

Concrete Dualities

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Abstract In this article we give an overview of the two-step procedure that leads to important classical dualities, such as Stone, Gelfand-Naimark, and Pontrjagin-van Kampen Duality. Thus, in the context of concrete categories, we describe how *schizophrenic objects* induce dual adjunctions which then, by restriction, yield dualities. This procedure is illustrated by various examples.

Introduction

A *duality* — sometimes also called a *dual equivalence* — is an equivalence between a category \mathbf{B} and the dual of some category \mathbf{A} , i.e., it is given by a pair of contravariant functors $S: \mathbf{B} \rightarrow \mathbf{A}$ and $T: \mathbf{A} \rightarrow \mathbf{B}$ and a pair of natural isomorphisms $\eta: 1_{\mathbf{B}} \rightarrow TS$ and $\epsilon: 1_{\mathbf{A}} \rightarrow ST$, which can be chosen such that for every \mathbf{A} -object A and every \mathbf{B} -object B the following equations hold:

$$T\epsilon_A \circ \eta_{TA} = 1_{TA} \quad \text{and} \quad S\eta_B \circ \epsilon_{SB} = 1_{SB}. \quad (1)$$

Of particular interest are dualities where the categories involved are equipped with faithful representable underlying functors

$$U: \mathbf{A} \rightarrow \mathbf{Set} \quad \text{and} \quad V: \mathbf{B} \rightarrow \mathbf{Set}.$$

Hence U and V are, up to natural isomorphisms, (covariant) hom-functors. This means that there are objects $A_0 \in \mathbf{A}$ and $B_0 \in \mathbf{B}$ — necessarily generators in their respective categories — such that

$$U \cong \mathbf{A}(A_0, -) \quad \text{and} \quad V \cong \mathbf{B}(B_0, -). \quad (2)$$

Note that these representing objects then are “free objects with one free generator”, and that they are available in many “everyday categories” (see [12, 30.3] for a list of examples). Throughout this paper we will only consider concrete categories of this type.

Every undergraduate student has encountered a duality of this type: the construction of the dual space of a finite dimensional K -vector space yields a duality between the category $\mathbf{K}\text{-Vec}_{fin}$ of finite dimensional vector spaces over some field K and itself. On the other hand, even nice categories \mathbf{A} may not admit a duality of this type, because they don't have a cogenerator, as the category \mathbf{Grp} of all groups [1, 7.18(8)]. Note that, for a given category \mathbf{A} , the existence of a duality with some concrete category \mathbf{B} might give considerable additional information about \mathbf{A} : if e.g. \mathbf{B} has limits — often quite obvious constructions in concrete categories

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— the category \mathbf{A} automatically will have colimits which, moreover, can be described explicitly (for \mathbf{A} algebraic usually a difficult task) as S -images of limits in \mathbf{B} .

Dualities typically arise in representation theory, that is in the study of the question whether a given structure (i.e. an arbitrary object of some category \mathbf{A}) can be represented via other familiar structures (i.e. objects of some well understood category \mathbf{B}) by means of some explicit construction. For example, the question whether every Boolean algebra can be represented as a collection of subsets of some set leads to *Stone Duality* (see Section 4). Also our leading example (see Section 1-A) will make this idea explicit.

The leading example will show another typical feature of how dualities might arise: the construction one starts with in order to obtain the desired representation of \mathbf{A} -objects via objects of \mathbf{B} might yield a *dual adjunction* only, i.e., functors S and T and natural transformations η and ϵ fulfilling the equations (1), but lacking the property of being isomorphisms. It turns out, however, that every dual adjunction induces a “maximal” dual equivalence between a pair of subcategories of \mathbf{A} and \mathbf{B} respectively (see Section 2-A). For this reason we start our exposition by studying dual adjunctions.

First we focus on the striking similarity of all known duality constructions: they can always be described by “dualizing objects” (the scalar field K in the case of finite dimensional vector spaces, and the two-element chain or space respectively in the case of Stone Duality). This observation is explained — even on the more general level of dual adjunctions — in Section 1-B: dual adjunctions between a pair of concrete categories \mathbf{A} and \mathbf{B} *have to be represented* by a pair of objects (\tilde{A}, \tilde{B}) of the respective categories with the same underlying set C . Observed by various authors in the late sixties or early seventies, this situation has been described by calling (\tilde{A}, \tilde{B}) objects *keeping summer and winter homes* (Isbell [14]) or calling C an object *sitting in both \mathbf{A} and \mathbf{B}* (Lawvere — see [19]) or a *schizophrenic object* (Simmons [30]).

Next (Section 1-C) we describe how to identify such objects, i.e., we formulate a set of conditions which let a pair (\tilde{A}, \tilde{B}) with the same underlying set really induce a dual adjunction. We will use the term *schizophrenic object* only for pairs satisfying these conditions, which are — as opposed to earlier published ones — of a nature sufficiently general in order to be applied to “algebraic” as well as to “topological” situations. Dual adjunctions not arising this way are discussed in Section 1-D.

In Section 2 we describe the mechanism of restricting a dual adjunction to a duality, with particular emphasis to the cogenerator properties a schizophrenic object enjoys. Finally we complete the discussion of our leading example.

We then (Section 3) briefly discuss the situation where the categories involved are particularly nice (e.g. algebraic or monotopological) before giving quite a number of examples. In a last section we give a summary of results concerning uniqueness of dualities and non-existence of dualities between algebraic categories with rank.

Our reference list cannot be seen as a complete account of the previous extensive categorical studies on dualities. Nevertheless it covers a substantial number of these.

1 Dual Adjunctions and Schizophrenic Objects

1-A The Leading Example

In the same way as the notion of a Boolean algebra arises as an abstraction of the power set $\mathcal{P}(X)$ of a set X , one gets the notion of a frame as an abstraction from the topology $\xi \subset \mathcal{P}(X)$ of a topological space (X, ξ) : a *frame* is a complete lattice which satisfies the distributive law

$a \wedge \bigvee b_i = \bigvee a \wedge b_i$ ¹. Let us denote by **Frm** the category with frames as objects and those maps as morphisms which preserve arbitrary sups and finite infs. **Frm** is a monadic category (see [15, II 1.2])². By **Top** we denote the category of topological spaces. Both **Frm** and **Top** are equipped with representable underlying functors: any singleton space may serve as representing object B_0 of $V: \mathbf{Top} \rightarrow \mathbf{Set}$, while the underlying functor $U: \mathbf{Frm} \rightarrow \mathbf{Set}$ is represented by the free frame with one free generator a , i.e., by the 3-chain $\frac{1}{a}$.

