CANTOR'S DIAGONAL PRINCIPLE FOR CATEGORIES

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TRANSLATOR'S NOTE: This paper contains several remarkable results, which played a crucial role at the time (1978). Amongst these: In Herrlich's definition of $(\mathcal{E}, \mathcal{M})$-factorization structures for sources ([3], 1974) the assumption that $\mathcal{E}$ consists of epimorphisms is redundant; the proofs that topological functors, $(\mathcal{E}, \mathcal{M})$-topological functors and solid functors are faithful can be given without assuming small hom-sets.

The next few pages contain an English translation, annotated here and there with details which the authors regarded as unnecessary. In a few places, for the sake of brevity, I used the term "source" which was not used by the authors.

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1. Four seemingly independent theorems

(1) Cantor's proof [1] of the uncountability of the reals means more generally that forming the power set of a set leads to a properly higher potency. In this paper we shall prove a categorical theorem that contains as special cases, besides Cantor's Theorem, the following three categorical theorems, of which only the first two seem to be known.

(2) Among the well known results of Freyd [2] there is the theorem that a small category with direct products (or coproducts) is a preordered set. Examining the proof shows at once that it is sufficient to require the existence of "weak" (co-)powers.

(3) Hoffmann [4] and Herrlich [3] proved that topological, and more generally, $(\mathcal{E}, \mathcal{M})$-topological functors are faithful. For their proofs it was essential to assume that the categories in question have small Hom classes. From our theorem it follows that this assumption is redundant and that one can prove the same result more generally even for semitopological functors [6].

> TRANSLATOR'S NOTE: Semitopological functors are nowadays called solid functors.

(4) When in a category one wishes to form the pushout of a class-indexed family of epimorphisms with common domain, it suffices—provided the category is well-powered,—to restrict oneself to a representative set-indexed family of epimorphisms

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and form their pushout, provided it exists. We shall show, conversely, that the consideration of pushouts of morphisms indexed by a proper class is only sensible for epimorphisms. Indeed, if in a category there exists a pushout of a proper class of occurrences of a single morphism, then the latter is necessarily epimorphic.

2. Cantor's Theorem for categories

For tow classes I and J we shall write $|J| \leq |I|$ if $J = \emptyset$ or there exists a surjection $\sigma : I \longrightarrow J$. Let $\mathcal{K}$ be a category and $(A_i)_{i\in I}$ an I-indexed class of objects of $\mathcal{K}$, and $\mathcal{B}$ a class of objects of $\mathcal{K}$; then we call a family of morphisms $(j_i : A_i \longrightarrow C)_{i\in I}$ a *weak coproduct (coproduct) of* $(A_i)_{i\in I}$ *with respect to* $\mathcal{B}$, if for each family $(h_i : A_i \longrightarrow B)_{i\in I}$ with $B \in \mathcal{B}$ there exists (unique) $h : C \longrightarrow B$ with $hj_i = h_i$ for all $i \in I$. The reference to $\mathcal{B}$ is omitted when $\mathcal{B}$ is the whole object class. (Weak) products (with respect to $\mathcal{B}$) are defined dually. We talk of a *(weak) Ith (co-)power of A (with respect to* $\mathcal{B}$*)* if all A_i equal A. For objects X and Y of $\mathcal{K}$ we denote by $\mathcal{K}(X, Y)$ their hom-class.

(5) **Theorem.** *Let B, A_i $(i \in I)$ be objects of the category $\mathcal{K}$, and let there exist a weak coproduct $(j_i : A_i \longrightarrow C)_{i\in I}$ of $(A_i)_{i\in I}$ with respect to $\{B\}$ with $|\mathcal{K}(C, B)| \leq |I|$. Then for every two morphisms $f, g \in \mathcal{K}(C, B)$ there exists an index $i_0 \in I$ with $fj_{i_0} = gj_{i_0}$.*

Proof. In case $f = g$ there is nothing to prove. Else we consider the following subclass J of $\mathcal{K}(C, B)$:

$$J = \big\{\, h \in \mathcal{K}(C, B) \mid (\forall i \in I)\ hj_i \in \{\, fj_i,\ gj_i \,\} \big\}.$$

Then $|J| \leq |I|$ and $J \neq \emptyset$, so that there exists a surjective map $\sigma : I \longrightarrow J$.

