories, just as he had considered metric spaces as V-categories, he in fact triggered the combination of his and Barr's work that we then pursued in the paper [9] with Clementino. Taking advantage of the results that had just been obtained by her with Hofmann in [8], the paper initiated the development of $(\mathbb{T}, \mathsf{V})$-*categories*, only a first account of which has been given in [17], but predecessors of which reach as far back as to Burroni's [6] elegant work on *T-categories* and include aspects of many later papers, such as Hermida's work [14] on multicategories.

The syntax of a $(\mathbb{T}, \mathsf{V})$-category in the form first described by Seal [35] and adopted in [17] involves a **Set**-monad $\mathbb{T}$ and a quantale V which interact via a lax extension of $\mathbb{T}$ from maps to V-valued relations of sets. In terms of the V-powerset (or discrete V-presheaf) monad $\mathbb{P}_{\mathsf{V}}$, this interaction is equivalently (and more elegantly) described by a Beck [4] distributive law of $\mathbb{T}$ over $\mathbb{P}_{\mathsf{V}}$. We stressed this point in [17, 24], where we also extended the general setting, by allowing

Email address: `tholen@mathstat.yorku.ca` (Walter Tholen)

[1]Partial financial assistance by the Natural Sciences and Engineering Research Council of Canada (Discovery Grant 501260) is gratefully acknowledged.

V to be replaced by a quantaloid $\mathcal{Q}$. In this way, $(\mathbb{T}, \mathcal{Q})$-categories include, among other things, categories enriched in an easy type of bicategory, as first considered by Walters [42].

In this paper we propose a further extension of the setting that should facilitate the investigation of categories of interest in topology, and beyond. The notion of *topological platform* as given in Section 3 involves two monads $\mathbb{T}$ and $\mathbb{P}$ on a category $\mathcal{K}$ tied by a distributive law. "Topologicity" enters through the requirement that the Kleisli category of $\mathbb{P}$ be a quantaloid—a natural requirement, since the category of sets with V-valued relations is the Kleisli category of the role model $\mathbb{P}_{\mathsf{V}}$. By contrast, $\mathbb{T}$ should be considered the "algebraic" part of the notion. Having briefly considered strict $(\mathbb{T}, \mathbb{P})$-algebras in Section 2, which include categories of *partial* $\mathbb{T}$*-algebras*, we proceed to state the fundamental fact that, for every topological platform, the category of *lax* $(\mathbb{T}, \mathbb{P})$*-algebras* is *topological* [15, 1] over $\mathcal{K}$. Despite its immediate proof, this fact leaves open a host of questions, some of them already for considerable time even when considered in narrower contexts, which we mention at the end of the paper.

Section 4 explains how previous work in monoidal topology fits under the $(\mathbb{T}, \mathbb{P})$-umbrella, and in Section 5 we invoke a result proved in [40] to make the point that the $(\mathbb{T}, \mathsf{V})$-setting considered previously should be seen as representing the monad $\mathbb{P}$ by a quantale V or, more generally by a quantaloid $\mathcal{Q}$, under only minor loss of generality.

There remains then the question of how to provide a useful representation on the side of $\mathbb{T}$, more precisely, a representation for the lax extension of $\mathbb{T}$, or for the corresponding distributive law. Such a representation was first provided by Hofmann [16], in the form of a lax $\mathbb{T}$-algebraic structure on the quantale V. In [10] we clarified to which extent the lax extensions induced by Hofmann's *topological theories* are special amongst all others, giving a complete characterization of them, and in [41] we showed that every lax extension actually induces a weaker form of Hofmann's topological theory, even when the quantale V is traded for a quantaloid. In Section 6 we propose a further generalization of the notion introduced in [41], which we call *quantalic topological theory*, and show how it entails the previous versions.

The new concepts introduced in this paper were presented at the conference on *Category Theory* (CT 2017) held at the University of British Columbia in July 2017. Their more thorough discussion and application than given here must still be undertaken.

2. Distributive laws and their algebras

Recall that, for monads $\mathbb{T} = (T, m, e)$, $\mathbb{P} = (P, s, y)$ on a category $\mathcal{K}$, a *distributive law* of $\mathbb{T}$ over $\mathbb{P}$ is a natural transformation $\lambda : TP \longrightarrow PT$ which is compatible with the monad operations; that is, the conditions

(1) $\lambda \cdot Ty = yT, \quad \lambda \cdot Ts = sT \cdot P\lambda \cdot \lambda P,$

(2) $\lambda \cdot eP = Pe, \quad \lambda \cdot mP = Pm \cdot \lambda T \cdot T\lambda$

most hold. It is well known (see, for example, [32, 17]) that distributive laws λ of $\mathbb{T}$ over $\mathbb{P}$ are in bijective correspondence with each of the following:

- *Extensions* $\hat{\mathbb{T}} = (\hat{T}, \hat{m}, \hat{e})$ of the monad $\mathbb{T}$ along the left-adjoint functor $F_{\mathbb{P}} : \mathcal{K} \longrightarrow \mathsf{Kl}(\mathbb{P})$ to the Kleisli category of $\mathbb{P}$; these are monads $\hat{\mathbb{T}}$ on $\mathsf{Kl}(\mathbb{P})$ satisfying

$$\hat{T}F_{\mathbb{P}} = F_{\mathbb{P}}T, \quad \hat{m}F_{\mathbb{P}} = F_{\mathbb{P}}m, \quad \hat{e}F_{\mathbb{P}} = F_{\mathbb{P}}e;$$

  equivalently, since $F_{\mathbb{P}}$ is identical on objects, they are given by endofunctors $\hat{T}$ of $\mathsf{Kl}(\mathbb{P})$ with $\hat{T}F_{\mathbb{P}} = F_{\mathbb{P}}T$, such that $\hat{m}_X = F_{\mathbb{P}}m_X$ and $\hat{e}_X = F_{\mathbb{P}}e_X$ define natural transformations $\hat{m} : \hat{T}\hat{T} \longrightarrow \hat{T}$ and $\hat{e} : 1_{\mathsf{Kl}(\mathbb{P})} \longrightarrow \hat{T}$, respectively.

