Met-like categories amongst concrete topological categories

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Dedicated in gratitude to Bob Lowen, founder of *Applied Categorical Structures*

Abstract

When replacing the non-negative real numbers with their addition by a commutative quantale V, under a metric lens one may then view small V-categories as sets that come with a V-valued distance function. The ensuing category V-**Cat** is well known to be a concrete topological category that is symmetric monoidal closed. In this paper we show which concrete symmetric monoidal-closed topological categories may be fully and bireflectively embedded into V-**Cat**, for some V.

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1. Introduction

By a **Met**-like category we mean a concrete category that, as such, is fully and reflectively embeddable into the symmetric monoidal-closed category V-**Cat** of small V-categories, for a commutative and unital *quantale* $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$ (see [29]); here k is neutral with respect to the binary monoidal structure $\otimes$ which, in each variable, distributes over arbitrary joins in the complete lattice V. The prototypical quantale is the Lawvere [21] quantale $\mathbb{R}_+ = (([0,\infty], \geq), +, 0)$, which produces the category **Met** $= \mathbb{R}_+$-**Cat** of generalized metric spaces (X, d), *i.e.*, of sets X which come with a distance function $d : X \times X \longrightarrow [0,\infty]$ satisfying the triangle inequality and making self-distances zero; morphisms $f : (X, d) \longrightarrow (Y, d')$ are contractive maps: $d'(fx, fy) \leq d(x, y)$, for all $x, y \in X$.

For any quantale V, the forgetful functor V-**Cat**$\longrightarrow$**Set** is easily seen to be *topological*, regardless of which flavour of the notion ([14, 1, 2, 11]) one chooses. The principal goal of this paper is to answer the question of how to recognize full bireflective subcategories of such quantale-induced topological categories amongst all those concrete topological categories which, like V-**Cat**, come with a *symmetric monoidal-closed structure* (see [20]). Assuming that singleton sets carry only one structure in the given concrete topological category $\mathcal{X}$, we find that the $\mathcal{X}$-fibre of a doubleton set plays a pivotal role in trying to answer the question. The fibre $\mathcal{X}(A)$ of a doubleton A inherits from $\mathcal{X}$ a monoidal structure, making it a quantale whenever a rather special compatibility property for the tensor product and suprema holds in that fibre. The fibre $\mathcal{X}(A)$ contains a subquantale, $\mathcal{X}_A$, defined by a symmetry condition, which, in the case $\mathcal{X} = \mathsf{V}$-**Cat**, fully recovers the quantale V from the category. In this way we obtain the first of four theorems about the interaction of quantales and “convenient” concrete topological categories, stating in a categorically satisfactory manner that the totality of the latter entities contain the former as a (pseudo-)retract (see Theorem 4.1).

But the “global” functors involved in this presentation of quantales as a retract of topological categories fail to be adjoint. In Theorem 5.5 we show that this defect may be overcome by

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restricting the array of topological categories to those satisfying a natural transitivity condition, at the expense of having to consider the entire fibre $\mathcal{X}(A)$ of the doubleton A, rather than its nicer subquantale $\mathcal{X}_A$. Avoidance of the transitivity condition is possible by consideration of the passage from a quantale V to the larger topological category V-**Gph** of V-*graphs*, rather than to V-**Cat**. In Theorem 6.2 we show that, when composing this adjunction with the coreflector that assigns to a topological category its *transitive core*, one recoups the principal adjunction as considered earlier in Theorem 5.5.

The units of that principal adjunction give us a way of comparing the topological category $\mathcal{X}$ with the category $\mathcal{X}(A)$-**Cat**, which leads us to the Representation Theorem 7.2. Amongst the topological categories $\mathcal{X}$ considered previously, the ones that are bireflectively embeddable into a category V-**Cat** for some V are characterized as those for which the fibre $\mathcal{X}(A)$ of a doubleton set A is *finally dense* in $\mathcal{X}$, so that every object is the codomain of a final family of morphisms whose domains are two-element objects. Furthermore, the fibre $\mathcal{X}(A)$ may be traded for the subquantale $\mathcal{X}_A$ when $\mathcal{X}$ satisfies a natural symmetry condition.

The results offered in this article can only be the beginning of a much larger program which aims at characterizing the quantale-monad-enriched categories $(\mathbb{T},\mathsf{V})$-**Cat** as studied in, for example, [9, 7, 8, 30, 15, 17, 31]. The most prominent and essential example of these is the category **App** of *approach spaces* (see [23]), introduced and intensively studied by Bob Lowen and his school and collaborators, in a large array of books, theses and research articles.

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2. Quantales induce concrete symmetric monoidal-closed topological categories

We consider a category $\mathcal{X}$ which comes with a faithful and amnestic functor to **Set** and allow ourselves to write the $\mathcal{X}$-objects suggestively as pairs (X,ξ), where X is the underlying set and ξ the "structure" of the object, which is —strictly speaking— the $\mathcal{X}$-object itself. In this notation, the *fibres*

$$\mathcal{X}(X)=\{\xi\,|\,(X,\xi)\in\mathcal{X}\}$$

of the faithful **Set**-functor of $\mathcal{X}$ carry a preorder $\leq$, whereby one writes $\xi\leq\upsilon$ when $\mathrm{id}_X:(X,\xi)\longrightarrow(X,\upsilon)$ is a morphism, and its amnesticity amounts precisely to $\leq$ being antisymmetric. Recall that $\mathcal{X}$ is *(concretely) topological* if, for every (possibly large or empty) family of **Set**-maps $f_i:X\longrightarrow Y_i$ and $\mathcal{X}$-structures $\upsilon_i\in\mathcal{X}(Y_i)$ $(i\in I)$, there is $\xi\in\mathcal{X}(X)$ (the *initial* $\mathcal{X}$-structure on X w.r.t. the given data) such that, for all $h:Z\longrightarrow X$ and $\zeta\in\mathcal{X}(Z)$, one has

$$h:(Z,\zeta)\longrightarrow(X,\xi)\text{ in }\mathcal{X}\iff\forall i\in I:(f_i\cdot h:(Z,\zeta)\longrightarrow(Y_i,\upsilon_i)\text{ in }\mathcal{X}.$$

Topologicity is known to be equivalent to the faithful **Set**-functor being a *bifibration* and having large-complete fibres, and it is therefore in particular a self-dual property (see, for example, [17]). Consequently, for a map $f:X\longrightarrow Y$ one has a pair of adjoint monotone maps

$$f_!\dashv f^*:\mathcal{X}(Y)\longrightarrow\mathcal{X}(X),$$

with f^* assigning to $\upsilon\in\mathcal{X}(Y)$ the (initial or) *Cartesian* $\mathcal{X}$-structure on X, and with $f_!$ performing the dual operation. In this notation, a **Set**-map $f:X\longrightarrow Y$ gives an $\mathcal{X}$-morphism $f:(X,\xi)\longrightarrow(Y,\upsilon)$ if, and only if, $\xi\leq f^*(\upsilon)$. Note also that $\mathcal{X}\longrightarrow$ **Set** has both, a right and a left adjoint, obtained by assigning to a set X, considered as an empty family of morphisms with domain or codomain X, its initial or *final* (= dual to initial) structure, respectively. In order to avoid the engagement into any further size considerations, we will assume throughout that *all of our concrete topological categories have small fibres.*

The concrete topological category $\mathcal{X}$ is *(concretely) symmetric monoidal closed* if it carries a symmetric monoidal structure which is closed and mapped to the Cartesian structure of **Set** by the topological functor $\mathcal{X} \longrightarrow \mathbf{Set}$; moreover, the unit (E, η) with respect to the binary monoidal structure $\otimes$ on $\mathcal{X}$ must be discrete, *i.e.*, be obtained by applying the left adjoint of $\mathcal{X} \longrightarrow \mathbf{Set}$ to the singleton set $E = \{*\}$. As a consequence of this last provision, every point $x : E \longrightarrow X$ of the set X corresponds to a morphism $x : (E, \eta) \longrightarrow (X, \xi)$. Using square brackets to denote the internal hom-objects in $\mathcal{X}$, we adopt the notation

$$(X, \xi) \otimes (Y, \upsilon) = (X \times Y, \xi \otimes \upsilon) \text{ and } [(X, \xi), (Y, \upsilon)] = (\mathcal{X}((X, \xi), (Y, \upsilon)), [\xi, \upsilon]);$$

note that the discreteness of the $\otimes$-unit (E, η) indeed allows us to take as the underlying set of the internal hom-object $[(X, \xi), (Y, \upsilon)]$ the set of all those maps $X \longrightarrow Y$ which give morphisms $(X, \xi) \longrightarrow (Y, \upsilon)$.