One can easily check that the assignment $B = (X; \xi) \mapsto \xi$ extends to a contravariant functor $\bar{S}: \mathbf{Top} \rightarrow \mathbf{Frm}$ if one defines $\bar{S}f$ by $O \mapsto f^{-1}[O]$ for a continuous map $f: (X, \xi) \rightarrow (Y, \nu)$ and $O \in \nu$. Observe, that — via characteristic functions — $\bar{S}B$ is naturally equivalent to the functor S , where SB is the hom-set $\mathbf{Top}(B, \mathbb{S})$ (\mathbb{S} denotes the Sierpinski space $\{0,1\}$ with $\{1\}$ open) provided with the *pointwise order*: $u \leq v$ iff $u(y) \leq v(y)$ in the initial frame $\mathbb{2} = \{0,1\}$ for all $y \in B$, and where Sf maps a continuous map ν to $\nu \circ f$.

It is now a reasonable question to ask whether any frame can be represented by a topological space, i.e. if, for any frame A , we can find a topological space TA such that A and STA are isomorphic in **Frm**.

In order to define $T: \mathbf{Frm} \rightarrow \mathbf{Top}$, let $\mathbf{Frm}(A, \mathbb{2})$ be underlying set of TA (this is motivated e.g. by the results of Section 1-B). As a topology on $\mathbf{Frm}(A, \mathbb{2})$ we choose the set $\{ \{p \in \mathbf{Frm}(A, \mathbb{2}) \mid p(x) = 1\} \mid x \in A \}$. It is worthwhile mentioning that this topology is nothing but the *initial topology* on $\mathbf{Frm}(A, \mathbb{2})$ with respect to the maps

$$\epsilon_A(x): \mathbf{Frm}(A, \mathbb{2}) \rightarrow \mathbb{S} \text{ with } p \mapsto p(x), x \in A. \tag{3}$$

Indeed, one easily checks that a map $h: Z \rightarrow \mathbf{Frm}(A, \mathbb{2})$ from any topological space Z is continuous iff all composites $\epsilon_A(x) \circ h$ are.

Moreover, from (3) one obtains a frame morphism $\epsilon_A: A \rightarrow STA$. In order to verify functoriality of our construction T and its adjointness to S , it suffices to show that, for every space B and every frame morphism $f: A \rightarrow SB$, there is a unique continuous map $g: B \rightarrow TA$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\epsilon_A} & STA \\ & \searrow f & \downarrow Sg \\ & & SB \end{array}$$

This is easily done (see [15, p.42]).

One now might check explicitly that the units $\eta_B: B \rightarrow TSB$ (for a space B) are given by

$$\eta_B(y): \mathbf{Top}(B, \mathbb{S}) \rightarrow \mathbb{2} \text{ with } u \mapsto u(y), y \in B, \tag{4}$$

but this will follow automatically from the general results of 1-B. However, we would like to stress the fact, that the frame structure on $\mathbf{Top}(B, \mathbb{S})$ chosen above in order to define SB makes

¹Remember that in the set ξ of open sets of a space (X, ξ) — ordered by inclusion — the sup of a subset S simply is the union of its members, while the inf is the intersection of the members of S only for a finite S — in general it will be the interior of the intersection.

²Observe that **Frm** is the dual of the category of locales as discussed in [16] elsewhere in this volume.

the family $(\eta_B(y))_{y \in B}$ an initial family in \mathbf{Frm} : a map $j: P \rightarrow \mathbf{Top}(B, \mathbb{S})$ is a frame morphism iff all composites $\eta_B(y) \circ j$ are.

We summarize these results as follows:

1.1 Proposition *The concrete categories \mathbf{Frm} and \mathbf{Top} have representable underlying functors U and V respectively. They admit a dual adjunction, given by functors $S: \mathbf{Top} \rightarrow \mathbf{Frm}$ and $T: \mathbf{Frm} \rightarrow \mathbf{Top}$ and natural transformations $\eta: 1_{\mathbf{Top}} \rightarrow TS$ and $\epsilon: 1_{\mathbf{Frm}} \rightarrow ST$ such that the following hold:*

1. S and T are "structured hom-functors" in the sense that $US = \mathbf{Top}(-, \mathbb{S})$ and $VT = \mathbf{Frm}(-, \mathbb{2})$.
2. The "representing objects" \mathbb{S} and $\mathbb{2}$ of S and T respectively have the same underlying sets.
3. The units η_B and ϵ_A are given "by evaluation" as in (4) and (3) respectively.
4. The families $(\epsilon_A(x): \mathbf{Frm}(A, \mathbb{2}) \rightarrow \mathbb{S})_{x \in A}$ and $(\eta_B(y): \mathbf{Top}(B, \mathbb{S}) \rightarrow \mathbb{2})_{y \in B}$ are U - and V -initial respectively.

1-B The Structure of Dual Adjunctions

We consider a pair of concrete categories with representable underlying functors

$$A(A_0, -) \cong U: A \rightarrow \mathbf{Set} \quad \text{and} \quad B(B_0, -) \cong V: B \rightarrow \mathbf{Set}$$

and a dual adjunction

$$S: B \rightarrow A, \quad T: A \rightarrow B; \quad \eta: 1_B \rightarrow TS, \quad \epsilon: 1_A \rightarrow ST$$

such that the equations (1) hold; then certainly there will be an isomorphism, natural in $A \in A$ and $B \in B$,

$$A(A, SB) \cong B(B, TA) \tag{5}$$

We will refer to the situation explained above as to the *basic situation*. We will often use the following simplified notation for images under U or V : instead of $Uf: UA \rightarrow UA'$ for a given A -morphism $f: A \rightarrow A'$ we will write $[f]: [A] \rightarrow [A']$; similarly $[g] = Vg$ for a B -morphism g .