- *Liftings* $\tilde{\mathbb{P}} = (\tilde{P}, \tilde{s}, \tilde{y})$ of the monad $\mathbb{P}$ through the right-adjoint functor $G^{\mathbb{T}} : \mathsf{EM}(\mathbb{T}) \longrightarrow \mathcal{K}$ of the Eilenberg-Moore category of $\mathbb{T}$; these are monads $\tilde{\mathbb{P}}$ on $\mathsf{EM}(\mathbb{T})$ satisfying

$$G^{\mathbb{T}}\tilde{P} = PG^{\mathbb{T}}, \quad G^{\mathbb{T}}\tilde{s} = sG^{\mathbb{T}}, \quad G^{\mathbb{T}}\tilde{y} = yG^{\mathbb{T}};$$

equivalently, since $G^{\mathbb{T}}$ is faithful, they are given by endofunctors $\tilde{P}$ of $\mathsf{EM}(\mathbb{T})$ with $G^{\mathbb{T}}\tilde{P} = PG^{\mathbb{T}}$ such that, for all $\mathbb{T}$-algebras $(X, \xi : TX \longrightarrow X)$, the $\mathcal{K}$-morphisms s_X and y_X give $\mathbb{T}$-homomorphisms $\tilde{P}\tilde{P}(X,\xi) \longrightarrow \tilde{P}(X,\xi)$ and $(X,\xi) \longrightarrow \tilde{P}(X,\xi)$, respectively.

The bijective correspondences

$$\lambda \mapsto \hat{\mathbb{T}} \quad \text{and} \quad \lambda \mapsto \tilde{\mathbb{P}}$$

come about in a similar fashion, except that, to define $\hat{T}$, it suffices to say what $\hat{T}$ does to morphisms, while for $\tilde{P}$ only the definition on objects matters:

$$\hat{T} : (\varphi : X \nrightarrow Y) \mapsto (\lambda_Y \cdot T\varphi : TX \nrightarrow TY) \quad \text{and} \quad \tilde{P} : (X, \xi) \mapsto (PX, P\xi \cdot \lambda_X).$$

Our notational convention here is to write a $\mathcal{K}$-morphism $\varphi : X \longrightarrow PY$ as an arrow $X \nrightarrow Y$ when considered as a morphism in $\mathsf{Kl}(\mathbb{P})$; its Kleisli composite with $\psi : Y \nrightarrow Z$ is denoted by

$$\psi \circ \varphi = s_Z \cdot P\psi \cdot \varphi : X \nrightarrow Z.$$

In what follows, we will also use the abbreviation

$$f_* = F_{\mathbb{P}} f = y_Y \cdot f : X \nrightarrow Y$$

for all $f : X \longrightarrow Y$ in $\mathcal{K}$; in particular, $(1_X)_* = y_X$ is the identity morphism on X in $\mathsf{Kl}(\mathbb{P})$.

Remark 2.1. There is a third equivalent description of distributive laws λ of $\mathbb{T}$ over $\mathbb{P}$, namely via compatible natural transformations $w : PTPT \longrightarrow PT$ which give us *composite monads* $\mathbb{PT} = (PT, w, yT \cdot e)$; the bijective correspondence with distributive laws is given by $\lambda \mapsto w = sT \cdot PPm \cdot P\lambda T$. As we are not using this correspondence in what follows, the interested reader is referred to the literature (such as [17]) for details.

For the remainder of this section, we consider a distributive law λ of a monad $\mathbb{T}$ over a monad $\mathbb{P}$ on $\mathcal{K}$, equivalently described by the corresponding lax extension $\hat{\mathbb{T}}$ to $\mathsf{Kl}(\mathbb{P})$.

Definition 2.2. A *(strict) λ-algebra* (X, α) is given by an object X and a morphism $\alpha : TX \longrightarrow PX$ in $\mathcal{K}$ satisfying

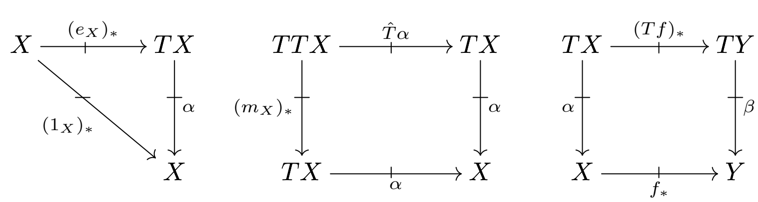
$$\alpha \cdot m_X = s_X \cdot P\alpha \cdot \lambda_X \cdot T\alpha \quad \text{and} \quad \alpha \cdot e_X = y_X.$$

Defining *(strict) λ-homomorphisms* $f : (X, \alpha) \longrightarrow (Y, \beta)$ to be $\mathcal{K}$-morphisms $f : X \longrightarrow Y$ satisfying

$$Pf \cdot \alpha = \beta \cdot Tf,$$

we obtain the category λ-$\mathbf{Alg}^{=}$.

Considering a λ-algebra structure $\alpha : TX \longrightarrow PX$ on X as a morphism $TX \nrightarrow X$ in $\mathsf{Kl}(\mathbb{P})$, one easily confirms that a λ-algebra (X, α) is nothing but an Eilenberg-Moore $\hat{\mathbb{T}}$-algebra; and for $f : (X, \alpha) \longrightarrow (Y, \beta)$ to be a λ-homomorphism amounts to f_* being a $\hat{\mathbb{T}}$-homomorphism.



When the meaning of $\hat{\mathbb{T}}$ is clear from the context, we often write $(\mathbb{T}, \mathbb{P})$-$\mathbf{Alg}^{=}$ instead of λ-$\mathbf{Alg}^{=}$. Note that, when $\mathbb{P}$ is the identity monad, $(\mathbb{T}, \mathbb{P})$-$\mathbf{Alg}^{=}$ is just the Eilenberg-Moore category $\mathsf{EM}(\mathbb{T})$, and when $\mathbb{T}$ is the identity monad (identically extended), $(\mathbb{T}, \mathbb{P})$-$\mathbf{Alg}^{=}$ is the Kleisli category $\mathsf{Kl}(\mathbb{P})$. Because of this double role, we don't expect $(\mathbb{T}, \mathbb{P})$-$\mathbf{Alg}^{=}$ to inherit good categorical properties from $\mathcal{K}$. Nevertheless, let us look at a rather natural example.