Remark 2.1. (1) The condition that the monoidal structure of our concretely symmetric monoidal-closed category should rest on the Cartesian structure of **Set** entails no restriction of generality. In fact, while [4] refers to an unpublished paper by J. Niederle in this context, the papers [22, 26] show that every (not necessarily symmetric) monoidal-closed structure on a large array of concrete categories, including all topological categories in which constant maps are morphisms, must rest on the Cartesian structure of **Set**. Consequently, we may drop the prefix "concretely" when referring to monoidal-closed structures in the context of our topological categories over **Set**.

(2) We note that, while **Top** admits only one symmetric monoidal-closed structure (see Example 3.2(3) below), there is a proper class of non-symmetric closed structures on **Top** (see [12, 13]). Alternative construction techniques for monoidal-closed structures on **Top** are presented in [25]. For subsequent contributions see [6], which, among other things, gives an example of a coreflective subcategory of the category of Hausdorff spaces with a proper class of symmetric monoidal-closed structures.

(3) Here is an easy example of a Cartesian closed and concretely topological category, **Sub**, for which the discrete structure on the singleton set E does not give a terminal (and, hence, $\times$-neutral) object. As a consequence, the underlying set of an internal hom-object is generally not given by the hom-set in the category: objects of **Sub** are pairs (X, U) of sets wth $U \subseteq X$, and morphisms $f : (X, U) \longrightarrow (Y, V)$ are maps $f : X \longrightarrow Y$ with $f(U) \subseteq V$; then $[(X, U), (Y, V)] = (\mathbf{Set}(X, Y), \mathbf{Sub}((X, U), (Y, V)))$.

For an object (A, α) in the concretely symmetric monoidal-closed category $\mathcal{X}$, the left-adjoint functor $(A, \alpha) \otimes (-) : \mathcal{X} \longrightarrow \mathcal{X}$ generally fails to preserve final families, even though it preserves all colimits; but it does preserve finality when all underlying maps of the family are identity maps, so that final structures are given by joins in the fibres, as follows:

Proposition 2.2. *For all sets A, X and every $\alpha \in \mathcal{X}(A)$, the map $\alpha \otimes (\text{-}) : \mathcal{X}(X) \longrightarrow \mathcal{X}(A \times X)$ preserves arbitrary joins.*

Proof. For $\xi = \bigvee_{i \in I} \xi_i$ in $\mathcal{X}(X)$, one concludes $\bigvee_{i \in I} \alpha \otimes \xi_i = \alpha \otimes \xi$ in $\mathcal{X}(A \times X)$ by adjunction:

$$\alpha \otimes \xi_i \leq \upsilon \ (i \in I) \iff \xi_i \leq [\alpha, \upsilon] \ (i \in I) \iff \xi \leq [\alpha, \upsilon] \iff \alpha \otimes \xi \leq \upsilon.$$

□

Quantales as alluded to in the Introduction provide the fundamental examples of concretely symmetric monoidal-closed topological categories, as follows.

Example 2.3. For a unital and commutative quantale $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$, the objects (X, a) of the category V-**Cat** of (small) V-*categories* and V-*functors* are sets X with a map $a : X \times X \longrightarrow \mathsf{V}$ satisfying the reflexivity and transitivity conditions

$$\mathsf{k} \leq a(x, x) \quad \text{and} \quad a(x, y) \otimes a(y, z) \leq a(x, z)$$

for all $x, y, z \in \mathcal{X}$; morphisms $f : (X, a) \longrightarrow (Y, b)$ are maps $f : X \longrightarrow Y$ satisfying $a(x, y) \leq b(fx, fy)$, for all $x, y \in X$. As standard examples, with V the two-element chain $\mathsf{2}$ and the Lawvere quantale, one obtains the categories **Ord** and **Met** of (pre)ordered sets and of generalized metric spaces, respectively. We refer to the extensive literature on other relevant quantales V and their categories V-**Cat** of small V-enriched categories, such as the quantale $\boldsymbol{\Delta}$ of left-continuous distribution functions with its category $\boldsymbol{\Delta}$-V-**Cat** = **ProbMet** of *probabilistic metric spaces*; see, for example [3, 17, 16, 19, 31]. Note also, that every frame (= complete lattice in which finite infima distribute over arbitrary suprema) becomes a quantale, with $\otimes = \wedge$; in the case of the lattice $\mathsf{V} = ([0, \infty], \geq)$ one obtains the category V-**Cat** = **UMet** of generalized ultrametric spaces.

In general, the category V-**Cat** is known to be topological, with the initial structure a on X with respect to a family $(f_i : X \longrightarrow Y_i,\, b_i \in \mathsf{V}\text{-}\mathbf{Cat}(Y_i))_{i \in I}$ given by

$$a(x, y) = \bigwedge_{i \in I} b_i(f_i x, f_i y).$$

The symmetric monoidal-closed structure of V-**Cat** is fully described by

$$(X, a) \otimes (Y, b) = (X \times Y, a \otimes b), \quad (a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y'),$$

$$[(X, a), (Y, b)] = (\mathsf{V}\text{-}\mathbf{Cat}((X, a), (Y, b)), [a, b]), \quad [a, b](f, g) = \bigwedge_{z \in X} b(fz, gz),$$

with the discrete (E, k) serving as the $\otimes$-neutral object in V-**Cat**.

Here are two properties of V-**Cat** which are generally not available for concretely symmetric monoidal-closed topological categories:

Proposition 2.4. *Let* V *be a unital and commutative quantale and* $(X, a) \in \mathsf{V}\text{-}\mathbf{Cat}$. *Then:*

(1) *The functor* $(X, a) \otimes (-) : \mathsf{V}\text{-}\mathbf{Cat} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat}$ *preserves Cartesian morphisms.*

(2) *If* X *has at most two elements and* δ *denotes the diagonal map* $X \longrightarrow X \times X$, *then*

$$\delta^*(a \otimes \bigvee_{i \in I} b_i) = \bigvee_{i \in I} \delta^*(a \otimes b_i) \qquad (*)$$

for all $b_i \in \mathsf{V}\text{-}\mathbf{Cat}(X), i \in I$.

Proof. (1) For $f : (Y, b) \longrightarrow (Z, c)$ with $b = f^*(c)$ one has

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes c(fy, fy') = (a \otimes c)((\mathrm{id}_X \times f)(x, y), (\mathrm{id}_X \times f)(x', y'))$$

for all $x, x' \in X$, $y, y' \in Y$ and, hence, $a \otimes b = (\mathrm{id}_X \times f)^*(a \otimes c)$.