Of crucial importance for our further studies are the objects

$$\tilde{A} := SB_0 \quad \text{and} \quad \tilde{B} := TA_0.$$

We now can explain the observations 1. and 2. of Proposition 1.1 as follows:

1.2 Proposition *Given the basic situation, the following hold:*

1. *The contravariant functors $VT: A \rightarrow \mathbf{Set}$ and $US: B \rightarrow \mathbf{Set}$ are representable functors, represented by \tilde{A} and \tilde{B} respectively, i.e., there exist natural isomorphisms*

$$VF \cong A(-, \tilde{A}) \quad \text{and} \quad US \cong B(-, \tilde{B}).$$

2. *\tilde{A} and \tilde{B} have the same underlying sets, up to isomorphism, i.e., there exists a bijective map $V\tilde{B} \cong U\tilde{A}$.*

Proof Representability of V (2) and adjointness of S and T (5) yield the following (natural) isomorphisms for any $A \in \mathbf{A}$: $VT A \cong \mathbf{B}(B_0, TA) \cong \mathbf{A}(A, SB_0) = \mathbf{A}(A, \tilde{A})$; similarly $USB \cong \mathbf{B}(B, \tilde{B})$ for every $B \in \mathbf{B}$. Now, with $A = \tilde{A}$ one obtains in particular $V\tilde{B} = VT A_0 \cong \mathbf{A}(A_0, SB_0) \cong USB_0 = U\tilde{A}$. \diamond

1.3 Remark If, in the situation of Proposition 1.2, there are even strict identities

$$VT = \mathbf{A}(-, \tilde{A}) \quad \text{and} \quad US = \mathbf{B}(-, \tilde{B})$$

we say that the adjunction of S and T is *strictly represented* by (\tilde{A}, \tilde{B}) . This may, without loss of generality, always be assumed, if the functors U and V are *uniquely transportable* (i.e., in the case of U if, for every Set-isomorphism $x: X \rightarrow UA'$, there is a unique \mathbf{A} -object A and an \mathbf{A} -isomorphism $f: A \rightarrow A'$ with $UA = X$ and $Uf = x$. For "everyday concrete categories" transportability is no restriction at all (see also [1, 5.28 ff]). Therefore unique transportability of all underlying functors will be assumed throughout this paper.³

Next we want to show that, for a basic situation with the dual adjunction strictly represented by any pair of objects (\tilde{A}, \tilde{B}) , the units and co-units are necessarily given "by evaluation" as in Proposition 1.1(3.), up to the canonical bijection $[\tilde{A}] \cong [\tilde{B}]$ (which happens to be the identity in the setting of our leading example). To make this precise, let us introduce the following notation. For $A \in \mathbf{A}$ and $x \in [A]$, there is an *evaluation map*

$$\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{A}] \quad \text{with} \quad s \mapsto [s](x). \quad (6)$$

Symmetrically, for $B \in \mathbf{B}$ and $y \in [B]$, there is a map

$$\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{B}] \quad \text{with} \quad t \mapsto [t](y). \quad (7)$$

Furthermore, there are canonical maps

$$\tau: [\tilde{A}] \rightarrow [\tilde{B}] \quad \text{with} \quad \tilde{x} \mapsto [[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}), \quad (8)$$

$$\sigma: [\tilde{B}] \rightarrow [\tilde{A}] \quad \text{with} \quad \tilde{y} \mapsto [[\eta_{\tilde{B}}](\tilde{y})](1_{\tilde{B}}). \quad (9)$$

The reader is advised to verify that, in the setting of our leading example, τ and σ are identity maps.

1.4 Proposition τ and σ are inverse to each other, and the following identities hold

$$[[\epsilon_A](x)] = \tau \varphi_{A,x} \quad \text{and} \quad [[\eta_B](y)] = \sigma \psi_{B,y}.$$

Proof

$$\begin{aligned} \tau \varphi_{A,x} &= \tau([s](x)) && \text{(definition of } \varphi_{A,x}) \\ &= [[\epsilon_{\tilde{A}}][s](x)](1_{\tilde{A}}) && \text{(definition of } \tau) \\ &= [[STs][\epsilon_A](x)](1_{\tilde{A}}) && \text{(naturality of } \epsilon) \\ &= [[\epsilon_A](x) \cdot Ts](1_{\tilde{A}}) && (US \cong \mathbf{B}(-, \tilde{B})) \\ &= [[\epsilon_A](x)](s) && (VT \cong \mathbf{A}(-, \tilde{A})); \end{aligned}$$

³The uniqueness requirement is not really important; it is included only for the convenience of the reader using [1] as a general reference.

symmetrically the second identity follows.

A particular instance of the first identity is $[[\epsilon_{S\tilde{B}}](1_{\tilde{B}})](s) = \tau\varphi_{S\tilde{B},1_{\tilde{B}}}(s)$ for all $s: S\tilde{B} \rightarrow \tilde{A}$. For $s = [\eta_{\tilde{B}}](\tilde{y})$ (with $\tilde{y} \in [\tilde{B}]$), this means

$$\begin{aligned} \tau\sigma(\tilde{y}) &= \tau(\varphi_{S\tilde{B},1_{\tilde{B}}}(s)) && \text{(definition of } \sigma) \\ &= [[\epsilon_{S\tilde{B}}](1_{\tilde{B}})]([\eta_{\tilde{B}}](\tilde{y})) = \tilde{y}; \end{aligned}$$

the last identity is derived from $[S\eta_{\tilde{B}}][\epsilon_{S\tilde{B}}] = 1_{[S\tilde{B}]}$ which, when evaluated at $1_{\tilde{B}}$, gives $[\epsilon_{S\tilde{B}}](1_{\tilde{B}}) = 1_{\tilde{B}}$ since $US = \mathbf{B}(-, \tilde{B})$. Therefore $\tau\sigma = 1_{[\tilde{B}]}$, and $\sigma\tau = 1_{[\tilde{A}]}$ follows symmetrically. \diamond

1.5 Remark We observe that the families of maps $(\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{A}])_{x \in [A]}$ and $(\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{B}])_{y \in [B]}$ are point-separating families (see [1, 10.6]) in **Set** for every $A \in \mathbf{A}$ respectively $B \in \mathbf{B}$.

1-C Schizophrenic Objects Induce Natural Dual Adjunctions

We now give an answer to the question how, and under which conditions, we can establish a dual adjunction between two concrete categories $U: \mathbf{A} \rightarrow \mathbf{Set}$ and $V: \mathbf{B} \rightarrow \mathbf{Set}$, strictly represented by a given pair of objects (\tilde{A}, \tilde{B}) of \mathbf{A} and \mathbf{B} respectively. Certainly we have to assume that there is a bijection $[\tilde{A}] \xrightarrow{\tau} [\tilde{B}]$. But that hardly will suffice to enable us to define a suitable \mathbf{A} -structure on the hom-set $\mathbf{B}(B, \tilde{B})$ in order to define $SB \in \mathbf{A}$, and a suitable \mathbf{B} -structure on $\mathbf{A}(A, \tilde{A})$ in order to define $TA \in \mathbf{B}$. Observation (4.) of Proposition 1.1 in connection with Proposition 1.4 suggests an additional set of conditions which leads to the central definition in this context.