Translator's remark: The claim $|J| \leq |I|$ follows from $J \subset \mathcal{K}(C, B)$ and the hypothesis $|\mathcal{K}(C, B)| \leq |I|$, but it is only trivial if one interprets $|J|$, $|I|$, etc. as cardinal numbers and uses AC to gloss over the fact that this paper's relation "$\leq$" is defined in terms of surjections. If we stick closely to this paper's definitions and eschew AC, then the argument should run as follows:

$J \neq \emptyset$ since $f, g \in J$. Since $J \subseteq \mathcal{K}(C, B)$ and $J \neq \emptyset$, we have a surjection $J \xleftarrow{\alpha} \mathcal{K}(C, B)$ (map all elements of J to themselves and all other elements of $\mathcal{K}(C, B)$ to say f. We have to face the fact that the paper uses case distinctions based on a non-constructive "or"). Further, since $|\mathcal{K}(C, B)| \neq \emptyset$ it follows by definition from $|\mathcal{K}(C, B)| \leq |I|$ that we have a surjection $\mathcal{K}(C, B) \xleftarrow{\beta} I$. The composite $\alpha \circ \beta$ is the desired surjection $\sigma : I \longrightarrow J$.

By the weak coproduct property there exists an $h \in \mathcal{K}(C, B)$ such that $\forall i \in I$,

$$hj_i = \begin{cases} fj_i & \text{in case } \sigma(i)j_i = gj_i, \\ gj_i & \text{in case } \sigma(i)j_i = fj_i. \end{cases}$$

Since $h \in J$ and $\sigma : I \longrightarrow J$ is surjective, we have an $i_0 \in I$ with $h = \sigma(i_0)$. Now we have

$$\sigma(i_0)j_{i_0} = fj_{i_0} \iff \sigma(i_0)j_{i_0} = gj_{i_0}$$

and therefore necessarily $fj_{i_0} = gj_{i_0}$. □

(6) **Corollary.** *For $A, B \in \operatorname{ob}\mathcal{K}$, suppose there exists a weak Ith copower C of A with respect to $\{B\}$ or a weak Ith power D of B with respect to $\{A\}$. Then $|\mathcal{K}(A, B)| \le 1$.*

Proof. Consider $r, s \in \mathcal{K}(A, B)$. In this case of a weak copower $(j_i : A \longrightarrow C)_{i \in I}$ with respect to $\{B\}$ there exists $f \in \mathcal{K}(C, B)$ such that $(\forall i \in I)\ fj_i = r$, and likewise $g \in \mathcal{K}(C, B)$ such that $(\forall i \in I)\ gj_i = s$. Theorem (5) now gives $i_0 \in I$ with $fj_{i_0} = gj_{i_0}$. Thus $r = fj_{i_0} = gj_{i_0} = s$. The remaining claim (in the weak copower case) is dual. □

> Translator's note: In the above proof I have changed the paper's f, g, $\overline{f}$, $\overline{g}$ to r, s, f, g to make the lettering agree with Theorem (5).

(7) **Corollary.** *For each object A of $\mathcal{K}$ the following are equivalent:*
- *(i) For all objects B in $\mathcal{K}$, $|\mathcal{K}(A, B)| \le 1$.*
- *(ii) For each nonvoid class I there exists an Ith copower of A.*
- *(iii) For each nonvoid class I there exists a weak Ith copower of A.*
- *(iv) There exists a weak Ith copower of A with $I = \operatorname{Mor}\mathcal{K}$.*
- *(v) There exists a weak Ith copower C of A for a class I with $|\mathcal{K}(C, B)| \le |I|$ for all objects B in $\mathcal{K}$.*

Proof. (i) $\Longrightarrow$ (ii): Trivial: take the copower $(A \xrightarrow{1_A} A)_{i \in I}$.

Also trivially, (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v). Use Corollary (6) to show (v) $\Longrightarrow$ (i). □

3. Applications

We return to the theorems mentioned in Section 1.

Re (1): In (6) we take $\mathcal{K}$ to be the category of sets, $A = 1$, $B = 2$. Then the assumption $|2^X| \le |X|$ leads to the contradiction $2 \le 1$.

> Translator's comment: The output is here that there is no surjection from a set X to 2^X. This is indeed the straightforward content of the usual elementary proof of Cantor's Theorem.

Re (2): Freyd's Theorem follows directly from (7). One can formulate a sequence of variants. e.g. the following:

Any category with countable (co-)powers which allows a faithful functor into the category of finite sets, is a partially ordered class.