(2) For $|X| \leq 2$, the join of any family $(b_i)_{i \in I}$ in V-**Cat**(X) may be computed pointwise in the quantale V, where $\otimes$ distributes over joins. Consequently, for all $x, y \in X$ one has

$$\delta^*(a \otimes \bigvee_{i \in I} b_i)(x, y) = a(x, y) \otimes \bigvee_{i \in I} b_i(x, y) = \bigvee_{i \in I} a(x, y) \otimes b_i(x, y) = \bigvee_{i \in I} \delta^*(a \otimes b_i)(x, y).$$

□

Remark 2.5. Considering $\mathsf{V} = \mathsf{2}$ and $|I| = 2$, one easily sees that $(*)$ of Proposition 2.4(2) generally fails when X has at least three elements, since then the pointwise join of V-category structures on X will generally not be a V-category structure again. Likewise, even in the case $|X| = 2 = |I|$, the map δ^* generally does not commute with joins, *i.e.*, the identity $(*)$ fails if the structures $a \otimes b_i$ in $(*)$ get replaced by arbitrary V-**Cat**-structures c_i on $X \times X$.

3. Concretely symmetric monoidal-closed topological categories induce quantales

The properties observed in Proposition 2.4 turn out to be essential for recognizing quantale-induced topological categories. In what follows, we fix a two-element set $A = \{0,1\}$ and consider the natural maps

$$\delta : A \longrightarrow A \times A \quad \text{and} \quad ! : A \longrightarrow E.$$

Definition 3.1. We call a symmetric monoidal-closed topological category $\mathcal{X}$ *prequantalic* if, in the notation of Section 2,

(1) for every object $(X,\xi) \in \mathcal{X}$, the functor $(X,\xi)\otimes(-) : \mathcal{X}\longrightarrow\mathcal{X}$ preserves Cartesian morphisms,

(2) $\delta^*(\alpha \otimes \bigvee_{i\in I} \beta_i) = \bigvee_{i\in I} \delta^*(\alpha \otimes \beta_i)$, for all $\alpha, \beta_i \in \mathcal{X}(A), i \in I$.

We refer to property (2) as $\mathcal{X}$ satisfying the *doubleton property*; we note that, by Proposition 2.2, one always has $\alpha \otimes \bigvee_{i\in I} \beta_i = \bigvee_{i\in I} \alpha \otimes \beta_i$. Hence, (2) may be equivalently stated as (2') below. Also, the full strength of (1) will not be needed before Section 6; in what follows, we will exploit it only for $X = A$ and apply it to the Cartesian morphism $! : (A, !^*(\eta)) \longrightarrow (E,\eta)$. Since, with the identification $(A,\alpha)\otimes(E,\eta) \cong (A,\alpha)$, the map $\mathrm{id}_A \times !$ becomes the first projection $p_1 : A\times A \longrightarrow A$, the specialized version of (1) then reads as (1'):

(1') $p_1^*(\alpha) = \alpha \otimes (\,!^*(\eta))$,

(2') $\delta^*(\bigvee_{i\in I} \alpha \otimes \beta_i) = \bigvee_{i\in I} \delta^*(\alpha \otimes \beta_i)$, for all $\alpha, \beta_i \in \mathcal{X}(A), i \in I$.

Example 3.2. (1) By Proposition 2.4, for every commutative unital quantale V, the category V-**Cat** is prequantalic; in particular, **Ord, Met, UMet**, and **ProbMet** are.

(2) Let **Conv** denote the category of convergence spaces [10], *i.e.*, of sets X that come with a relation $\to$ from the set $\mathsf{F}X$ of filters on X to the set X, satisfying the conditions 1. $\dot{x} \to x$, 2. $(\mathsf{f} \to x, \mathsf{f} \subseteq \mathsf{g} \Longrightarrow \mathsf{g} \to x)$, 3. $(\mathsf{f} \to x, \mathsf{g} \to x \Longrightarrow \mathsf{f} \cap \mathsf{g} \to x)$, for all $x \in X$ and $\mathsf{f}, \mathsf{g} \in \mathsf{F}X$; morphisms are maps preserving the convergence relation. (Here $\dot{x} = \uparrow\{x\}$ is the principal filter on $x \in X$; we require a filter f on X to be proper, *i.e.*, we do not allow $\mathsf{f} \ni \emptyset$.) **Conv** is a concretely Cartesian closed topological category, with initial structures formed similarly as for V-**Cat**, and with function spaces carrying the pointwise convergence relation. Also, it is easy to see that the endofunctor $(X,\to)\times(-)$ preserves Cartesian structures. With the three proper filters on the two-element set A one can form four convergence structures on A, ordered in diamaond shape, and one then confirms routinely that the doubleton property holds as well, so that **Conv** is prequantalic.

(3) The topological category **Top** of topological spaces and continuous maps (and any of its full epireflective subcategories) allows for only one concretely symmetric monoidal-closed structure which, for spaces X, Y, endows the set $X \times Y$ with the topology of separate continuity, *i.e.*, with the coarsest topology which makes those functions on $X \times Y$ continuous that are continuous in each variable: see Remark 2.1(1) and [6]; for generalizations to concrete topological categories, and beyond, see [5, 27, 28]. One can prove that **Top**, with its unique symmetric monoidal-closed structure, satisfies condition (2) of Definition 3.1, but not condition (1). The same situation occurs for the topological category **Unif** of uniform spaces and uniformly continuous maps when endowed with the semi-uniform product: see [18], Chapter III. Hence, although satisfying all other requirements of our setting, **Top** and **Unif** fail to be prequantalic.

From a prequantalic category $\mathcal{X}$ as in Definition 3.1 we now extract a quantale, as follows. For the two-element set A, with $s : A \longrightarrow A$ denoting the non-identical bijection, the set

$$\mathcal{X}_A := \{\alpha \in \mathcal{X}(A) \mid s : (A,\alpha) \longrightarrow (A,\alpha) \text{ is a morphism in } \mathcal{X}\}$$

is easily seen to be closed under joins in $\mathcal{X}(A)$, and it therefore inherits the sup-lattice structure from $\mathcal{X}(A)$. (Note that when $s : (A,\alpha) \longrightarrow (A,\alpha)$ is a morphism in $\mathcal{X}$, it is necessarily an isomorphism.) We can now transfer the monoidal structure of $\mathcal{X}$ to $\mathcal{X}_A$, as follows: for $\alpha, \beta \in \mathcal{X}_A$, denoting as in Section 2 the structure of $(A,\alpha) \otimes (A,\beta)$ by $\alpha \otimes \beta$, we put

$$\alpha \boxtimes \beta := \delta^*(\alpha \otimes \beta) \quad \text{and} \quad \kappa := \,!^*(\eta),$$

where (E,η) continues to denote the $\otimes$-neutral object in $\mathcal{X}$.

Proposition 3.3. *$\mathcal{X}_A = (\mathcal{X}_A, \boxtimes, \kappa)$ is a commutative unital quantale.*

Proof. Clearly, when $s : (A, \alpha) \longrightarrow (A, \alpha)$ and $s : (A, \beta) \longrightarrow (A, \beta)$ are morphisms, so is $s \times s : (A \times A, \alpha \otimes \beta) \longrightarrow (A \times A, \alpha \otimes \beta)$, and then, since $\delta \cdot s = (s \times s) \cdot \delta$, also $s : (A, \alpha \boxtimes \beta) \longrightarrow (A, \alpha \boxtimes \beta)$ is a morphism. Likewise, since $! \cdot s = !$, also $s : (A, \kappa) \longrightarrow (A, \kappa)$ is a morphism. Hence, $\boxtimes$ and κ are well defined.