1.6 Definition A triple $(\tilde{A}, \tau, \tilde{B})$, with a pair of objects $(\tilde{A}, \tilde{B}) \in \mathbf{A} \times \mathbf{B}$ and a bijective map $\tau: [\tilde{A}] \rightarrow [\tilde{B}]$, is called a *schizophrenic object* (for the concrete categories \mathbf{A} and \mathbf{B}) if the following two conditions are satisfied⁴:

SO 1 For every $A \in \mathbf{A}$, the family $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{B}])_{x \in [A]}$ admits a V -initial lifting $(e_{A,x}: TA \rightarrow \tilde{B})_{x \in [A]}$.

SO 2 For every $B \in \mathbf{B}$, the family $(\sigma\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{A}])_{y \in [B]}$ with $\sigma = \tau^{-1}$ admits a U -initial lifting $(d_{B,y}: SB \rightarrow \tilde{A})_{y \in [B]}$.

A dual adjunction (S, T) strictly represented by (\tilde{A}, \tilde{B}) is called a *natural dual adjunction* if the families $([e_A](x): TA \rightarrow \tilde{B})_{x \in [A]}$ and $([\eta_B](y): SB \rightarrow \tilde{A})_{y \in [B]}$ are initial with respect to U and V respectively, for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$.

It is clear that the dual adjunction between **Frm** and **Top** in the leading example is natural and that $(\mathbf{2}, \mathbf{1}, \mathbf{S})$ is a schizophrenic object for **Frm** and **Top**.

We now can prove the statement of this section's title.

1.7 Theorem ([5]) For every schizophrenic object $(\tilde{A}, \tau, \tilde{B})$, there is a natural dual adjunction strictly represented by (\tilde{A}, \tilde{B}) , such that τ and $\sigma = \tau^{-1}$ are the canonical maps in the sense of Section 1-B.

⁴Recall that the requirement of SO 1 means the following: there exists a \mathbf{B} -morphism $e_{A,x}: TA \rightarrow \tilde{B}$ such that $[TA] = \mathbf{A}(A, \tilde{A})$ and $[e_{A,x}] = \tau\varphi_{A,x}$ and, whenever one has $Z \in \mathbf{B}$ and a map $h: [Z] \rightarrow \mathbf{A}(A, \tilde{A})$ such that all composites $\tau\varphi_{A,x}h$ are the underlying maps of \mathbf{B} -morphisms $Z \rightarrow \tilde{B}$, then h is the underlying map of a \mathbf{B} -morphism $Z \rightarrow TA$.

Proof T and S are already defined on objects by Conditions (SO 1) and (SO 2) respectively. Given now some \mathbf{A} -morphism $f: A \rightarrow A'$, we must find a \mathbf{B} -morphism $Tf: TA' \rightarrow TA$ with $VTf = \mathbf{A}(f, \tilde{A}): \mathbf{A}(A', \tilde{A}) \rightarrow \mathbf{A}(A, \tilde{A})$. To show that $\mathbf{A}(f, \tilde{A})$ is actually the underlying map of a \mathbf{B} -morphism, by (SO 1) it suffices to show, that all composites $\tau\varphi_{A,x} \circ \mathbf{A}(f, \tilde{A})$, $x \in [A]$, are underlying maps of \mathbf{B} -morphisms. This follows from

$$\tau\varphi_{A,x} \circ \mathbf{A}(f, \tilde{A})(s) = \tau\varphi_{A,x}(sf) = \tau\varphi_{A',[f](x)}(s) = [e_{A',[f](x)}](s).$$

Obviously, T is a functor. S can be defined symmetrically.

It remains to be shown that S and T are adjoint. For that we construct the units η_B and ϵ_A as follows. First, in order to establish the existence of $\eta_B: B \rightarrow TSB$, we only need to define a map $[\eta_B]: [B] \rightarrow [TSB] = \mathbf{A}(SB, \tilde{A})$ and then show that each composite $\tau\varphi_{SB,t} \circ [\eta_B]$, $t \in [SB] = \mathbf{B}(B, \tilde{B})$, can be lifted along V . Proposition 1.4 in connection with (SO 2) forces us to put

$$[\eta_B]: [B] \rightarrow \mathbf{A}(SB, \tilde{A}), \quad y \mapsto d_{B,y}. \tag{10}$$

Now, for all $y \in [B]$,

$$\begin{aligned} \tau\varphi_{SB,t} \circ [\eta_B](y) &= \tau[d_{B,y}](t) \quad (10), (6) \\ &= \tau\sigma\psi_{B,y}(t) \quad (\text{SO2}) \\ &= [t](y). \quad (7) \end{aligned}$$

This proves not only the existence of η_B . In addition, it shows the equations

$$e_{SB,t} \circ \eta_B = t \quad \text{for all } t \in [SB]. \tag{11}$$

Symmetrically one obtains morphisms $\epsilon_A: A \rightarrow STA$ with

$$[\epsilon_A]: [A] \rightarrow \mathbf{B}(TA, \tilde{B}), \quad x \mapsto e_{A,x}; \tag{12}$$

$$d_{TA,s} \circ \epsilon_A = s \quad \text{for all } s \in [TA]. \tag{13}$$

Naturality of η and ϵ is easily verified. Since by definition of S and (11) $[S\eta_B][\epsilon_{SB}](t) = \mathbf{B}(\eta_B, \tilde{B})(e_{SB,t}) = t$ for all $t \in \mathbf{B}(B, \tilde{B})$, the second of the identities (1) holds, since U is faithful. The other identity follows symmetrically; hence we have an adjunction.