Example 2.3. The adjunction whose right adjoint is the underlying **Set**-functor of the category of pointed sets, induces the monad $\mathbb{P} = (P, m, e)$ on **Set** with $PX = X + 1$ and the obvious natural maps $X + 1 + 1 \longrightarrow X + 1$ and $X \longrightarrow X + 1$, for every set X. Its Kleisli category is equivalently described as the category **ParSet** of sets and partial(ly defined) maps, whose morphisms $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ are spans $(X \hookleftarrow A \xrightarrow{\varphi} Y)$ in **Set**; the composite with $(Y \hookleftarrow B \xrightarrow{\psi} Z)$ is the span

$$\psi \circ \varphi = (X \hookleftarrow \varphi^{-1}(B) \xrightarrow{\psi\cdot\varphi|_{\varphi^{-1}(B)}} Z).$$

Given any monad $\mathbb{T}$ on **Set**, since T preserves monomorphisms, for $A \hookrightarrow X$ we may assume $TA \hookrightarrow TX$ and define an extension $\hat{\mathbb{T}}$ to **ParSet** by

$$\hat{T} : (X \hookleftarrow A \xrightarrow{\varphi} Y) \mapsto (TX \hookleftarrow TA \xrightarrow{T\varphi} TY).$$

The corresponding category $(\mathbb{T}, \mathbb{P})$-$\mathbf{Alg}^{=}$ may be described as the category of *strong partial* $\mathbb{T}$-*algebras* and *strong* $\mathbb{T}$-*homomorphisms*. Its objects are sets X equipped with an operation α which assigns to *some* terms in TX a value in X, subject to the two Eilenberg-Moore-algebra laws, which must be read carefully: for all $x \in X$, $\alpha(e_X(x))$ is defined and equals x; and for all $\tau \in TTX$, $\alpha(m_X(\tau))$ is defined *precisely* when $\alpha(T\alpha(\tau))$ is defined, and then the two values are equal. Similarly for morphisms $f : (X, \alpha) \longrightarrow (Y, \beta)$: these are maps $f : X \longrightarrow Y$ with the property that, for all $t \in TX$, $\alpha(t)$ is defined *precisely* when $\beta(Tf(t))$ is defined, which then equals $f(\alpha(t))$.

It may be worth noting that, in the example above, **Set** may be replaced by any *lextensive category* $\mathcal{K}$ (in the sense of [7]). The subobjects $A \hookrightarrow X$ are then given by coproduct injections, *i.e.*, by direct summands A of X, and the monad $\mathbb{T}$ needs to preserve finite coproducts, so that TA becomes a direct summand of TX.

Realizing that **ParSet** carries a natural order, making it a 2-category (even when we replace **Set** by a lextensive category), we may define the larger (and more natural) category of *partial* $\mathbb{T}$-*algebras* and their $\mathbb{T}$-*homomorphisms*, thus foregoing "strength", by relaxing the strict commutativity conditions for the diagrams above, as follows:

$$\begin{array}{ccc} X & \xrightarrow{(e_X)_*} & TX \\ & \searrow^{(1_X)_*} \quad \le & \downarrow \alpha \\ & & X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{\hat{T}\alpha} & TX \\ {\scriptstyle (m_X)_*}\downarrow & \ge & \downarrow \alpha \\ TX & \xrightarrow[\alpha]{} & X \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{(Tf)_*} & TY \\ {\scriptstyle \alpha}\downarrow & \le & \downarrow \beta \\ X & \xrightarrow[f_*]{} & Y \end{array}$$

Still, the resulting category has poor properties; it's lacking even a terminal object. However, these diagrams do provide a good way of defining a category $(\mathbb{T}, \mathbb{P})$-**Alg** in full generality; but we need to relax the conditions on the extension $\hat{\mathbb{T}}$, and strengthen the conditions on the 2-categorical structure of $\mathsf{Kl}(\mathbb{P})$ to arrive at a satisfactorily behaved category. This is our starting point for the next section.

3. Topological platforms

We continue to work with monads $\mathbb{T}$ and $\mathbb{P}$ on a category $\mathcal{K}$. These may be assumed to be 2-monads on a 2-category later on, with the 2-cells given by (pre)order.

Definition 3.1. $(\mathbb{T}, \mathbb{P})$ is a *topological platform* over $\mathcal{K}$ if

(sup) $\mathsf{Kl}(\mathbb{P})$ is a *quantaloid*, *i.e.*, a category enriched in the category **Sup** of complete lattices and suprema-preserving maps; in particular, with the order on its hom-sets, $\mathsf{Kl}(\mathbb{P})$ is a 2-category;

(map) for every $\mathcal{K}$ morphism $f : X \longrightarrow Y$, its image f_* under $F_{\mathbb{P}}$ in $\mathsf{Kl}(\mathbb{P})$ is a *map* in Lawvere's sense, *i.e.*, f_* has a right adjoint $f^* : Y \longrightarrow\!\!\!\!\!\!\mapsto X$;

(ext) $\mathbb{T}$ comes with a *lax extension* $\hat{\mathbb{T}} = (\hat{T}, \hat{m}, \hat{e})$ of $\mathbb{T}$ along $F_{\mathbb{P}}$ to $\mathsf{Kl}(\mathbb{P})$, that is:

- $\hat{T} : \mathsf{Kl}(\mathbb{P}) \longrightarrow \mathsf{Kl}(\mathbb{P})$ is a lax functor, coinciding with T on objects;
- $\hat{m}_X = (m_X)_*$ and $\hat{e}_X = (e_X)_*$ define oplax natural transformations $\hat{m} : \hat{T}\hat{T} \longrightarrow \hat{T}$ and $\hat{e} : 1_{\mathsf{Kl}(\mathbb{P})} \longrightarrow \hat{T}$, respectively;

(whi) $\hat{T}$ satisfies the *right-whiskering* condition: $\hat{T}(\psi \circ f_*) = \hat{T}(\psi) \circ (Tf)_*$, for all $f : X \longrightarrow Y$ in $\mathcal{K}$ and $\psi : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ in $\mathsf{Kl}(\mathbb{P})$.

The category $\mathsf{Alg}(\mathbb{T}, \mathbb{P})$ of *lax* $(\mathbb{T}, \mathbb{P})$*-algebras* (X, α) and their *lax* $(\mathbb{T}, \mathbb{P})$*-homomorphisms* is defined by the last set of lax-commutative diagrams of the previous section.

Remark 3.2. (1) It is important to keep in mind that $\hat{\mathbb{T}}$ is an integral part of the given data. Whenever needed, one should write more accurately $\mathsf{Alg}(\mathbb{T}, \mathbb{P}, \hat{\mathbb{T}})$ instead of $\mathsf{Alg}(\mathbb{T}, \mathbb{P})$.

(2) Since, as one readily sees, the right adjoint $(1_X)^*$ necessarily coincides with $(1_X)_*$, it is convenient to denote the identity morphism on X in $\mathsf{Kl}(\mathbb{P})$ by 1_X^*.