An application of condition (1) above gives

$$\alpha \boxtimes \kappa = \delta^*(\alpha \otimes \kappa) = \delta^*((\mathrm{id}_A \times !)^*(\alpha \times \eta)) = \delta^*(p_1^*(\alpha)) = \alpha$$

for all $\alpha \in \mathcal{X}_A$. Hence, we have a commutative monoid $\mathcal{X}_A = (\mathcal{X}_A, \boxtimes, \kappa)$. Finally, the fact that $\alpha \boxtimes (-)$ distributes over joins in $\mathcal{X}_A$ is a direct consequence of condition (2) above and affirms the claim of the Proposition. $\square$

Example 3.4. (1) For a a commutative unital quantale V, the above construction returns the same quantale, provided that V is *integral*, *i.e.*, if the $\otimes$-neutral element k is the top element in V:

$$((\mathsf{V}\text{-}\mathbf{Cat})_A, \boxtimes, \kappa) \cong (\mathsf{V}, \otimes, \mathsf{k}).$$

Indeed, for a V-$\mathbf{Cat}$-structure $a : A \times A \longrightarrow \mathsf{V}$ on $A = \{0, 1\}$ we necessarily have $a(0,0) = a(1,1) = \mathsf{k}$ when V is integral, and when $s : (A, a) \longrightarrow (A, a)$ is a V-functor, necessarily $a(0,1) \leq a(1,0) \leq a(0,1)$. Hence,

$$(\mathsf{V}\text{-}\mathbf{Cat})_A \longrightarrow \mathsf{V}, \quad a \mapsto a(0,1),$$

is a bijection, in fact: an order isomorphism. It also preserves the monoidal structure, since $!^*(\mathsf{k})$ is the structure on A with constant value k and, for all $a, b \in (\mathsf{V}\text{-}\mathbf{Cat})_A$,

$$(a \boxtimes b)(0,1) = (a \otimes b)((0,0),(1,1)) = a(0,1) \otimes b(0,1).$$

(2) Of the four convergence structures on A (see Example 3.2(2)), only the least and the largest one make the switch map $s : A \longrightarrow A$ a morphism in $\mathbf{Conv}$. Consequently, $\mathbf{Conv}_A$ is the 2-element frame, considered as a quantale (see Example 2.3).

4. Quantales as a pseudo-retract of topological categories

Not surprisingly, the assignments

$$\mathsf{V} \mapsto \mathsf{V}\text{-}\mathbf{Cat} \quad \text{and} \quad \mathcal{X} \mapsto \mathcal{X}_A \qquad (+)$$

as described in the previous sections are functorial, but care must be given to the choice of morphisms connecting the topological categories at issue. There is no question, however, about the two relevant types of morphisms of quantales in our context. We denote by

$$\mathbf{Qnt}$$

the category of unital, commutative and integral (see Example 3.4(1)) quantales and their *lax homomorphisms* $\varphi : (\mathsf{V}, \otimes, \mathsf{k}) \longrightarrow (\mathsf{W}, \otimes, \mathsf{l})$; these are maps with

$$\bigvee \varphi(u_i) \leq \varphi(\bigvee u_i), \quad \varphi(u) \otimes \varphi(v) \leq \varphi(u \otimes v), \quad \mathsf{l} \leq \varphi(\mathsf{k}),$$

for all $u, v, u_i \; (i \in I)$ in V. Of course, the first of these inequalities just amounts to monotonicity of φ, and the last one is necessarily an equality, by integrality of W. If the first two inequalities may also be replaced by equalities, φ is a *(strict)homomorphism*, and the resulting category is denoted by $\mathbf{Qnt}^{=}$.

The (very large) category

$$\mathbf{TOP/Set}$$

has as objects prequantalic and *integral* topological categories, where integral refers to the requirement that the $\otimes$-neutral object (E, η) of the monoidal category $\mathcal{X}$ must be terminal in $\mathcal{X}$. Since we already demanded from the outset that η must be the discrete structure on E, this amounts to asking η to be both least and largest in $\mathcal{X}(E)$, whence the condition may be read as $|\mathcal{X}(E)| = 1$. A morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ in **TOP/Set** is a functor which commutes with the underlying **Set** functors, preserves Cartesian morphisms, and which is a lax homomorphism of monoidal categories. In the notation

$$(f : (X, \xi) \longrightarrow (Y, \upsilon)) \mapsto (f : (X, F\xi) \longrightarrow (Y, F\upsilon))$$

for F, this means

$$F(f^*(\upsilon)) = f^*(F\upsilon), \quad F\xi \otimes F\upsilon \le F(\xi \otimes \upsilon), \quad \varepsilon \le F\eta,$$

for all $\xi \in \mathcal{X}(X), \upsilon \in \mathcal{X}(Y)$; here ε denotes the $\otimes$-neutral structure on E in $\mathcal{Y}$, and, of course, the last inequality is, in fact, an equality, by integrality of $\mathcal{Y}$. The morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ in **TOP/Set** is *strict* if it preserves the tensor products strictly, as well as all joins in the fibres of *two*-element sets:

$$F(\bigvee \alpha_i) = \bigvee F(\alpha_i),$$

for all α_i $(i \in I)$ in $\mathcal{X}(A)$, with $A = \{0, 1\}$; these are the morphism of $\mathbf{TOP}^{=}/\mathbf{Set}$.

Theorem 4.1. *The assignments* (+) *are the object parts of functors*

$$\mathbf{Qnt} \longrightarrow \mathbf{TOP}/\mathbf{Set} \longrightarrow \mathbf{Qnt} \quad \textit{and} \quad \mathbf{Qnt}^{=} \longrightarrow \mathbf{TOP}^{=}/\mathbf{Set} \longrightarrow \mathbf{Qnt}^{=},$$

with the composite functors being isomorphic to $\mathrm{Id}_{\mathbf{Qnt}}$ *and* $\mathrm{Id}_{\mathbf{Qnt}^{=}}$*, respectively.*

Proof. For $\varphi : \mathsf{V} \longrightarrow \mathsf{W}$ in **Qnt** one has the *change-of-base functor* (see [17])

$$B_\varphi : \mathsf{V}\text{-}\mathbf{Cat} \longrightarrow \mathsf{W}\text{-}\mathbf{Cat}, \ (X, a) \mapsto (X, \varphi a),$$

with $(\varphi a)(x, y) = \varphi(a(x, y))$ for all $x, y \in X$. A straightforward verification shows that B_φ is a morphism in **TOP/Set**, even in $\mathbf{TOP}^{=}/\mathbf{Set}$ when φ is in $\mathbf{Qnt}^{=}$.

For $F : \mathcal{X} \longrightarrow \mathcal{Y}$ in **TOP/Set**, we have the restriction

$$F_A : \mathcal{X}_A \longrightarrow \mathcal{Y}_A, \ \alpha \mapsto F\alpha,$$

which, by functoriality of F, is well defined and monotone. Furthermore, since the lax homomorphism F preserves Cartesian morphisms,

$$F\alpha \boxtimes F\beta = \delta^*(F\alpha \otimes F\beta) \le \delta^*(F(\alpha \otimes \beta)) = F(\delta^*(\alpha \otimes \beta)) = F(\alpha \boxtimes \beta),$$

for all $\alpha, \beta \in \mathcal{X}_A$; likewise, in the notation used earlier, $\lambda := !^*(\varepsilon) \le !^*(F\eta) = F(!^*(\varepsilon)) = F\kappa$. Clearly, the two inequality signs in the last two chains get replaced by equality signs when F is strict, in which case F_A preserves also all suprema.