Finally we must show that our given τ is induced by this adjunction as described in Section 1-B, i.e. that it satisfies (8). Indeed, for every $\tilde{x} \in [\tilde{A}]$ one has $[[\epsilon_{\tilde{A}}](\tilde{x})](1_{\tilde{A}}) = \tau\varphi_{\tilde{A},\tilde{x}}(1_{\tilde{A}}) = \tau([1_{\tilde{A}}](\tilde{x})) = \tau(\tilde{x})$. \diamond

1.8 Remark We will end this section with an interpretation of Condition SO 1 when the category \mathbf{B} has products which are preserved by V . Given any $A \in \mathbf{A}$, the family $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{A}])_{x \in [A]}$ induces a unique map Φ_A such that the following diagram commutes for every $x \in [A]$:

$$\begin{array}{ccc} \mathbf{A}(A, \tilde{A}) = [TA] & \xrightarrow{\Phi_A} & [\tilde{B}]^{[A]} = [\tilde{B}^{[A]}] \\ & \searrow \tau\varphi_{A,x} & \downarrow \pi_x \\ & & [\tilde{B}] \end{array}$$

Now the following is obvious:

- Φ_A admits a lifting along V iff the family $\tau\varphi_{A,x}$ admits a lifting along V .
- Φ_A becomes an embedding iff SO 1 holds.

Hence, in the presence of concrete products, a dual adjunction (S, T) is natural with schizophrenic object $(\tilde{A}, \tau, \tilde{B})$ iff for every $A \in \mathbf{A}$ and every $B \in \mathbf{B}$ there are (natural) embeddings $T \rightarrow \tilde{B}^{[-]}$ and $S \rightarrow \tilde{A}^{[-]}$.

It is clear from Proposition 1.4 that for *any* dual adjunction in the basic situation the sources discussed in SO 1 and SO 2 have to have liftings along U and V respectively. Hence in the presence of products even for non-natural dual adjunctions there will at least be (natural) monomorphisms $T \rightarrow \tilde{B}^{[-]}$ and $S \rightarrow \tilde{A}^{[-]}$. This generalizes what has been known before for dual equivalences (see [1, 10N] or [25]).

1-D Non-Natural Dual Adjunctions

It can happen — see 3-B and 4-B for examples — that a dual adjunction is *not* natural, i.e., is not induced by a schizophrenic object in the sense of the previous section. In this section we will briefly discuss *arbitrary* dual adjunctions in their relation to natural ones.

It is clear from Section 1-B that any basic situation, i.e., an arbitrary dual adjunction (S', T') between concrete categories \mathbf{A} and \mathbf{B} , determines a triple $(\tilde{A}, \tau, \tilde{B})$, with a pair of objects $(\tilde{A}, \tilde{B}) \in \mathbf{A} \times \mathbf{B}$ and a bijective map $\tau: [\tilde{A}] \rightarrow [\tilde{B}]$, such that the following weakened versions of the Conditions SO 1 and SO 2 are fulfilled:

WSO 1 For every $A \in \mathbf{A}$, the family $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{B}])_{x \in [A]}$ admits a lifting $(e'_{A,x}: T'A \rightarrow \tilde{B})_{x \in [A]}$ along V which extends functorially (i.e., for every $A \xrightarrow{f} A'$ in \mathbf{A} there exists a \mathbf{B} -morphism $T'A' \xrightarrow{T'f} T'A$ with $[T'f] = \mathbf{A}(f, \tilde{A})$).

WSO 2 For every $B \in \mathbf{B}$, the family $(\sigma\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{A}])_{y \in [B]}$ with $\sigma = \tau^{-1}$ admits a lifting $(d'_{B,y}: S'B \rightarrow \tilde{A})_{y \in [B]}$ along U which extends functorially.

Under the additional assumption that the functors U and V are solid in the sense of [1] and that \mathbf{A} and \mathbf{B} have (Epi, Monosource)-factorizations (this means in particular that U and V respectively can be decomposed as $M \circ A$ with an essentially algebraic functor A and a topological functor M (see [1, 26.3])), we can now prove that any dual adjunction is a "natural refinement" of a natural adjunction, i.e., there is the following proposition:

1.9 Proposition *Let (\mathbf{A}, U) and (\mathbf{B}, V) be concrete categories with (Epi, Monosource)-factorizations and solid functors U and V .*

For any triple $(\tilde{A}, \tau, \tilde{B})$ satisfying Conditions WSO1 and WSO 2 the following hold:

1. *The triple $(\tilde{A}, \tau, \tilde{B})$ is a schizophrenic object for \mathbf{A} and \mathbf{B} , and hence induces a natural dual adjunction (S, T) .*
2. *The functors S' and T' constitute a dual adjunction, strictly represented by (\tilde{A}, \tilde{B}) , such that τ is as in (8).*
3. *The natural dual adjunction (S, T) is larger than the dual adjunction (S', T') , in the sense that there are natural transformations $\kappa: S' \rightarrow S$ and $\lambda: T' \rightarrow T$ with $U\kappa = 1_{\mathbf{B}(-, \tilde{B})}$ and $V\lambda = 1_{\mathbf{A}(-, \tilde{A})}$.*

Proof Our assumptions on U and V guarantee that the families $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{B}])_{x \in [A]}$ and $(\sigma\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{A}])_{y \in [B]}$ with $\sigma = \tau^{-1}$ admit U - and V -initial lifts respectively, if they admit liftings at all. Now the first statement is obvious. Since $(e_{A,x}: TA \rightarrow \tilde{B})_{x \in [A]}$ is a V -initial lifting and $(e'_{A,x}: T'A \rightarrow \tilde{B})_{x \in [A]}$ is a lifting along V of the same source $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{B}])_{x \in [A]}$, the identity map $1_{\mathbf{A}(A, \tilde{A})}$ can be lifted along V to a \mathbf{B} -morphism $\lambda_A: T'A \rightarrow TA$; similarly, the identity map $1_{\mathbf{B}(B, \tilde{B})}$ lifts to an \mathbf{A} -morphism $\kappa_B: S'B \rightarrow SB$. Naturality of κ and λ then is obvious. Hence only the existence of natural transformations η' and ϵ' fulfilling (1) remains to be checked. By 1.7 we know that there are the units η and ϵ of the adjunction (S, T) . Define $\eta' := \kappa \circ \eta$ and $\epsilon' := \lambda \circ \epsilon$. Since the equations (1) hold for η and ϵ and the underlying functors are faithful, they also hold for η' and ϵ' . \diamond