Re (3): At the basis of the considerations in [3, 4, 6] one has the following consequence of (6):

(8) **Corollary.** *Consider a functor $T : \mathcal{A} \longrightarrow \mathcal{X}$ and morphisms $f, g : A \longrightarrow B$ in $\mathcal{A}$ with $Tf = Tg =: x$. Assume that we have a class I, a source $(D \xrightarrow{m_i} B)_{i \in I}$ in $\mathcal{A}$ and a morphism $e : TA \longrightarrow TD$ in $\mathcal{X}$ with the following properties:*

(a) $Tm_i)e = x$ *for all* $i \in I$.
(b) $J = \{\, h \in \mathcal{A}(A, D) \mid Th = e \,\}$ *satisfies* $|J| \le |I|$.
(c) To each source $(\xrightarrow{a_i} B)_{i \in I}$ *in* $\mathcal{A}$ *with* $(\forall i \in I)(Ta_i = x)$ *there exists* $h \in J$ *with* $m_i h = a_i$ *for all* $i \in I$.

Then $f = g$.

Proof. Let $\mathcal{K}$ be the comma category $\langle TA, T\rangle$, that is, having objects all pairs (y, E) with $E \in \operatorname{ob}\mathcal{A}$ and $y \in \mathcal{X}(TA, TE)$, and having morphisms $k : (y, E) \longrightarrow (z, F)$ given by $k \in \mathcal{A}(E, F)$ with $(Tk)y = z$. The conditions 1.–3. now ensure the existence of a weak Ith power (e, D) of (x, B) with respect to $\{(1_{TA}, A)\}$ with $\bigl|\mathcal{K}\bigl((1_{TA}, A), (e, D)\bigr)\bigr| \le |I|$ and make Corollary (6) applicable.

> TRANSLATOR'S NOTE: In the second line above, I have twice written $(1_{TA}, A)$ where the paper writes (TA, A), for ease of reading. Of course TA and 1_{TA} are the same thing.
> *Quick key to understanding the proof:* One has $J = \mathcal{K}\bigl((1_{TA}, A), (e, D)\bigr)$ and $J \neq \emptyset$ because $f, g \in J$.
> The weak Ith power of (x, B) with respect to $\{(1_{TA}, A)\}$ is, in more detail, the source
> $$\bigl((e, D) \xrightarrow{m_i} (x, B)\bigr)_{i \in I}$$
> which clearly lies in $\mathcal{K}$.

$\square$

Re (4): A (weak) pushout of a source $(a_i : A \longrightarrow B_i)_{i \in I}$ in $\mathcal{K}$ is a (weak) coproduct in the comma category $\langle A, \mathcal{K}\rangle$ whose objects are morphisms with domain A, in other words a diagram

(*)
$$\begin{tikzcd} & B_i \arrow[dr, "p_i"] & \\ A \arrow[ur, "a_i"] \arrow[rr, "p"'] & & C \end{tikzcd}$$

with the usual (weak) universal property.

(9) **Corollary.** *If [the above diagram] (*) is a weak pushout with* $|\mathcal{K}(C, B)| \le |I|$ *for all objects* B *in* $\mathcal{K}$*, then the following are equivalent:*
(i) p *is an epimorphism.*
(ii) For all $i \in I$, p_i *is an epimorphism.*

Proof. (i) $\Longrightarrow$ (ii) is trivial, while (ii) $\Longrightarrow$ (i) follows from Theorem (5). $\square$

Application of Corollary (7) yields the result which we mentioned in (4):

(10) **Corollary.** *If [the above diagram] (*) is a weak pushout with* $a_i = f$ *for all* $i \in I$ *and* $I = \operatorname{Mor}\mathcal{K}$*, then* f *is an epimorphism.*

Corollary (10) comprises a surprising consequence: If each (possibly large) cone in a category $\mathcal{A}$ possesses an $(\mathcal{E}, \mathcal{M})$-factorization fulfilling Isbell's diagonal condition, with $\mathcal{E}$ being a subclass of $\mathrm{Mor}\,\mathcal{A}$ and $\mathcal{M}$ a class of cones in $\mathcal{A}$, then $\mathcal{E}$ necessarily consists just of epimorphisms. Indeed, in this case there exist in $\mathcal{A}$ pushouts (co-intersections) of arbitrarily large families of morphisms from $\mathcal{E}$.

> Translator's note: The paper is slightly ambiguous about the "class of cones" $\mathcal{M}$. In fact one should speak of a conglomerate.

In particular it implies that in Herrlich's definition of an $(\mathcal{E}, \mathcal{M})$-category (compare [3]) the assumption that $\mathcal{E}$ consists only of epimorphisms is no restriction in generality. The corresponding fact holds more generally for cone factorizations relative to a functor (compare [5]).

It is clear that in Corollary (10) instead of $I = \mathrm{Mor}\,\mathcal{K}$ one can take any proper class, provided one presupposes a set theory which implies the equipotency of any two proper classes. One achieves this if one can well-order the universal class, which in turn can be guaranteed by a Strong Axiom of Choice: There should exist a function which selects one element from every non-void set. We are indebted to Professor J. Diller for this remark.

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