We already noted in Example 3.4(1) that the composite object assignment renders isomorphic quantales. One easily checks that the relevant isomorphisms live in $\mathbf{Qnt}^{=}$, and that the morphism assignments $\varphi \mapsto B_\varphi$ and $F \mapsto F_A$ make them natural. □

5. The adjunction beween quantales and transitive topopolgical categories

We now show that, after a restriction of their codomains, the functors $\mathbf{Qnt} \longrightarrow \mathbf{TOP}/\mathbf{Set}$ and $\mathbf{Qnt}^{=} \longrightarrow \mathbf{TOP}^{=}/\mathbf{Set}$ of Theorem 4.1 (sending V to V-**Cat**) become right adjoint. Their left adjoints assign to a topological category $\mathcal{X}$ which enjoys a certain additional property, the quantale

$$(\mathcal{X}(A), \boxtimes, \kappa),$$

rather than its subquantale $\mathcal{X}_A$, where $\boxtimes$ and κ are defined exactly as in the smaller environment $\mathcal{X}_A$. To define the needed additional property of $\mathcal{X}$, we consider for $(X, \xi) \in \mathcal{X}$ and all $x, y \in X$ the $\mathcal{X}$-structure

$$\xi_{x,y} := e^*_{x,y}(\xi) \quad \text{with} \quad e_{x,y} : A \longrightarrow X, \ 0 \mapsto x, \ 1 \mapsto y,$$

on $A = \{0, 1\}$.

Definition 5.1. A prequantalic and integral topological category $\mathcal{X}$ is *transitive* if every object (X, ξ) in $\mathcal{X}$ is transitive, that is, if for all $x, y, z \in X$, one has

$$\xi_{x,y} \boxtimes \xi_{y,z} \leq \xi_{x,z}.$$

Remark 5.2. (1) Every prequantalic and integral topological category $\mathcal{X}$ is automatically *reflexive*, in the sense that, for every object (X, ξ) and all $x \in X$, one has $\kappa \leq \xi_{x,x}$. Indeed, considering the map $x : E \longrightarrow X$ and the fact that E carries only one $\mathcal{X}$-structure, we have

$$\kappa = !^*(\eta) = !^*(x^*(\xi)) = e^*_{x,x}(\xi) = \xi_{x,x}.$$

(2) It is easy to see that, for a unital, commutative and integral quantale V, the prequantalic and integral topological category V-**Gph** of V-*graphs* $(X, a : X \times X \longrightarrow \mathsf{V})$ (the only constraint for which is $\mathsf{k} \leq a(x, x)$ for all $x \in X$), with morphisms defined as in its full subcategory V-**Cat**, is reflexive, but generally fails to be transitive.

(3) $(X, \to))$ in **Conv** is transitive if, and only if, the relation $(x \leq y \Leftrightarrow \dot{x} \to y)$ is transitive, *i.e.*, a preorder on X. It is easy to construct a convergence structure on a 3-element set that fails to be transitive.

Proposition 5.3. *Given* $\mathsf{V} \in$ **Qnt** *and* $\mathcal{X} \in$ **TOP**/**Set**, $\mathcal{X}$ *transitive, there is a natural bijective correspondence between morphisms*

$$\varphi : \mathcal{X}(A) \longrightarrow \mathsf{V} \quad \textit{and} \quad F : \mathcal{X} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat}$$

in **Qnt** *and* $\mathcal{X} \in$ **TOP**/**Set**, *which may be restricted to morphisms of* $\mathbf{Qnt}^{=}$ *and* $\mathbf{TOP}^{=}/\mathbf{Set}$, *respectively. These bijections become order isomorphisms when we order the hom-sets of* **Qnt** *pointwise and, for* $G, H : \mathcal{X} \longrightarrow \mathcal{Y}$ *in* **TOP**/**Set**, *define* $G \leq H$ *by letting the monotone maps* $G_X, H_X : \mathcal{X}(X) \longrightarrow \mathcal{Y}(X)$ *satisfy* $G_X \leq H_X$, *for all sets* X.

Proof. To $F : \mathcal{X} \longrightarrow \mathsf{V}$-**Cat** one assigns the monotone map

$$\varphi^F : \mathcal{X}(A) \longrightarrow \mathsf{V}, \quad \alpha \mapsto (F\alpha)(0, 1),$$

with the V-category structure $F\alpha$ on A. Since F preserves Cartesianess and is a lax homomorphism, so is φ^F. Indeed, for all $\alpha, \beta \in \mathcal{X}(A)$ one has

$$\varphi^F(\alpha \boxtimes \beta) = F(\delta^*(\alpha \otimes \beta))(0, 1) = \delta^*(F(\alpha \otimes \beta))(0, 1) \geq \delta^*(F\alpha \otimes F\beta)(0, 1) = \varphi^F(\alpha) \otimes \varphi^F(\beta),$$

with "=" holding when F is a homomorphism. Likewise,

$$\varphi^F(\kappa) = F(\,!^*(\eta))(0, 1) = (\,!^*(F\eta))(0, 1) = \mathsf{k}.$$

Also, since F preserves joins in $\mathcal{X}(A)$ when F is strict, obviously φ^F enjoys the same property.

Conversely, given $\varphi : \mathcal{X}(A) \longrightarrow \mathsf{V}$, we define the functor

$$F^\varphi : \mathcal{X} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat},\ (X, \xi) \mapsto (X, F^\varphi \xi), \quad (F^\varphi \xi)(x, y) := \varphi(\xi_{x,y}),$$

for all $x, y \in X$. Since (X, ξ) is transitive, the lax homomorphism φ makes $(X, F^\varphi \xi)$ indeed a V-category:

$$(F^\varphi \xi)(x, y) \otimes (F^\varphi \xi)(y, z) = \varphi(\xi_{x,y}) \otimes \varphi(\xi_{y,z}) \leq \varphi(\xi_{x,y} \boxtimes \xi_{y,z}) \leq \varphi(\xi_{x,z}) = (F^\varphi \xi)(x, z),$$

$$(F^\varphi \xi)(x, x) = \varphi(\xi_{x,x}) = \varphi(\kappa) \geq \mathsf{k}.$$

For a morphism $f : (X, \xi) \longrightarrow (Y, \upsilon)$, since $f \cdot e_{x,y} = e_{fx,fy}$, one has

$$\xi_{x,y} = e^*_{x,y}(\xi) \leq e^*_{x,y}(f^*(\upsilon)) = e^*_{fx,fy}(\upsilon) = \upsilon_{fx,fy}$$

for $x, y \in X$, so that $f : (X, F^\varphi\xi) \longrightarrow (Y, F^\varphi\upsilon)$ is a V-functor:

$$(F^\varphi\xi)(x,y) = \varphi(\xi_{x,y}) \leq \varphi(\upsilon_{fx,fy}) = (F^\varphi\upsilon)(fx,fy).$$

Next we verify that F^φ is a lax homomorphism of monoidal categories. Indeed, for $(X,\xi),(Y,\upsilon)$ in $\mathcal{X}$ and all $x, x' \in X$, $y, y' \in Y$, one first notes that the two maps

$$e_{(x,y),(x',y')},\ (e_{x,x'} \times e_{y,y'}) \cdot \delta : A \longrightarrow X \times Y$$

coincide. The lax homomorphism φ then gives

$$\begin{aligned}(F^\varphi\xi \otimes F^\varphi\upsilon)((x,y),(x',y')) &= (F^\varphi\xi)(x,x') \otimes (F^\varphi\upsilon)(y,y')\\ &= \varphi(\xi_{x,x'}) \otimes \varphi(\upsilon_{y,y'})\\ &\leq \varphi(\xi_{x,x'} \boxtimes \upsilon_{y,y'})\\ &= \varphi(\delta^*(e^*_{x,x'}(\xi) \otimes e^*_{y,y'}(\upsilon))\\ &= \varphi(e^*_{(x,y),(x',y')}(\xi \otimes \upsilon))\\ &= F^\varphi(\xi \otimes \upsilon)((x,y),(x',y')),\end{aligned}$$

with equality holding when φ is strict. In conjunction with $F^\varphi\eta = \varphi(\kappa) \geq \mathsf{k}$ (since $e_{*,*} = ! : A \longrightarrow E$), this proves the claim about F^φ. Clearly then, when φ is strict, so is F^φ.