2 Dualities Arising From Dual Adjunctions

2-A The Maximal Duality Induced by an Adjunction

Throughout this section we will always assume that there is given a dual adjunction $S: \mathbf{B} \rightarrow \mathbf{A}$ and $T: \mathbf{A} \rightarrow \mathbf{B}$, strictly represented by (\tilde{A}, \tilde{B}) . Naturality is not required in general. From equation (1) it follows immediately that, if ϵ_A is an isomorphism in \mathbf{A} for some $A \in \mathbf{A}$, then also η_{TA} is an isomorphism in \mathbf{B} ; similarly ϵ_{SB} will be an isomorphism if η_B is. Hence the functors S and T can be restricted to the full subcategories

$$\text{Fix}\epsilon := \{A \in \mathbf{A} \mid \epsilon_A \text{ is an isomorphism}\} \quad \text{and} \quad \text{Fix}\eta := \{B \in \mathbf{B} \mid \eta_B \text{ is an isomorphism}\}$$

where they induce a duality $S_{\text{fix}}: \text{Fix}\eta \rightarrow \text{Fix}\epsilon$, $T_{\text{fix}}: \text{Fix}\epsilon \rightarrow \text{Fix}\eta$. Obviously these *fixed subcategories* are the largest subcategories of \mathbf{A} and \mathbf{B} respectively, to which S and T can be restricted in order to obtain a duality. In general determining the fixed subcategories of a given dual adjunction can be quite difficult; it therefore will be convenient to have at hand a mechanism describing them "stepwise" by means of the following intermediate full subcategories of \mathbf{A} and \mathbf{B} respectively⁵:

$$\begin{aligned} \mathbf{A}_1 &= \{A \in \mathbf{A} \mid \epsilon_A \text{ is an embedding}\} & \mathbf{B}_1 &= \{B \in \mathbf{B} \mid \eta_B \text{ is an embedding}\} \\ \mathbf{A}_2 &= \{A \in \mathbf{A} \mid [\epsilon_A] \text{ is surjective}\} & \mathbf{B}_2 &= \{B \in \mathbf{B} \mid [\eta_B] \text{ is surjective}\} \\ \text{Im}S &= \{A \in \mathbf{A} \mid \exists B \in \mathbf{B} : A \cong SB\} & \text{Im}T &= \{B \in \mathbf{B} \mid \exists A \in \mathbf{A} : B \cong TA\} \end{aligned}$$

There are the following obvious relations between these various subcategories:

$$2.1 \text{ Lemma} \quad 1. \mathbf{A}_1 \cap \mathbf{A}_2 = \text{Fix}\epsilon \quad \text{and} \quad \mathbf{B}_1 \cap \mathbf{B}_2 = \text{Fix}\eta$$

$$2. \text{Im}S \subset \mathbf{A}_1 \quad \text{and} \quad \text{Im}T \subset \mathbf{B}_1$$

$$3. A \in \mathbf{A}_2 \implies TA \in \text{Fix}\eta \quad \text{and} \quad B \in \mathbf{B}_2 \implies SB \in \text{Fix}\epsilon$$

Proof Clearly the isomorphisms in \mathbf{A} and \mathbf{B} respectively are precisely the initial morphisms with underlying bijections (use e.g. [1, 8.14] and the fact that the underlying functors — admitting left adjoints — preserve monomorphisms).

By (1), ϵ_{SB} is a section, hence an embedding (see [1, 8.7]), for every $B \in \mathbf{B}$. Therefore $\text{Im}S \subset \mathbf{A}_1$, and, similarly $\text{Im}T \subset \mathbf{B}_1$.

For $A \in \mathbf{A}_2$, the counit ϵ_A is an epimorphism since the faithful underlying functor U reflects

⁵Recall that an *embedding* in \mathbf{A} is a U -initial monomorphism.

epimorphisms. Hence $T\epsilon_A$ is a monomorphism in \mathbf{B} by adjointness. By (1) $T\epsilon_A$ is also a retraction, thus an isomorphism. \diamond

We summarize as follows.

2.2 Theorem *Given our basic situation with a dual adjunction, one obtains, by restriction, dual adjunctions*

$$S_i: \mathbf{B}_i \rightarrow \mathbf{A}_i \quad \text{and} \quad T_i: \mathbf{A}_i \rightarrow \mathbf{B}_i$$

for $i = 1, 2$ as well as a duality

$$S_{fix}: \mathbf{Fix}\eta \rightarrow \mathbf{Fix}\epsilon, \quad T_{fix}: \mathbf{Fix}\epsilon \rightarrow \mathbf{Fix}\eta.$$

Moreover the following holds:

$$\begin{aligned} \mathbf{A}_1 \cap \mathbf{A}_2 &= \mathbf{Fix}\epsilon = \mathbf{Im}S_{fix} = \mathbf{Im}S_2 \subset \mathbf{Im}S_1 \subset \mathbf{Im}S \subset \mathbf{A}_1, \\ \mathbf{B}_1 \cap \mathbf{B}_2 &= \mathbf{Fix}\eta = \mathbf{Im}T_{fix} = \mathbf{Im}T_2 \subset \mathbf{Im}T_1 \subset \mathbf{Im}T \subset \mathbf{B}_1 \end{aligned}$$

Proof By the second statement of the lemma, there is the restricted adjunction for $i = 1$. The third statement of the lemma establishes the restriction for $i = 2$ as well as the inclusions $\mathbf{Im}T_2 \subset \mathbf{Fix}\eta$ and $\mathbf{Im}S_2 \subset \mathbf{Fix}\epsilon$. One has $\mathbf{Fix}\epsilon \subset \mathbf{Im}S_{fix}$ and, dually, $\mathbf{Fix}\eta \subset \mathbf{Im}T_{fix}$, since for an isomorphism ϵ_A one has $A \cong S(TA)$ with $TA \in \mathbf{Fix}\eta$. The rest is trivial. \diamond

The main task for establishing a duality in a concrete situation is now to identify the categories \mathbf{A}_i and \mathbf{B}_i . This can be a very hard problem, and this is where categorical guidance comes to an end. It even happens that complete solutions are not known but only sufficient conditions, guaranteeing that an object will belong to the subcategory under consideration (see. e.g. example 4-B in Section 4). However in this context the following results can be helpful — stated here for the subcategory \mathbf{A}_1 only, but which analogously then hold for \mathbf{B}_1 .