(3) By adjunction, the oplax naturality conditions

$$(m_Y)_* \circ \hat{T}\hat{T}\varphi \leq \hat{T} \circ (m_X)_* \text{ and } (e_Y)_* \circ \varphi \leq \hat{T}\varphi \circ (e_X)_*$$

may be written equivalently as the lax naturality conditions

$$\hat{T}\hat{T}\varphi \circ m_X^* \leq m_Y^* \circ \hat{T}\varphi \text{ and } \varphi \circ e_X^* \leq e_Y^* \circ \hat{T}\varphi$$

for all $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathsf{Kl}(\mathbb{P})$, so that there are lax natural transformations $m^* : \hat{T} \longrightarrow \hat{T}\hat{T}$ and $e^* : \hat{T} \longrightarrow 1_{\mathsf{Kl}(\mathbb{P})}$.

(4) Given that $\hat{T}$ is a lax functor and, hence, preserves the order of the hom-sets, the right whiskering condition may be equivalently stated as the left-whiskering condition $\hat{T}(h^* \circ \varphi) = (Th)^* \circ \hat{T}\varphi$, for all $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathsf{Kl}(\mathbb{P})$ and $h : Z \longrightarrow Y$ in $\mathcal{K}$. This is best proved by showing that each of the two whiskering conditions is equivalent to the condition that

$$(Tf)_* \leq \hat{T}(f_*) \text{ and } (Tf)^* \leq \hat{T}(f^*),$$

for all $f : X \longrightarrow Y$ in $\mathcal{K}$. (The proof proceeds as the proofs of Prop. III.1.4.3 in [17] and of Prop. 6.3 in [41]).

Unlike $\mathsf{Alg}(\mathbb{T}, \mathbb{P})^{=}$ of the previous section, $\mathsf{Alg}(\mathbb{T}, \mathbb{P})$ inherits many of the good standard properties that $\mathcal{K}$ may have, including (total) (co)completeness (but excluding cartesian closedness, of course), for the simple reason that has been noted repeatedly in the literature in narrower contexts, as follows.

Proposition 3.3. *The forgetful functor* $\mathsf{Alg}(\mathbb{T}, \mathbb{P}) \longrightarrow \mathcal{K}$ *is topological.*

Proof. Since the condition $f_* \circ \alpha \leq \beta \circ (Tf)_*$ for a lax $(\mathbb{T}, \mathbb{P})$-homomorphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ may be rewritten equivalently as $\alpha \leq f^* \circ \beta \circ (Tf)_*$, one sees easily that enforcing equality in the last inequality defines cartesian liftings for a single morphism. Now, the quantalic structure of $\mathsf{Kl}(\mathbb{P})$ allows us to do the same simultaneously for any family $f_i : X \longrightarrow Y_i$ of morphisms in $\mathcal{K}$ with common domain, with every Y_i carrying a $(\mathbb{T}, \mathbb{P})$-structure β_i; one simply puts

$$\alpha := \bigwedge_{i \in I} f_i^* \circ \beta_i \circ (Tf_i)_*$$

to obtain the initial structure on X, thus showing topologicity. □

Remark 3.4. (1) The functor of Proposition 3.3 has also the following additional two properties, sometimes required by some authors (see, for example, [15, 1, 5]) for functors to qualify as topological: it has *small fibres* (granted that $\mathcal{K}$ has small hom-sets), and it is *amnestic*, so that isomorphisms in the domain of the functor that are being mapped to identity morphisms must be identity morphisms themselves.

(2) Definition 3.1 describes topological platforms in terms of lax extensions $\hat{\mathbb{T}}$ to $\mathsf{Kl}(\mathbb{P})$. But it is important to note that the bijective correspondences of *strict* extensions with distributive laws of $\mathbb{T}$ over $\mathbb{P}$ and liftings $\tilde{\mathbb{P}}$ as alluded to in Section 2 extend to the lax environment, as follows. Given monads $\mathbb{P}, \mathbb{T}$ on $\mathcal{K}$ satisfying conditions (sup) and (map), *lax extensions* $\hat{\mathbb{T}}$ satisfying (ext) and (whi) are in bijective correspondence with each of the following:

- *Lax distributive laws* $\lambda : TP \longrightarrow PT$, given by lax natural transformations λ satisfying conditions (1), (2) of Section 2 laxly, as well as a monotonicity condition; hence, when we consider $\mathcal{K}$-morphisms $\varphi : X \longrightarrow PY$ always as morphisms $X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathsf{Kl}(\mathbb{P})$, the morphisms λ_X must satisfy
  - (0) $PTf \cdot \lambda_X \leq \lambda_Y \cdot TPf$, for all $f : X \longrightarrow Y$ in $\mathcal{K}$;
  - (1) $y_{TX} \leq \lambda_X \cdot Ty_X, \quad s_{TX} \cdot P\lambda_X \cdot \lambda_{PX} \leq \lambda_X \cdot Ts_X$;
  - (2) $Pe_X \leq \lambda_X \cdot e_{PX}, \quad Pm_X \cdot \lambda_{TX} \cdot T\lambda_X \leq \lambda_X \cdot m_{PX}$;
  - (3) whenever $\varphi \leq \psi$, then $\lambda_Y \cdot T\varphi \leq \lambda_Y \cdot T\psi$, for all $\varphi, \psi : X \longrightarrow PY$ in $\mathcal{K}$.
- *Lax liftings* $\tilde{\mathbb{P}}$ of $\mathbb{P}$, given by an assignment $\tilde{P}$ which gives for every strict $\mathbb{T}$-algebra (X, ξ) lax $\mathbb{T}$-algebras $(PX, \tilde{\xi}), (PPX, \tilde{\tilde{\xi}})$ (to be understood as in the diagrams at the end of Section 2), such that
  - (a) $y_X : (X, \xi) \longrightarrow (PX, \tilde{\xi}),\ s_X : (PPX, \tilde{\tilde{\xi}}) \longrightarrow (P, \tilde{\xi})$ are lax $\mathbb{T}$-homomorphisms;
  - (b) for every strict $\mathbb{T}$-homomorphism $f : (X, \xi) \longrightarrow (Y, \upsilon),\ \tilde{P}$ makes $Pf : (PX, \tilde{\xi}) \longrightarrow (PY, \tilde{\upsilon})$ a lax $\mathbb{T}$-homomorphism;
  - (c) for $f = \xi : (TX, m_X) \longrightarrow (X, \xi)$ in (b), Pf remains strict, that is: $P\xi \cdot \widetilde{m_X} = \tilde{\xi} \cdot TP\xi$.

