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Birkhoff's Theorem for Categories

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Introduction

In this paper we investigate the relations between two basic properties of categories

(I) the existence of a cogenerating set of objects, and

(II) the existence of enough injective objects.

In the first part we prove the validity of implication (II) $\Rightarrow$ (I) under some mild side conditions, whereas (I) $\Rightarrow$ (II) is discussed in the second part. The implication (II) $\Rightarrow$ (I) is proved by generalizing

Birkhoff's Subdirect Representation Theorem (cf. [5]): Every (finitary) universal algebra is a subdirect product of subdirectly irreducible algebras.

By this theorem, varieties admitting only a set of non-isomorphic subdirectly irreducible algebras possess a cogenerating set. They are called residually small and are well characterized by Taylor [8] and Banaschewski and Nelson [3]. It will be shown that many of their results still hold in fairly general categories admitting a certain generating set. These categories include all wellpowered locally $\aleph_0$-presentable categories in the sense of Gabriel and Ulmer, in particular all Grothendieck categories with a generator and all quasi-varieties of (finitary) universal algebras.

The second part of the paper consists of a generalization of Barr's result [4] on the existence of injective effacements in coregular categories and of an application of Banaschewski's important result [1], [2] on the existence of injective hulls. By the generalization of Barr's Theorem we are able to show that property (I) is equivalent to a weakening of (II), namely the existence of so called local injective effacements, whereas Banaschewski's Theorem shows the equivalence between (II) and a strengthening of (I), namely the existence of a cogenerating set consisting of injective objects.

Throughout the paper, for the sake of brevity, let A be an abstract category with small hom-sets satisfying the following properties:

(A) A is complete and cocomplete,

(B) A is endowed with a proper (E,M)-factorization

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* I am indebted to B. Banaschewski for useful directions.

system such that A is E-cowellpowered and A is M-cowellpowered (cf. Freyd and Kelly [6]),

(C) A possesses a generating set G of objects (i.e., for every pair of different morphisms $g,h : A \to B$ there is a $G \in G$ and a morphism $f : G \to A$ with $gf \neq gh$) such that, for every $G \in G$, the hom-functor $A(G,-) : A \to Set$ preserves colimits of chains (i.e., direct limits indexed by some segment of the ordinals).

Most of the results given in this paper hold under weaker conditions. A careful analysis of the assumptions really needed at each single stage can be found in an extended version of this paper (cf. [10]). Note that all definitions are with respect to the chosen (E,M)-factorization system.

"(II) ⇒ (I)"

The first definition contains the basic notion of the paper:

Definition. (1) An object S of A is called subdirectly irreducible, if there exists an object X and two different morphisms $x,y : X \to S$ such that any morphism f with domain S and $fx \neq fy$ belongs to M.

(2) A is called residually small, if there is, up to isomorphisms, only a set of subdirectly irreducible objects.

One easily proves that S is subdirectly irreducible, iff any small mono-source $(e_i : S \to B_i)_{i \in I}$ (i.e., $e_i u = e_i v$ for all i only if $u = v$) with all e_i belonging to E contains at least one isomorphism. Since mono-sources correspond to monomorphisms into direct products the notion given above coincides with the classical concept of subdirect irreducibility. But, throughout this paper, we only need the description given in the Definition which avoids any use of direct products and which allows an immediate proof of the following

Proposition (Birkhoff's Subdirect Representation Theorem). For every object A of A there is a small mono-source $(e_i : A \to S_i)_{i \in I}$ with all e_i belonging to E and with all S_i being subdirectly irreducible.

Proof: For a given $G \in G$ and a pair $x,y : G \to A$ of different morphisms consider a representative set of E-morphisms with domain A which leave x and y different. By Zorn's Lemma, we find a maximal element $e_{xy} : A \to S_{xy}$ in this partially ordered set. For this purpose, let $(f_{\alpha\beta} : X_\alpha \to Y_\beta)_{0 \le \alpha \le \beta < \lambda}$ be any chain in this set with $X_0 = A$. Let $(h_\alpha : X_\alpha \to L)_{0 \le \alpha < \lambda}$ be its colimit which is preserved by $A(G,-)$. Therefore, as all $f_{0\alpha}$ satisfy the inequality $f_{0\alpha}x \neq f_{0\alpha}y$, we have $h_0 x \neq h_0 y$. Let $h_0 = me$ be an (E,M)-factorization. Then, as $me = h_\alpha f_{0\alpha}$ and by the diagonalization property, one recognizes e as an upper bound of the $f_{0\alpha}$'s.

Obviously, the family $(e_{xy})_{x \neq y,\, G \in \mathcal{G}}$ forms a small mono-source, since for every pair of different morphisms $u,v : B \to A$ one has a $G \in \mathcal{G}$ and a morphism $w : G \to B$ with $x := uw \neq vw =: y$, whence $e_{xy}x \neq e_{xy}y$ and thus $e_{xy}u \neq e_{xy}v$. It remains to be proved that every S_{xy} is subdirectly irreducible: Every g with $g e_{xy} x \neq g e_{xy} y$ can be (E,M)-factorized as $g = np$ and hence $p e_{xy} x \neq p e_{xy} y$. Since $p e_{xy} \in E$, by maximality of e_{xy}, p has to be an isomorphism. This means $g \cong n \in M$.