2.3 Lemma *The following statements are equivalent for every $A \in \mathbf{A}$:*

1. $(A \xrightarrow{s} \bar{A})_{s \in \mathbf{A}(A, \bar{A})}$ is a monosource in \mathbf{A} .
2. $([A] \xrightarrow{[s]} [\bar{A}])_{s \in \mathbf{A}(A, \bar{A})}$ is a point-separating source in \mathbf{Set} .
3. ϵ_A is a monomorphism in \mathbf{A} .
4. $[\epsilon_A]$ is an injective map.

Proof The equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4) follow from [1, 10.7]. To prove the equivalence (1) \Leftrightarrow (3) we observe first that, for every $B \in \mathbf{B}$, the family $(d_{B,y})_{y \in [B]}$ is jointly monomorphic as a lift of a point-separating family (see 1.5). With $B = TA$, the result now follows from equation (13) using [1, 10.9]. \diamond

Lemma 2.3 is, in view of the definition of \mathbf{A}_1 , complemented by the following result which is proven in the same way as the equivalence (1) \Leftrightarrow (3) of 2.3, using [1, 10.0] instead of [1, 10.9].

2.4 Lemma *The following statements are equivalent for every $A \in \mathbf{A}$:*

1. $(A \xrightarrow{s} \bar{A})_{s \in \mathbf{A}(A, \bar{A})}$ is a U -initial family,
2. ϵ_A is a U -initial morphism. \diamond

2-B Cogenerator-Properties of \tilde{A}

2.5 Remark Observe that, by the definition of cogenerator, the full subcategory $\mathbf{C}(\tilde{A})$ of \mathbf{A} consisting of those objects of \mathbf{A} for which the equivalent statements of 2.3 hold, is the largest subcategory of \mathbf{A} for which \tilde{A} is a cogenerator.

Recall that an object E in a category \mathbf{C} is called *extremal cogenerator*⁶ if, for every \mathbf{C} -object C , the family $\mathbf{C}(C, E)$ is an extremal monosource. In the presence of products this is equivalent to saying that, for every $C \in \mathbf{C}$, the canonical morphism $\mu_C: C \rightarrow E^{\mathbf{C}(C, E)}$ is an extremal monomorphism; i.e. the morphism μ_C making the following diagram commutative (with π_s the projection corresponding to $s \in \mathbf{C}(C, E)$)

$$\begin{array}{ccc} C & \xrightarrow{\mu_C} & E^{\mathbf{C}(C, E)} \\ & \searrow s & \downarrow \pi_s \\ & & E \end{array}$$

This weakens the concept of *regular cogenerator* where μ_C is required to be a regular monomorphism. In concrete categories the following concept will be useful:

2.6 Definition In a concrete category (\mathbf{C}, U) an object M is called *(U-) initial cogenerator* if, for every $C \in \mathbf{C}$, the family $\mathbf{C}(C, M)$ is a U -initial monosource⁷.

2.7 Remark By lemmas 2.3 and 2.4, \mathbf{A}_1 is the largest subcategory of \mathbf{A} for which \tilde{A} is an initial cogenerator.

It is useful to observe how the various notions of cogenerator are related. We mention here

- Every regular cogenerator is an extremal cogenerator.
- Every cogenerator is an extremal cogenerator if \mathbf{A} is balanced.
- Under very mild conditions, e.g., for any concrete category with products and a representable underlying functor, a regular cogenerator always is an initial cogenerator (see [5]).

Due to additional relations between regular (respectively extremal) monomorphisms and embeddings in particular situations (for definitions, see [1]) one has moreover:

- If (\mathbf{A}, U) is a topological category, then
 A is a regular cogenerator $\Leftrightarrow A$ is an extremal cogenerator $\Leftrightarrow A$ is an initial cogenerator.
- If (\mathbf{A}, U) is a monotopological category, then
 A is an extremal cogenerator $\Rightarrow A$ is an initial cogenerator.
 The converse does not hold e.g. for the category \mathbf{Haus} of Hausdorff spaces, but holds for the category \mathbf{POS} of partially ordered sets.

⁶sometimes also called a *strong cogenerator*; observe also that, unlike [1] and [15], we use *cogenerator* rather than *coseparator*.

⁷In the presence of products this is equivalent to the definition of a \mathcal{M} -cogenerator in the sense of [1] with \mathcal{M} being the class of all embeddings.

- If (\mathbf{A}, U) is an essentially algebraic category, then
 A is an extremal cogenerator $\Rightarrow A$ is an initial cogenerator $\Leftrightarrow A$ is a cogenerator
- If (\mathbf{A}, U) is a regularly algebraic category, then
 A is a regular cogenerator $\Leftrightarrow A$ is an extremal cogenerator

The following are simple consequences of this analysis and the observation that $\mathbf{A}(\tilde{A}, \tilde{A})$ contains $1_{\tilde{A}}$ and hence is an initial monosource.

2.8 Proposition *Given a basic situation with a dual adjunction, strictly represented by (\tilde{A}, \tilde{B}) , the following holds:*

1. ϵ_A is a monomorphism for every $A \in \mathbf{A}$ iff \tilde{A} is a cogenerator in \mathbf{A} .
2. \tilde{A} is a cogenerator in \mathbf{A}_1 .
3. $\mathbf{A}_1 = \mathbf{A}$ if \tilde{A} is an initial cogenerator in \mathbf{A} .

If (S, T) is even a natural dual adjunction, we have in addition:

4. \tilde{A} is an initial cogenerator of \mathbf{A}_1
5. $\mathbf{A}_1 = \mathbf{A}$ if and only if \tilde{A} is an initial cogenerator in \mathbf{A} .

The analogous statements hold for $\mathbf{B}, \mathbf{B}_1, \tilde{B}$. \diamond

2.9 Corollary *Assume a basic situation with a dual adjunction (S, T) strictly represented by (\tilde{A}, \tilde{B}) . Then*

1. $\mathbf{A}_1 = \mathbf{A}$, provided \tilde{A} is a regular cogenerator of \mathbf{A} .
2. If (S, T) is natural then \tilde{A} is a regular cogenerator of \mathbf{A}_1 , provided \mathbf{A} is either topological or regularly algebraic and balanced. \diamond

2.10 Corollary *Let (S, T) be a duality between \mathbf{A} and \mathbf{B} strictly represented by (\tilde{A}, \tilde{B}) . Then (S, T) is natural if and only if \tilde{A} and \tilde{B} are initial cogenerators in \mathbf{A} and \mathbf{B} respectively. \diamond*

2-C On the Reflectivity of the Fixed Subcategories

Next we investigate the question which closure properties the various subcategories enjoy and whether there are reflective embeddings between them. Unfortunately one cannot expect reflectivity of the fixed subcategories in \mathbf{A} or \mathbf{B} respectively, as is shown by the first example in Section 4; the following however is checked easily.