Proofs for these bijective correspondences require a very careful re-examination of the proofs for the strict cases, the details of which must be largely left to the reader. For the first correspondence, details are to be found in [41, 24], albeit in more special environments. We note that condition (3) comes for free in a 2-categorical setting when T is a 2-functor. Likewise, lax liftings may also be described slightly more compactly in that setting.

4. Principal examples

Throughout this section, let V be a (unital, but not necessarily commutative) *quantale*, *i.e.*, a one-object quantaloid; hence, V is a complete lattice that comes with a monoid structure, so that its binary operation $\otimes$ distributes over arbitrary joins from either side; we denote the $\otimes$-neutral element in V by k, and $\bot$ is its bottom element.

Example 4.1. Let $\mathbb{P} = \mathbb{P}_{\mathsf{V}} = (\mathsf{P}_{\mathsf{V}}, \mathsf{s}, \mathsf{y})$ be the V-*powerset monad* on $\mathcal{K} = \mathbf{Set}$, given by

$$\mathsf{P}_{\mathsf{V}}X = \mathsf{V}^X, \quad \mathsf{P}_{\mathsf{V}}(X \xrightarrow{f} Y) = f_! : \mathsf{V}^X \longrightarrow \mathsf{V}^Y \text{ with } f_!(\sigma)(y) = \bigvee_{x \in f^{-1}y} \sigma(x),$$

$$\mathsf{s}_X : \mathsf{V}^{\mathsf{V}^X} \longrightarrow \mathsf{V}^X \text{ with } \mathsf{s}_X(\Sigma)(x) = \bigvee_{\sigma \in \mathsf{V}^X} \Sigma(\sigma) \otimes \sigma(x),$$

$$\mathsf{y}_X : X \longrightarrow \mathsf{V}^X \text{ with } \mathsf{y}(x)(y) = \left\{ \begin{array}{ll} \mathsf{k} & \text{if } y = x \\ \bot & \text{else} \end{array} \right\}.$$

A morphism $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathsf{Kl}(\mathbb{P}_{\mathsf{V}})$ may be described equivalently as a map $X \times Y \longrightarrow \mathsf{V}$, *i.e.*, as a V-valued relation from X to Y, and its Kleisli composite with $\psi : Y \longrightarrow\!\!\!\!\!\!\mapsto Z$ may then be computed as

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \varphi(x, y) \otimes \psi(y, z). \qquad (+)$$

Hence, $\mathsf{Kl}(\mathbb{P}_{\mathsf{V}})$ is the category of sets and V-relations as defined in [17], which here we denote by V^*-**Rel**; when one orders its hom-sets pointwise as in V, it becomes a quantaloid. Trivially, every map $f : X \longrightarrow Y$ gives the adjunction $f_* \dashv f^*$ with

$$f_*(x,y) = \left\{ \begin{array}{ll} \mathsf{k} & \text{if } f(x) = y \\ \bot & \text{else} \end{array} \right\} = f^*(y,x).$$

Hence, conditions (sup) and (map) of Definition 3.1 hold, and for a **Set**-monad $\mathbb{T} = (T, m, e)$ to satisfy conditions (ext), (whi) means precisely that the monad $\mathbb{T}$ comes, in the terminology of [17], with a lax extension $\hat{\mathbb{T}}$ to V^*-**Rel**. Moreover,

$$\mathbf{Alg}(\mathbb{T}, \mathbb{P}_{\mathsf{V}}) = (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat}$$

is precisely the category of (small) $(\mathbb{T}, \mathsf{V})$-*categories* $(X, \alpha : TX \longrightarrow\!\!\!\!\!\!\mapsto X)$ and their $(\mathbb{T}, \mathsf{V})$-*functors* $f : (X,\alpha) \longrightarrow (Y, \beta)$ as defined in [17], *i.e.*, structured sets X, Y and maps $f : X \longrightarrow Y$ required to satisfy

$$\mathsf{k} \leq \alpha(e_X(x), x), \quad \hat{T}\alpha(\mathfrak{X}, \mathfrak{y}) \otimes \alpha(\mathfrak{y}, z) \leq \alpha(m_X(\mathfrak{X}), z), \quad \alpha(\mathfrak{x}, y) \leq \beta(Tf(\mathfrak{x}), f(y)),$$

for all $x, y, z \in X, \mathfrak{x}, \mathfrak{y} \in TX, \mathfrak{X} \in TTX$. Of course, for $\mathbb{T} = \mathbb{I}$ the identity monad (identically extended to V-**Rel**), $(\mathbb{T}, \mathsf{V})$-**Cat** $=$ V-**Cat**. Less trivially, when the complete lattice V is completely distributive and $\mathbb{T}$ is the ultrafilter monad $\mathbb{U}$ (as induced by the adjunction that has the forgetful **CompHaus**$\longrightarrow$**Set** as a right adjoint), then one has a lax extension $\overline{\mathbb{U}}$, usually named after Barr [3], whose lax functor $\overline{\mathsf{U}} : \mathsf{V}^*\text{-}\mathbf{Rel} \longrightarrow \mathsf{V}^*\text{-}\mathbf{Rel}$, given by

$$\overline{\mathsf{U}}\varphi(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{X}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} \varphi(x, y),$$

for all V-relations $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y, \mathfrak{x} \in \mathsf{U}X, \mathfrak{y} \in \mathsf{U}Y$, actually turns out to be a genuine functor; for proofs, see [22].

We mention here the protaganistic examples for V, which have been discussed earlier in various iterations in more restrictive contexts (see [17]), namely the two-element chain 2, the Lawvere quantale $([0, \infty], +, 0)$ (see [26]), and the quantale $\mathbf{\Delta}$ of all distribution functions $[0, \infty] \longrightarrow [0, 1]$ (which, as noted in [41], is nothing but the coproduct of two copies of the Lawvere quantale in the category of commutative quantales). For these three quantales, with $\mathbb{T} = \mathbb{I}$ one obtains respectively the categories **Ord, Met, ProbMet** of (pre)ordered sets, (generalized) metric spaces, and (generalized) probabilistic metric spaces, respectively, and with $\mathbb{T} = \mathbb{U}$ the categories **Top, App, ProbApp** of topological spaces, approach spaces ([28]), and probabilistic approach spaces, respectively. We refer to [22] for details.