Let us finally show that the assignments

$$\varphi \mapsto F^\varphi \quad \text{and} \quad F \mapsto \varphi^F$$

are inverse to each other. First, since $e_{0,1} : A \longrightarrow A$ is the identity map on A, for $\alpha \in \mathcal{X}(A)$ we have

$$\varphi^{F^\varphi}(\alpha) = (F^\varphi\alpha)(0,1) = \varphi(\alpha_{0,1}) = \varphi(e^*_{0,1}(\alpha)) = \varphi(\alpha),$$

whence $\varphi^{F^\varphi} = \varphi$. To confirm $F^{\varphi^F} = F$, it suffices to show that the two functors coincide on every object (X,ξ) in $\mathcal{X}$. Indeed, since F preserves Cartesianness, for all $x, y \in X$ one has

$$(F^{\varphi^F}\xi)(x,y) = \varphi^F(\xi_{x,y}) = F(e^*_{x,y}(\xi))(0,1) = e^*_{x,y}(F\xi)(0,1) = (F\xi)(x,y).$$

As the claims about order preservation are obvious from the definitions, this completes the proof. □

Remark 5.4. If the topological category $\mathcal{X}$, in addition to being prequantalic and transitive, is also *symmetric*, so that every object (X,ξ) is symmetric in the sense that $\xi_{x,y} = \xi_{y,x}$ for all $x, y \in X$, then all $\xi_{x,y}$ lie in $\mathcal{X}_A$, rather than just in $\mathcal{X}(A)$. Consequently, the assertion of Proposition 5.3 remains true when $\mathcal{X}(A)$ gets replaced by $\mathcal{X}_A$ and V-**Cat** by its full subcategory **Sym**-V-**Cat**, containing all *symmetric* V-categories (X,a), required to satisfy $a(x,y) = a(y,x)$ for all $x, y \in X$.

We note that *a topological category $\mathcal{X}$ is symmetric precisely when $\mathcal{X}(A) = \mathcal{X}_A$, that is, when every $\mathcal{X}$-structure α on the doubleton A makes the switch map a morphism $s : (A,\alpha) \longrightarrow (A,\alpha)$.*

We denote by

TTOP/Set

the full subcategory of **TOP/Set** of transitive prequantalic and integral topological categories, made into a 2-category as described in Proposition 5.3, and by $\mathbf{TTOP}^{=}$**/Set** the corresponding sub-2-category of $\mathbf{TOP}^{=}$**/Set**. The Proposition provides the proof of the following Theorem:

Theorem 5.5. *The 2-functors*

$$\mathbf{Qnt} \longrightarrow \mathbf{TTOP/Set} \quad \textit{and} \quad \mathbf{Qnt}^{=} \longrightarrow \mathbf{TTOP}^{=}/\mathbf{Set}, \quad \mathsf{V} \mapsto \mathsf{V}\text{-}\mathbf{Cat},$$

have left adjoints, both given by $\mathcal{X} \mapsto \mathcal{X}(A)$.

Remark 5.6. (1) The counit of the adjunction at $\mathsf{V} \in \mathbf{Qnt}$ is formally described by

$$(\mathsf{V}\text{-}\mathbf{Cat})(A) \longrightarrow \mathsf{V}, \quad \alpha \mapsto \alpha(0,1).$$

But since α is determined by any pair of values in V, the unit is more easily described as the first projection $\pi_1 : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ of the Cartesian product of V with itself in $\mathbf{Qnt}$. Since the right adjoint of the adjunction of Theorem 5.5 preserves the pullback $\mathsf{V} \times \mathsf{V}$ over the terminal quantale $\mathsf{1}$, one has the isomorphism

$$(\mathsf{V} \times \mathsf{V})\text{-}\mathbf{Cat} \cong \mathsf{V}\text{-}\mathbf{Cat} \times_{\mathbf{Set}} \mathsf{V}\text{-}\mathbf{Cat}$$

in $\mathbf{TTOP}^{=}/\mathbf{Set}$—which is also easily confirmed by direct inspection. Briefly then: the units of the adjunction are the first pullback projections, assigning to a set X equipped with two V-category structures a_1, a_2 (*i.e., a bi-*V*-category,*) the V-category (X, a_1).

(2) Letting $\mathbf{STTOP}^{(=)}/\mathbf{Set}$ denote the full subcategory of $\mathbf{TTOP}^{(=)}/\mathbf{Set}$ of symmetric categories (as defined in Remark 5.4), one has the 2-functor

$$\mathbf{Qnt}^{(=)} \longrightarrow \mathbf{STTOP}^{(=)}/\mathbf{Set}, \quad \mathsf{V} \mapsto \mathbf{Sym}\text{-}\mathsf{V}\text{-}\mathbf{Cat},$$

whose left adjoint is given by $\mathcal{X} \mapsto \mathcal{X}_A$. But now the counits

$$(\mathbf{Sym}\text{-}\mathsf{V}\text{-}\mathbf{Cat})_A \longrightarrow \mathsf{V}, \quad \alpha \mapsto \alpha(0,1),$$

become isomorphisms of quantales, making the left adjoint full and faithful and a pseudo-retraction.

6. Not assuming transitivity from the outset

In this section we give a version of Theorem 5.5 which avoids transitivity, for the price that the right adjoint assigns to a quantale V the category $\mathsf{V}\text{-}\mathbf{Gph}$ (see Remark 5.2(2)), rather than the more interesting category $\mathsf{V}\text{-}\mathbf{Cat}$, which is the full subcategory of transitive objects in $\mathsf{V}\text{-}\mathbf{Gph}$. We first focus on this last aspect and show more generally that, from every prequantalic topological category $\mathcal{X}$, one can extract its *transitive core*, $\mathsf{T}(\mathcal{X})$, as follows.

Proposition 6.1. $\mathbf{TTOP}^{(=)}/\mathbf{Set}$ *is a full coreflective subcategory of* $\mathbf{TOP}^{(=)}/\mathbf{Set}$*, with the coreflector assigning to a prequantalic topological category* $\mathcal{X}$ *its full subcategory* $\mathsf{T}(\mathcal{X})$ *of transitive objects in* $\mathcal{X}$*.*

Proof. To make sure that $\mathsf{T}(\mathcal{X})$ is an object in $\mathbf{TTOP}^{(=)}/\mathbf{Set}$, we should check that $\mathsf{T}(\mathcal{X})$ is closed under initial structures and tensor products in $\mathcal{X}$. While the latter task is a bit tedious, the former check proceeds straightforwardly. Indeed, given maps $f_i : X \longrightarrow Y_i$ and transitive structures $\upsilon_i \in \mathcal{X}(Y_i)$, $i \in I$, the initial structure $\xi = \bigwedge_{i\in I} f_i^*(\upsilon_i)$ is easily seen to be transitive again since, for $x, y, z \in X$, one has

$$\begin{aligned}
\xi_{x,y} \boxtimes \xi_{y,z} &= \delta^*(e^*_{x,y}(\bigwedge_i f_i^*(\upsilon_i)) \otimes e^*_{y,z}(\bigwedge_j f_j^*(\upsilon_j))) \\
&\le \bigwedge_i \bigwedge_j \delta^*(e^*_{f_i x, f_i y}(\upsilon_i) \otimes e^*_{f_j y, f_j z}(\upsilon_j)) \\
&\le \bigwedge_i e^*_{f_i x, f_i z}(\upsilon_i) \qquad \text{(by transitivity of } \upsilon_i) \\
&= \bigwedge_i e^*_{x,z}(f_i^*(\upsilon_i)) = e^*_{x,z}(\bigwedge_i f_i^*(\upsilon_i)) = \xi_{x,z}
\end{aligned}$$

where, at the end, we take advantage of the fact that the right-adjoint map $e^*_{x,z}$ preserves infima.