Remarks. (1) In this form the above Proposition was first announced in [9]. Two other categorical versions of the Subdirect Representation Theorem were known before, both being more restrictive than the above result: Wiegandt [12] is dealing with "group like" categories, and Vinarek [11] is working within certain concrete categories admitting a two-point cogenerator. Both theorems do not cover Birkhoff's original result (in contrast to the above Proposition).

(2) The proof of the Proposition shows which items of the general assumptions (A) - (C) are really needed. For instance, completeness is obviously not needed. However, condition (C) turns out to be essential even if one only deals with monadic categories over $\mathcal{Set}$: Let $\mathcal{A}$ be the category of compact Abelian groups which is dually equivalent to the category $\mathcal{Ab}$ of Abelian groups. The sphere $S^1 \in \mathcal{A}$ corresponds to $Z \in \mathcal{Ab}$. Since all subgroups of Z are not subdirectly irreducible in $\mathcal{Ab}^{op}$, S^1 has no subdirect representation in $\mathcal{A}$. This example is due to Wiegandt [12].

By the Proposition, every residually small category (satisfying (A) - (C)) possesses a cogenerating set consisting of subdirectly irreducible objects. It turns out that residual smallness is not only a sufficient but also a necessary condition for the existence of a cogenerating set:

Corollary 1. $\mathcal{A}$ possesses a cogenerating set if and only if $\mathcal{A}$ is residually small.

Proof: Let $\mathcal{C}$ be a cogenerating set of $\mathcal{A}$, and let S be subdirectly irreducible. We choose $x,y : X \to S$ as in the Definition and will find an $f : S \to C$ with $fx \neq fy$ and $C \in \mathcal{C}$. Since f must belong to M, S appears as a subobject of a member of $\mathcal{C}$, and there is, up to isomorphisms, only a set of those.

This proof generalizes earlier observations due to Isbell and Pareigis and Sweedler [7], namely that a category with a cogenerating set contains only a set of simple objects which are in particular subdirectly irreducible.

To complete the proof of (II) $\Rightarrow$ (I) we need two lemmas which are well known for universal algebras. The first one is based on the observation that a subdirectly irreducible object is representable as an essential extension of an object with two generators. As usual, an M-morphism m is called *essential*, if for all f one has $fm \in M$ only if $f \in M$. M^* denotes the class of all essential M-morphisms.

Lemma 1. If A is M^*-cowellpowered, then A is residually small.

Proof: For every subdirectly irreducible object S one can choose a $G \in G$ and different morphisms $g,h : G \to S$ such that any $f : S \to A$ with $fg \neq fh$ belongs to M. With i,j being canonical injections one has a $t : G \quad G \to S$ with $ti = g$ and $tj = h$ which can be (E,M)-factorized: $t = me$. We show that $m \in M^*$. Assume $fm \in M$; since $fg = fh$ would imply $fmei = fmej$, hence $ei = ej$ and so $g = h$, we have immediately $f \in M$. Hence we proved that the subdirectly irreducible objects appear as M^*-extensions of E-quotients of twofold copowers of objects of G, and there is only a set of those.

Now, property (II) is easily seen to imply M^*-cowellpoweredness. In fact, one needs less than enough injectives. Recall that an *injective effacement* (cf. Zimmermann [13]) of an object X is an M-morphism $u : X \to I$ such that every diagram

$$\begin{array}{ccc} X & \xrightarrow{\;u\;} & I \\ {\scriptstyle f}\uparrow & & \\ Y & \xrightarrow{\;m\;} & Z \end{array}$$

with $m \in M$ can be completed to a commutative square by a morphism $f' : Z \to I$; if this property is assumed to hold only for $f = 1$ we call u a *local injective effacement* of X. A is said to have (local) injective effacements, if every object of A has a (local) injective effacement.

Lemma 2. If A has local injective effacements, A is M^*-cowellpowered.

Proof: For a given object X of A one chooses a local injective effacement $u : X \to I$. Then u factors over every essential extension of X; moreover, it factors by an M-morphism. Therefore, the essential extensions of X appear as subobjects of I.

Corollary 2. (II) $\Rightarrow$ (I).

"(I) $\Rightarrow$ (II)"

As property (I) is implied by a weakening of (II) one cannot expect that (I) $\Rightarrow$ (II) holds in general. One can, however, expect that this weakening is a necessary condition of (I). This turns out to be true up to the condition that A has cointersections of M-morphisms, i.e., multiple pushouts of M-morphisms belong to M, provided one uses a stronger notion of a cogenerating set: A set C of A-objects is called M-cogenerating, iff for every object A of A the canonical morphism

$$A \to \Pi_{C \in \mathcal{C}} \Pi_{f \in \mathcal{A}(A,C)} C$$

belongs to $\mathcal{M}$. For $\mathcal{M}$ = monomorphisms this means nothing new, and for $\mathcal{M}$ = external monomorphisms one has the known notion of a strong cogenerating set. Analyzing a result due to Barr [4] one now gets:

Lemma 3. Let $\mathcal{A}$ possess an $\mathcal{M}$-cogenerating set. $\mathcal{A}$ then has local injective effacements if and only if $\mathcal{A}$ has cointersections of $\mathcal{M}$-morphisms.