Remark 4.2. In [41] and other recent papers, V-**Rel** denotes the category of sets and V-relations with the composition rule

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \otimes \varphi(x, y). \qquad (*)$$

With $*$ referring to the interchange of the arguments in the tensor product of V, one then has

$$\mathsf{V}^*\text{-}\mathbf{Rel} \cong (\mathsf{V}\text{-}\mathbf{Rel})^{\mathrm{op}}.$$

Using $(*)$ rather than $(+)$ of 4.1 makes it more natural to consider $\varphi(x, y), \psi(y, z)$ as morphisms and $\otimes$ as a category composition, a point that becomes relevant when one replaces the quantale V more generally by a quantaloid $\mathcal{Q}$, as has been done in [41] and elsewhere. Hence in, what follows, we will use $(*)$, rather than $(+)$, *i.e.*, work with V-**Rel** rather than V^*-**Rel**. Of course, this makes no difference to the concrete quantales considered in 4.1, namely $2, [0, \infty]$ and $\mathbf{\Delta}$, as these are all commutative.

We are ready to consider the non-discrete version of Example 4.1.

Example 4.3. Let $\mathcal{K}$ be the 2-category V-**Cat**, and let V-**Mod** denote the category of (small) V-categories and their V-*(bi)modules* $\varphi : (X,\alpha) \to\!\!\!\!\!\!\!\;+\; (Y,\beta)$, *i.e.*, the V-relations $\varphi : X \to\!\!\!\!\!\!\!\;+\; Y$ with $\beta\circ\varphi\circ\alpha \leq \varphi$. With the V-relational composition $(*)$ one obtains a quantaloid, in which the identity morphism 1^*_X on $X = (X,\alpha)$ is given by α. More generally, for a V-functor $f : (X,\alpha) \longrightarrow (Y,\beta)$, one has the adjunction $f_* \dashv f^* : (Y,\beta) \to\!\!\!\!\!\!\!\;+\; (X,\alpha)$ in V-**Mod**, where

$$f_*(x,y) = \beta(fx,y) \quad \text{and} \quad f^*(y,x) = \beta(y,fx),$$

for all $x \in X, y \in Y$. The quantale V itself becomes a V-category with its internal hom $[v,w]$ as the V-structure, defined by $(u \leq [v,w] \iff u \otimes v \leq w)$, for all $u,v,w \in \mathsf{V}$. Every hom-set V-**Mod**(X,Y) inherits that structure when one puts

$$[\varphi,\psi] = \bigwedge_{x\in X, y\in Y} [\varphi(x,y),\psi(x,y)].$$

In particular, for $Y = \mathsf{E}$ the singleton-set generator $(\{*\},\mathsf{k})$ of V-**Cat**, we obtain the V-category

$$\mathsf{P}X = \mathsf{P}_{\mathsf{V}}X = \mathsf{V}\text{-}\mathbf{Mod}(X,\mathsf{E})$$

of V-*presheaves* σ on $X = (X,\alpha)$, *i.e.*, of those $\sigma \in \mathsf{V}^X$ satisfying $\sigma(y) \otimes \alpha(x,y) \leq \sigma(x)$, for all $x,y \in X$. In a natural way, P becomes an endo-2-functor of V-**Cat**, with $\mathsf{P}f = f_! : \mathsf{P}X \longrightarrow \mathsf{P}Y, \sigma \mapsto \sigma\circ f^*$ for all $f : X \longrightarrow Y$, and even a 2-monad $\mathbb{P} = (\mathsf{P},\mathsf{s},\mathsf{y})$, with

$$\mathsf{y}_X : X \longrightarrow \mathsf{P}X, \quad x \mapsto 1^*_X(-,x), \text{ and } \mathsf{s}_X : \mathsf{P}\mathsf{P}X \longrightarrow \mathsf{P}X, \quad \Sigma \mapsto \Sigma\circ(\mathsf{y}_X)_*.$$

It is well known (and verified directly in [40]) that the Kleisli category of $\mathbb{P}$ is precisely V^*-**Mod**, the dual of V-**Mod**.

In [23] we studied lax extensions of 2-monads $\mathbb{T}$ on V-**Cat** to V-**Mod**, required to satisfy the conditions (ext) and (whi) of Definition 3.1, and showed that there is a rich supply of these. To start with, given any **Set**-monad with a lax extension to V-**Rel** as in Example 4.1, one may extend that monad to a 2-monad on V-**Cat** (by applying the lax extension to the structure of the V-category, see [39, 17]), which then itself allows for a lax extension to V-**Mod**, satisfying (ext) and (whi). Beyond these examples that arise from the discrete setting, in [24] we have given four examples of 2-monads $\mathbb{T}$ on V-**Cat** which allow for lax extension to V-**Mod**, describing also the categories $\mathsf{Alg}(\mathbb{T},\mathbb{P})$ in each case, namely for $\mathbb{T}$ the V-presheaf 2-monad $\mathbb{P}$ itself, the copresheaf 2-monad $\mathbb{P}^\dagger$, the double presheaf 2-monad $\mathbb{P}\mathbb{P}^\dagger$, or the double copresheaf 2-monad $\mathbb{P}^\dagger\mathbb{P}$. Moreover, in [23], we showed that the restriction to conical (co)presheaves, which defines the *Hausdorff* submonads of $\mathbb{P},\mathbb{P}^\dagger$ as previously considered in [2, 38], admit lax extensions as well; and when V is completely distributive, such lax extensions exist also for the two iterated monads.

Remark 4.4. *All statements of* Example 4.3 *remain valid if the quantale* V *is traded for a (small) quantaloid* $\mathcal{Q}$. For details we refer to [24, 23].

5. Representing $\mathbb{P}$

Our next goal is to simplify (and thereby specialize) the notion of topological platform, by representing the monad $\mathbb{P}$ and the lax extension $\hat{\mathbb{T}}$ in a convenient manner. With respect to a representation of $\mathbb{P}$, we are guided by Example 4.3, where V-**Mod**$(\mathsf{E},\mathsf{E}) \cong \mathsf{V}$, with E the singleton-set generator of V-**Cat**. The following theorem, adapted from [40], gives general conditions under which Example 4.3 turns out to be rather all-encompassing.

Theorem 5.1. *Let* $\mathbb{P} = (P,m,e)$ *be a monad on a category* $\mathcal{K}$ *satisfying conditions* (sup) *and* (map) *of* Definition 3.1. *If there is an object* E *in* $\mathcal{K}$ *with* $|\mathcal{K}(E,E)| = 1$ *and*

$$\bigvee_{x\in\mathcal{K}(E,X)} x_*\circ x^* = 1^*_X$$

for all X in $\mathcal{K}$, then there is a quantale V and a functor $|\text{-}| : \mathcal{K} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat}$ which may be extended to a full and faithful homomorphism $|\text{-}| : \mathsf{Kl}(\mathbb{P}) \longrightarrow \mathsf{V}\text{-}\mathbf{Mod}$ of quantaloids such that

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\;|\text{-}|\;} & \mathsf{V}\text{-}\mathbf{Cat} \\ {\scriptstyle (\text{-})_*}\downarrow & & \downarrow{\scriptstyle (\text{-})_*} \\ \mathsf{Kl}(\mathbb{P}) & \xrightarrow[\;|\text{-}|\;]{} & \mathsf{V}\text{-}\mathbf{Mod} \end{array}$$

commutes.