For transitive objects $(X, \xi), (Y \upsilon)$, we must now show that $\xi \otimes \upsilon$ is a transitive structure on $X \times Y$, that is:

$$(\xi \otimes \upsilon)_{(x,y),(u,v)} \boxtimes (\xi \otimes \upsilon)_{(u,v),(z,w)} \le (\xi \otimes \upsilon)_{(x,y),(z,w)}, \qquad (\#)$$

for all $x, u, z \in X, y, v, w \in Y$. First, with σ denoting the bijective middle-interchange map, we take note of the commutative diagram

$$\begin{array}{ccccc}
A \times A & \xrightarrow{\delta\times\delta} & (A\times A)\times(A\times A) & \xrightarrow{(e_{x,u}\times e_{u,z})\times(e_{y,v}\times e_{v,w})} & (X\times X)\times(Y\times Y) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\sigma} \\
A & \xrightarrow{\delta} & A\times A & \xrightarrow{e_{(x,y),(u,v)}\times e_{(u,v),(z,w)}} & (X\times Y)\times(X\times Y).
\end{array}$$

Now the following calculation confirms (#), where the first identity follows from the fact that tensor products of Cartesian morphisms are Cartesian:

$$\begin{aligned}
&\delta^*(e^*_{(x,y),(u,v)}(\xi\otimes\upsilon)\otimes e^*_{(u,v),(z,w)}(\xi\otimes\upsilon)) \\
&= \delta^*((e_{(x,y),(u,v)}\times e_{(u,v),(z,w)})^*((\xi\otimes\upsilon)\otimes(\xi\otimes\upsilon))) \\
&= \delta^*((\delta\times\delta)^*((e_{x,u}\times e_{u,z})\times(e_{y,v}\times e_{v,w}))^*((\xi\otimes\xi)\otimes(\upsilon\otimes\upsilon))) \\
&= \delta^*(\delta^*(\xi_{x,u}\otimes\xi_{u,z})\otimes\delta^*(\upsilon_{y,v}\otimes\upsilon_{v,w})) \\
&= (\xi_{x,u}\boxtimes\xi_{u,z})\boxtimes(\upsilon_{y,v}\boxtimes\upsilon_{v,w}) \\
&\leq \xi_{x,z}\boxtimes\upsilon_{y,w} \quad \text{(by transitivity of } \xi \text{ and } \upsilon) \\
&= \delta^*(e^*_{x,z}(\xi)\otimes e^*_{y,w}(\upsilon)) = e^*_{(x,y),(z,w)}(\xi\otimes\upsilon).
\end{aligned}$$

To finally validate the coreflectivity claim, it suffices to show that a morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ in **TOP**/**Set** with $\mathcal{X}$ transitive takes its values in $\mathsf{T}(\mathcal{Y})$, that is: if (X,ξ) is transitive in $\mathcal{X}$, then we must show the transitivity of $(X, F\xi)$ in $\mathcal{Y}$. But with F preserving Cartesian morphisms, for all $x, y, z \in X$ we have

$$\begin{aligned}
(F\xi)_{x,z} = e^*_{x,z}(F\xi) = F(e^*_{x,z}(\xi)) = F\xi_{x,z} &\geq F(\xi_{x,y}\boxtimes\xi_{y,z}) \\
&= F(\delta^*(\xi_{x,y}\otimes\xi_{y,z})) \\
&= \delta^*(F(\xi_{x,y}\otimes\xi_{y,z})) \\
&\geq \delta^*(F\xi_{x,y}\otimes F\xi_{y,z}) \\
&= \delta^*(Fe^*_{x,y}(\xi)\otimes Fe^*_{y,z}(\xi)) \\
&= \delta^*(e^*_{x,y}(F\xi)\otimes e^*_{y,z}(F\xi)) = (F\xi)_{x,y}\boxtimes(F\xi)_{y,z}.
\end{aligned}$$

□

Theorem 6.2. *There are right-adjoint 2-functors*

$$\mathbf{Qnt}^{(=)} \longrightarrow \mathbf{TOP}^{(=)}/\mathbf{Set},\ \mathsf{V}\mapsto \mathsf{V}\text{-}\mathbf{Gph}, \quad \textit{and} \quad \mathbf{TOP}^{(=)}/\mathbf{Set} \longrightarrow \mathbf{TTOP}^{(=)}/\mathbf{Set},\ \mathcal{X}\mapsto\mathsf{T}(\mathcal{X}),$$

whose composite functor is the right-adjoint of the adjunction of Theorem 5.5*, regardless of whether one reads the lax or the strict version.*

Proof. Right adjointness of $\mathsf{V}\mapsto\mathsf{V}$-**Gph** follows from a trimmed-down version of Theorem 5.5, and right adjointness of $\mathcal{X}\mapsto\mathsf{T}(\mathcal{X})$ was proved in Proposition 6.1. Since trivially $\mathsf{T}(\mathsf{V}\text{-}\mathbf{Gph}) = \mathsf{V}\text{-}\mathbf{Cat}$, the proof is complete. □

Remark 6.3. By assigning to a topological category $\mathcal{X}$ its *symmetric core*, defined as its full subcategory $\mathsf{S}(\mathcal{X})$ formed by the symmetric objects in $\mathcal{X}$, one can follow the pattern of the proof of Proposition 6.1 and show that $\mathbf{STTOP}^{(=)}/\mathbf{Set}$ lies coreflectively in $\mathbf{TTOP}^{(=)}/\mathbf{Set}$. Precomposing this coreflector S with the right adjoint of Theorem 5.5 (*i.e.*, with the composite right adjoint of Theorem 6.2), returns the right adjoint 2-functor

$$\mathbf{Qnt}^{(=)} \longrightarrow \mathbf{STTOP}^{(=)}/\mathbf{Set}, \quad \mathsf{V}\mapsto\mathbf{Sym}\text{-}\mathsf{V}\text{-}\mathbf{Cat},$$

of Remark 5.6(2).

7. The representation theorem

Returning to the adjunctions of Theorem 5.5, from Proposition 5.3 we have that the unit of these adjunctions at $\mathcal{X} \in \mathbf{TTOP/Set}$ is given by the *comparison functor*

$$H_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X}(A)\text{-}\mathbf{Cat}, \quad (X, \xi) \mapsto (X, (\xi_{x,y})_{x,y \in X}).$$

In order to see when this functor is full, we recall that a class $\mathcal{A}$ of objects in the topological category $\mathcal{X}$ is *finally dense* in $\mathcal{X}$ if every object in $\mathcal{X}$ is the common codomain of some final family of morphisms whose domains lie in $\mathcal{A}$. One calls the full subcategory of $\mathcal{A}$ of $\mathcal{X}$ *bireflective* if the inclusion functor has a left adjoint whose units are bijections; equivalently, if there is a concrete functor that is left adjoint to the inclusion functor.