Proof: The necessity of the cointersection condition is obvious. Let us therefore construct a local injective effacement of an object X assuming this condition. We consider the set H_X of all morphisms with domain X and with codomain in the cogenerating set $\mathcal{C}$. For each subset $F \subseteq H_X$ let $\hat{F}$ be the induced morphism

$$X \to \Pi_{f \in F} \text{codomain}(f) .$$

Take now $u : X \to I$ to be the multiple pushout of those $\hat{F}$'s which belong to $\mathcal{M}$. In order to see that u is a local injective effacement consider any $\mathcal{M}$-morphism $m : X \to Y$ for which, by assumption on $\mathcal{C}$ there exists some $\mathcal{M}$-morphism $n : Y \to \Pi_{i \in I} C_i$ with $C_i \in \mathcal{C}$. The $\mathcal{M}$-morphism nm factorizes over $\hat{F}$ with $F = \{p_i n m \mid i \in I\}$ (p_i being projections) such that $\hat{F}$ belongs to $\mathcal{M}$. On the other hand, $\hat{F}$ also factorizes over nm, and, therefore, u factorizes over m.

In order to apply Lemma 3 in connection with the preceding results we have to restrict ourselves to the case $\mathcal{M}$ = monomorphisms. This is assumed for the rest of the paper.

Theorem 1. If $\mathcal{A}$ has cointersections, the following conditions are equivalent:

(i) $\mathcal{A}$ has a cogenerating set ($\Longleftrightarrow$ (I)),

(ii) $\mathcal{A}$ is residually small,

(iii) $\mathcal{A}$ is cowellpowered with respect to essential extensions,

(iv) $\mathcal{A}$ has local injective effacements.

Proof: (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) follows by Lemma 2, Lemma 1, the Proposition, and Lemma 3 successively.

One easily checks that $\mathcal{A}$ has injective effacements, iff $\mathcal{A}$ has local injective effacements and $\mathcal{M}$ is couniversal i.e., $\mathcal{M}$-morphisms are preserved by pushouts. Therefore, by adding this last condition to each of the equivalent conditions of Theorem 1 one obtains characterizations for the existence of injective effacements. The assumption that $\mathcal{A}$ has to have cointersections is then automatically satisfied since cointersections can be formed by transfinite induction using (twofold) pushouts and colimits of chains of monomorphisms which are again monomorphic by condition (C).

The surprising fact is that in this way not only injective effacements can be obtained, but extensions into injective objects, even essential extensions into injective objects, i.e., injective hulls. In order to prove this we apply Banaschewski's existence theorem on injective hulls (cf. [2]). We have to check his conditions (E1) to (E6) on M, of which (E1) and (E2) are automatically fulfilled for any class M belonging to a factorization system. (E4) means couniversality, (E5) means closedness under colimits of chains and is implied by (C), and (E6) is M^*-cowellpoweredness. It remains to be proved, that (E3) is always fulfilled within the present context. This is done by the following lemma (which holds for arbitrary M):

Lemma 4. For any $m \in M$ there is an $e \in E$ with em being M-essential.

Proof: Very similarly to the corresponding part of the proof of the Proposition one chooses a maximal element in a representative set of all E-morphisms whose domain is the codomain of m and whose composition with m belongs still to M.

We are now ready to state

Theorem 2. The following conditions are equivalent:

(i) A has a cogenerating set consisting of injective objects,

(ii) A is residually small, and monomorphisms are couniversal,

(iii) A is cowellpowered with respect to essential extensions, and monomorphisms are couniversal,

(iv) A has injective effacements,

(v) A has enough injective objects ($\Longleftrightarrow$ (II)),

(vi) A has injective hulls.

Proof: (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) follows from Theorem 1. (ii) $\Rightarrow$ (vi) follows from Lemma 4 and Banaschewski's Theorem. (v) $\Rightarrow$ (i) follows from the Proposition and the trivial observation, that in a category with enough injectives and a cogenerating set the latter can be choosen as in condition (i). (i) $\Rightarrow$ (v) is wellknown (and trivial).

Remark. From the esthetic point of view it is a little disappointing that, in the last part of the paper, we had to restrict ourselves to the case M = monomorphisms. The reason for this lies in the fact that the Proposition only yields the existence of a cogenerating set instead of an M-cogenerating set. One therefore wishes to solve the following problem: Is there a generalized version of the notions "subdirectly irreducible object" and "residual smallness" such that Corollary 1 holds with "$\mathcal{M}$ cogenerating" instead of "cogenerating"?

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