Proof. (Sketch—for details see [40], Theorem 3.1.) Since $\mathsf{Kl}(\mathbb{P})$ is a quantaloid, the monoid $\mathsf{V} := \mathsf{Kl}(\mathbb{P})(E, E)$ becomes a quantale. The hom-functor $|\text{-}| := \mathcal{K}(E, \text{-})$ takes values in $\mathsf{V}\text{-}\mathbf{Cat}$ when one puts $1^*_{|X|}(x, y) = y^* \circ x_*$ for all $x, y \in |X|$, and it extends to a homomorphism of quantaloids when one puts $|\varphi|(x, y) := y^* \circ \varphi \circ x_*$ for all $\varphi : X \longrightarrow\!\!\!\!\!\!\!\!\circ\;\; Y$ in $\mathsf{Kl}(\mathbb{P})$ and $x \in |X|, y \in |Y|$. The homomorphism turns out to be fully faithful with respect to both, 1-cells and 2-cells. □

Remark 5.2. (1) The paper [40] elaborates on contexts in which the assignment $(\mathbb{P} \mapsto \mathsf{V})$ as produced by Theorem 5.1 may be considered as a retraction to a functor which assigns to every quantale V its presheaf monad $\mathbb{P}_{\mathsf{V}}$. In fact, the paper produces restricted environments in which the former assignment gives a left adjoint to the latter.

(2) The consideration of an object E with the properties required by Theorem 5.1 (in particular that $\mathcal{K}(E, E)$ must be a singleton) is somewhat restrictive. For example, the Theorem may generally not be applied to the presheaf monad on $\mathcal{K} = \mathcal{Q}\text{-}\mathbf{Cat}$, for a small quantaloid $\mathcal{Q}$ that is not a quantale (see the note at the end of Section 4). However, there is a natural way of generalizing the Theorem, as follows. In an ordered category $\mathcal{K}$ with a terminal object 1 and a monad $\mathbb{P}$ satisfying (sup) and (map), instead of a single object E one may assume to have a representative set of *atoms* of 1, that is, of the minimal strong subobjects of 1, and treat that set as a small full subquantaloid $\mathcal{Q}$ of the Kleisli category of P. If the identity morphisms of $\mathsf{Kl}(\mathbb{P})$ allow for a sup-representation as in the Theorem, but with x now ranging over all $\mathcal{K}$-morphisms $A \longrightarrow X$, where A is an atom of 1, then the assertion of the Theorem remains true in "quantaloidic form". The details are to be given in joint work with H. Lai.

6. Representing $\mathbb{T}$

Theorem 5.1 affirms that, with a rather special object E in $\mathcal{K}$, the monad $\mathbb{P} = (P, s, y)$ of a topological platform may be largely retrieved from the quantalic structure on the set $|PE| = \mathcal{K}(E, PE)$. We propose the following definition, in order to achieve a similar concentration of information for the lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$, keeping in mind that such extensions correspond to certain lax liftings $\tilde{\mathbb{P}}$ of $\mathbb{P}$, as described in Remark 3.4. As has been done there, we consider $\mathcal{K}$-morphisms $\varphi : X \longrightarrow PY$ always as morphisms $X \longrightarrow\!\!\!\!\!\!\!\!\circ\;\; Y$ in $\mathsf{Kl}(\mathbb{P})$, and conversely. For $f : X \longrightarrow Y$ in $\mathcal{K}$, under conditions (sup) and (map) of Definition 3.1 we put

$$f^! := s_X \cdot (Pf)^* : PY \longrightarrow PX.$$

Definition 6.1. A *quantalic topological theory* on a category $\mathcal{K}$ is given by monads $\mathbb{T} = (T, m, e)$ and $\mathbb{P} = (P, s, y)$ on $\mathcal{K}$ satisfying (sup) and (map), and an object E equipped with a morphism $\tau : PE \longrightarrow E$ in $\mathcal{K}$, such that

- E carries a strict $\mathbb{T}$-algebra structure ζ, and PE a lax $\mathbb{T}$-algebra structure ξ, making $\tau : (PE, \xi) \longrightarrow (E, \zeta)$ a strict $\mathbb{T}$-homomorphism;
- $y_E : (E, \zeta) \longrightarrow (PE, \xi)$ and $s_E : (PPE, \theta) \longrightarrow (PE, \xi)$ are lax $\mathbb{T}$-homomorphisms, with the induced lax $\mathbb{T}$-algebra structure $\theta := P\xi \cdot (\zeta \cdot T\tau)^! \cdot \xi \cdot TP\tau$ on PPE;

- whenever $\varphi \le \psi$, then $\xi \cdot T\varphi \le \xi \cdot T\psi$, for all $\varphi, \psi : X \longrightarrow PE$ in $\mathcal{K}$.

Let us discuss this definition in the context of Example 4.1.

Example 6.2. For a quantale $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$, let $\mathbb{P} = \mathbb{P}_{\mathsf{V}}$ as in 4.1. With $E = \{*\}$ terminal in $\mathcal{K} = \mathbf{Set}$, for any monad $\mathbb{T}$, E carries a unique $\mathbb{T}$-algebra structure, and there is a unique map $\tau : \mathsf{V} = \mathsf{P}_{\mathsf{V}}E \longrightarrow E$. Now the provision of a quantalic topological theory requires giving just a map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ such that, in the pointwise order inherited from V, one has:

- (V, ξ) is a lax $\mathbb{T}$-algebra, that is: $1_{\mathsf{V}} \le \xi \cdot e_{\mathsf{V}},\ \xi \cdot T\xi \le \xi \cdot m_{\mathsf{V}}$;
- the maps $\mathsf{y}_E = \mathsf{k} : E \longrightarrow \mathsf{V},\ * \mapsto \mathsf{k}$, and $\mathsf{s}_E : \mathsf{V}^{\mathsf{V}} \longrightarrow \mathsf{V},\ \Sigma \mapsto \bigvee_{u \in \mathsf{V}} \Sigma(u) \otimes u$, are lax $\mathbb{T}$-homomorphisms, where V^{V} is provided with the lax $\mathbb{T}$-algebra structure
$$\theta = \xi_! \cdot (\zeta \cdot T\tau)^! \cdot \xi \cdot T(\tau_!) : T(\mathsf{V}^{\mathsf{V}}) \longrightarrow \mathsf{V}^{\mathsf{V}};$$
- whenever $\varphi \le \psi$, then $\xi \cdot T\varphi \le \xi \cdot T\psi$, for all maps $\varphi, \psi : X \longrightarrow \mathsf{V}$.