Proposition 7.1. *The following conditions are equivalent for $\mathcal{X} \in \mathbf{TTOP/Set}$:*

(i) *the set $\{(A, \alpha) \mid \alpha \in \mathcal{X}(A)\}$ is finally dense in $\mathcal{X}$;*

(ii) *for all $(X, \xi) \in \mathcal{X}$, the family $e_{x,y} : (A, \xi_{x,y}) \longrightarrow (X, \xi)\ (x, y \in X)$ is final;*

(iii) *the functor $H_{\mathcal{X}}$ is full;*

(iv) *the functor $H_{\mathcal{X}}$ is a full bireflective embedding.*

Proof. (i) $\Longleftrightarrow$ (ii): It is clear that, for (X, ξ) being the codomain of some final family of arrows with domains in a full subcategory $\mathcal{A}$, is equivalent to the family of all arrows with codomain (X, ξ) and domains in $\mathcal{A}$ being final. But then, amongst all arrows, it suffices to consider those which, as individual morphisms, are initial. In the case of $\mathcal{A} = \mathcal{X}(A)$, this leads precisely to statement (ii).

(ii) $\Longleftrightarrow$ (iii): For every map $f : X \longrightarrow Y$ and all $(X, \xi), (Y, \upsilon)$ in $\mathcal{X}$ one has the equivalences

$$\begin{aligned}
& f : H_{\mathcal{X}}(X, \xi) \longrightarrow H_{\mathcal{X}}(Y, \upsilon) \text{ in } \mathcal{X}(A)\text{-}\mathbf{Cat} \\
\Longleftrightarrow\ & \forall x, y \in X : \xi_{x,y} \leq \upsilon_{fx,fy} = e^*_{fx,fy}(\upsilon) \text{ in } \mathcal{X}(A) \\
\Longleftrightarrow\ & \forall x, y \in X : f \cdot e_{x,y} = e_{fx,fy} : (A, \xi_{x,y}) \longrightarrow (Y, \upsilon) \text{ in } \mathcal{X}.
\end{aligned}$$

Fullness of $H_{\mathcal{X}}$ requires f to be a morphism $(X, \xi) \longrightarrow (Y, \upsilon)$ whenever the first of the equivalent statements holds, and finality of the family of (ii) requires the same whenever the last one holds.

(ii)$\Longrightarrow$(iv): Injectivity on objects for $H_{\mathcal{X}}$ follows readily with the above equivalences when one considers $f = \mathrm{id}_X$ and $(X, \xi), (X, \upsilon) \in \mathcal{X}$, which show that (ii) makes $f : (X, \xi) \longrightarrow (X, \upsilon)$ an isomorphism; $\xi = \upsilon$ follows. Now it suffices to construct a concrete left adjoint $K_{\mathcal{X}} \dashv H_{\mathcal{X}}$, as follows: for $(X, c) \in \mathcal{X}(A)\text{-}\mathbf{Cat}$, let $K_{\mathcal{X}}(X, c) =: (X, \xi^c)$ be defined by the requirement that the family

$$e_{x,y} : (A, c(x, y)) \longrightarrow (X, \xi^c),\ x, y \in X,$$

be final. In particular, this structure makes all $e_{x,y}$ morphisms in $\mathcal{X}$, so that $c(x, y) \leq e^*_{x,y}(\xi^c) = \xi^c_{x,y}$, which gives us the $\mathcal{X}(A)$-functor $\mathrm{id}_X : (X, c) \longrightarrow H_{\mathcal{X}}(X, \xi^c)$. This morphism serves as the unit of the adjunction at (X, c). Indeed, every $\mathcal{X}(A)$-functor $f : (X, c) \longrightarrow H_{\mathcal{X}}(Y, \upsilon)$ satisfies $c(x, y) \leq \upsilon_{fx,fy} = e^*_{fx,fy}(\upsilon)$ and, therefore, gives $\mathcal{X}$-morphisms $e_{fx,fy} : (A, c(x, y)) \longrightarrow (Y, \upsilon)$, for all $x, y \in X$. Finality of ξ^c now shows that $f : K_{\mathcal{X}}(X, c) \longrightarrow (X, \xi)$ is a morphism in $\mathcal{X}$. $\square$

The Proposition provides the main part of the proof of the following Representation Theorem:

Theorem 7.2. *Let the concrete category $\mathcal{X}$ satisfy the following three conditions:*

1. *$\mathcal{X}$ is a fibre-small topological category admitting only one structure on a singleton set;*

2. *$\mathcal{X}$ is concretely symmetric monoidal closed, such that the tensor product preserves Cartesianness of morphisms;*

3. *$\mathcal{X}$ satisfies the doubleton property and is transitive.*

Then $\mathcal{X}$ is fully and bireflectively embeddable into V-**Cat** *for some commutative, unital and integral quantale* V *if, and only if, the set $\mathcal{X}(A)$ of $\mathcal{X}$-objects over a two-element set A is finally dense in $\mathcal{X}$.*

Proof. Conditions 1-3 just rephrase the meaning of $\mathcal{X}$ being an object of **TTOP/Set**. The assertion (i)⇒(iv) of Proposition 7.1 provides the proof for the "if"-part of the claim of the Theorem. To prove the necessity of the condition, we may assume that $\mathcal{X}$ is a full bireflective subcategory of V-**Cat**, for some commutative, unital and integral quantale V. We first note that, given a V-category (X, a), the family

$$e_{x,y} : (A, a_{x,y}) \longrightarrow (X, a)\,, \quad \text{with} \;\; a_{x,y} = e^*_{x,y}(a)\,(x, y \in X),$$

is indeed final in V-**Cat**, since $a_{x,y}(0, 1) = a(x, y)$. Now, one instantly sees that, when $\tilde{a}_{x,y}$ denotes the bireflective modification of $a_{x,y}$ and (X, a) lies in $\mathcal{X}$, the family

$$e_{x,y} : (A, \tilde{a}_{x,y}) \longrightarrow (X, a)\,(x, y \in X),$$

is final in $\mathcal{X}$. □

Remark 7.3. Condition 1 of the Theorem is also a necessary condition for $\mathcal{X}$ to be embeddable into V-**Cat** as a full and bireflective subcategory for some quantale V, simply because such subcategories inherit the topologicity from their parent category, with initial structures in the subcategory to be formed as in the parent category.

After an easy inspection of the proof of Proposition 7.1, Remarks 5.6(2) and 6.3 allow us to formulate the following "symmetric version" of Theorem 7.2:

Corollary 7.4. *Let the concrete category $\mathcal{X}$ satisfy the hypotheses* 1 *and* 2 *of Theorem* 7.2, *as well as*

3'. *$\mathcal{X}$ satisfies the doubleton property and is transitive and symmetric.*

Then $\mathcal{X}$ is fully and bireflectively embeddable into **Sym**-V-**Cat** *for some commutative, unital and integral quantale* V *if, and only if, the set $\mathcal{X}_A$ of $\mathcal{X}$-objects over a two-element set A is finally dense in $\mathcal{X}$.*

Remark 7.5. The question remains when a category $\mathcal{X}$ satisfying all the hypotheses of Corollary 7.4 is actualy equivalent to a category of the form **Sym**-V-**Cat**, which reduces to the question when the full bireflective embedding

$$H_{\mathcal{X}} : \mathcal{X} \longrightarrow \textbf{Sym-}\mathcal{X}_A\textbf{-Cat}$$

is actually an equivalence of categories. Clearly, this will be the case when the units of the adjunction $K_{\mathcal{X}} \dashv H_{\mathcal{X}}$ are isomorphisms. Their construction in the proof of Proposition 7.1 easily reveals that, being bijections, they will be isomorphisms exactly when the left adjoint $K_{\mathcal{X}}$ preserves Cartesian morphisms.

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