One calls such quantalic topological theory *natural* if

- $\xi_X(\varphi) := \xi \cdot T\varphi \quad (\varphi \in \mathsf{P}_{\mathsf{V}}X = \mathsf{V}^X)$ defines a natural transformation $\mathsf{P}_{\mathsf{V}} \longrightarrow \mathsf{P}_{\mathsf{V}}T$.

It is time to compare the natural quantalic topological theories with Hofmann's [16] notion of topological theory. Given the monad $\mathbb{T}$ on **Set** and a *commutative* quantale V, he requires the map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ to satisfy the same conditions as above, except that the somewhat cumbersome condition that $\mathsf{s}_E : (\mathsf{V}^{\mathsf{V}}, \theta) \longrightarrow (\mathsf{V}, \xi)$ be a lax $\mathbb{T}$-homomorphism gets traded for the condition that the monoid operation $\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ be a lax $\mathbb{T}$-homomorphism; here the domain carries the standard product structure as inherited from (V, ξ). On first sight, this difference may appear to be minor, but conceptually it is not. While the notion of natural quantalic topological theory emerges from a setting which emphasizes the role of the presheaf monad induced by V, Hofmann's notion emphasizes the role of the monoid structure of V from a universal-algebraic perspective.

It is therefore not surprising that proving the equivalence of the two concepts takes considerable effort, and it requires in any case some degree of well-behaviour of the monad $\mathbb{T}$ vis-a-vis cartesian structures: the endofunctor T of $\mathbb{T}$ needs to satisfy the *Beck-Chevalley Condition (BC)*, which here means that the **Set**-functor T needs to preserve weak pullback diagrams. One obtains the following equivalence theorem, established in [41]:

Theorem 6.3. *For a commutative quantale* V *and a* **Set***-monad* $\mathbb{T} = (T, m, e)$ *with* T *satisfying BC, the natural quantalic topological theories are precisely Hofmann's topological theories.*

Proof. (Sketch—for details see Theorem 8.2 of [41].) One can prove rather directly (and with modest effort) that, when $s_E : \mathsf{V}^{\mathsf{V}} \longrightarrow \mathsf{V}$ is a lax $\mathbb{T}$-homomorphism, so is $\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$, by employing the map

$$\chi : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}^{\mathsf{V}}, \quad \chi(u, v)(w) = u \otimes (\mathsf{y}_{\mathsf{V}} v)(w) = \left\{ \begin{array}{ll} u & \text{if } w = v, \\ \bot & \text{else} \end{array} \right\},$$

which satisfies $\mathsf{s}_E \cdot \chi = \otimes$. For the converse proposition, starting with a Hofmann theory ξ, one considers the lax extension $\mathbb{T}_\xi$ of $\mathbb{T}$ to V-**Rel**, as given by Hofmann [16], Definition 3.4:

$$(T_\xi \varphi)(\mathfrak{x}, \mathfrak{y}) = \bigvee \{(\xi \cdot T|\varphi|)(\mathfrak{w}) \mid \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y}\}, \qquad (\dagger)$$

for all V-relations $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y,\ \mathfrak{x} \in TX, \mathfrak{y} \in TY$, with product projections π_1, π_2 and $|\varphi| : X \times Y \longrightarrow \mathsf{V}$ denoting the underlying map of φ. But, as shown in [41], Theorem 5.5, every lax extension of $\mathbb{T}$ (in the guise of its corresponding lax distributive law) induces a quantalic topological theory which, when the lax extension is $\mathbb{T}_\xi$, turns out to return the same ξ. $\square$

Example 6.4. In Example 6.2 one may trade the quantale V for a small quantaloid $\mathcal{Q}$ and consider again its (discrete) preseheaf monad $\mathbb{P} = \mathbb{P}_{\mathcal{Q}}$. With the set E now the object set of $\mathcal{Q}$, which must carry a strict $\mathbb{T}$-algebra structure ζ, and with τ the *extent* (or *type* map) of the $\mathcal{Q}$-category $\mathsf{P}E$, a quantic topological theory is again given by a map $\xi : T\mathsf{P}E \longrightarrow \mathsf{P}E$ over E satisfying the same conditions as in Example 6.2 (with V traded for $\mathsf{P}E$ and V^{V} for $\mathsf{PP}E$, and "map" to mean "map over E"), and making $\tau : \mathsf{P}E \longrightarrow E$ a strict $\mathbb{T}$-homomorphism. Theses are precisely the topological theories as defined in [41], Definition 5.4.

Remark 6.5. (1) As shown in [16], the Barr extension $\overline{\mathsf{U}}$ of the ultrafilter monad as discussed in Example 4.1 is often induced by a topological theory ξ via formula (†), so that $\overline{\mathsf{U}} = \mathsf{U}_{\xi}$, leading in particular to the presentation of **Top** and **App** as categories of (U, V)-categories. But, as clarified further by their characterizations given in [10, 41], lax extensions of type T_{ξ} are rather special.

(2) There does not seem to be a natural way of extending Hofmann's notion of topological theory from its monoidal context of a quantale to the bicategorical context of a quantaloid, as has been done for quantic topological theories when moving from Example 6.2 to Example 6.4.

(3) A discussion of quantalic topological theories in the context of Example 4.3 when $\mathcal{K} = \mathsf{V}$-**Cat** must appear elsewhere.

Let us finish by admitting that we do not know which topological categories over $\mathcal{K}$ are presentable in the form $\mathsf{Alg}(\mathbb{T}, \mathbb{P})$ for some topological platform $(\mathbb{T}, \mathbb{P})$, even when $\mathcal{K} = \mathbf{Set}$. Also in the much narrower context of Example 4.1 the question has been left unanswered for quite some time, except that one knows that, under a slight restriction of the admissible lax extensions of $\mathbb{T}$, the question which topological categories are presentable in the form $(\mathbb{T}, \mathsf{V})$-**Cat** may be reduced to the case $\mathsf{V} = 2$: see Corollary IV.3.2.3 of [17].

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