2.11 Lemma *Given our basic situation, for the statements*

1. ϵ_A is an epimorphism for every $A \in \mathbf{A}$, or η_B is an epimorphism for every $B \in \mathbf{B}$.
2. The dual adjunction (S, T) is idempotent, i.e., $\text{Im}S = \text{Fix}\epsilon$ and $\text{Im}T = \text{Fix}\eta$.
3. $\text{Fix}\epsilon$ and $\text{Fix}\eta$ are reflective subcategories of \mathbf{A} and \mathbf{B} with reflections ϵ and η respectively.

the implications $1. \Rightarrow 2. \Rightarrow 3.$ hold. \diamond

With \mathcal{M} the class of all initial monosources in \mathbf{A} , this leads to

2.12 Theorem Given a basic situation with a dual adjunction, strictly represented by (\tilde{A}, \tilde{B}) , the following hold:

1. The dual adjunction can be restricted to a duality between $\mathbf{A}_1 \cap \mathbf{A}_2$ and $\mathbf{B}_1 \cap \mathbf{B}_2$; here $\mathbf{A}_1 \cap \mathbf{A}_2$ and $\mathbf{B}_1 \cap \mathbf{B}_2$ are epireflective subcategories of \mathbf{A}_2 and \mathbf{B}_2 respectively.
2. \mathbf{A}_1 and \mathbf{B}_1 are closed under U - (respectively V -)initial monosources in \mathbf{A} and \mathbf{B} respectively.
3. \mathbf{A}_1 and \mathbf{B}_1 are closed under direct products and embeddings in \mathbf{A} and \mathbf{B} respectively.
4. If \mathbf{A} is an (E, \mathcal{M}) -category, and if the adjunction (S, T) is natural, then \mathbf{A}_1 is even E -reflective in \mathbf{A} and, in fact, the E -reflective hull of \tilde{A} in \mathbf{A} .
The analogous statement holds for $\mathbf{B}, \mathbf{B}_1, \tilde{B}$.

Proof The first statement follows from Lemma 2.11, since the restriction (S_2, T_2) of the given adjunction is idempotent again by Lemma 2.11.

For proving the second statement, let $(m_i: A \rightarrow A_i)_{i \in I}$ be a U -initial monosource in \mathbf{A} with all $A_i \in \mathbf{A}_1$. Then also ϵ_A is an embedding, since $(\epsilon_A, m_i)_{i \in I} = (STm_i \epsilon_A)_{i \in I}$. The third statement is now clear since products and embeddings are special instances of U -initial monosources.

The last statement is clear in view of [1, 16.22]. \diamond

2.13 Remark The reader not familiar with (E, \mathcal{M}) -categories might prefer the following version of statement 4.:

If \mathbf{A} is complete and wellpowered w.r.t. embeddings, and if the adjunction (S, T) is natural, then \mathbf{A}_1 is even epireflective in \mathbf{A} and, in fact, the epireflective hull of \tilde{A} in \mathbf{A} .

One cannot expect results similar to those of the previous theorem with respect to the fixed subcategories, which in fact can be empty, as the following example shows. Let $\mathbf{A} = \mathbf{B} = \mathbf{Top}$ and $\mathbf{2}$ the two-element discrete space. Then $(\mathbf{2}, 1_{\mathbf{2}}, \mathbf{2})$ is a schizophrenic object with resulting adjunction $S(X) = T(X) = \mathbf{2}^{[X]}$. There obviously is no space X with $X \cong \mathbf{2}^{[X]}$.

The situation changes completely, provided the adjunction is idempotent. Here we have

2.14 Proposition For an idempotent dual adjunction, strictly represented by (\tilde{A}, \tilde{B}) , the following hold:

1. $\mathbf{Fix}\epsilon$ is an epireflective subcategory of \mathbf{A}_1 containing \tilde{A} .
2. $\mathbf{Fix}\epsilon = \mathbf{A}_1$ if \mathbf{A} is an (E, \mathcal{M}) -category and the adjunction is natural.

The analogous statements hold for $\mathbf{B}, \mathbf{B}_1, \tilde{B}$.

Proof $\tilde{A} \cong SB_0 \in \mathbf{Fix}\epsilon$ follows by the definition of an idempotent adjunction. ϵ_A is a monic reflection morphism for $A \in \mathbf{A}_1$ by Lemma 2.11, hence an epimorphism (see [1, 16.3]).

For the last statement observe that $\mathbf{Fix}\epsilon$ now is an epireflective subcategory of \mathbf{A} and hence closed under products and embeddings. Now $\mathbf{A}_1 \subset \mathbf{Fix}\epsilon$ follows by Theorem 2.12. \diamond

2-D Following up the Leading Example

In our leading example with $\mathbf{A} = \mathbf{Frm}$ and $\mathbf{B} = \mathbf{Top}$, we have $\mathbf{A}_2 = \mathbf{A}$ (by definition of ϵ_A , see equation (3), since certainly every frame-morphism preserves 0 and 1). \mathbf{A}_1 then is the category of those frames A for which ϵ_A is injective, that is: if $a \not\leq b$ in A , then there is some $p: A \rightarrow \mathbf{2}$ with $p(a) = 1$ and $p(b) = 0$. These frames are called *spatial*. Hence $\mathbf{Fix}\epsilon = \mathbf{A}_1 = \mathbf{SpFrm}$, the full subcategory of spatial frames, and this category is epireflective in \mathbf{Frm} , in fact the epireflective hull of the chain $\mathbf{2}$, by Theorem 2.12.

For every topological space B , the map η_B is V -initial, as is easily shown. Moreover, η_B is injective iff B is a T_0 -space. Hence \mathbf{B}_1 is the category \mathbf{Top}_0 of T_0 -spaces (which is well known to be epireflective in \mathbf{Top} and, in fact, the epireflective hull of the Sierpinski space \mathbf{S}). A space B is called *sober* iff η_B is a bijection; so it follows that $\mathbf{Fix}\eta$ is the category \mathbf{Sob} of sober spaces which is epireflective in \mathbf{Top}_0 .

Hence we can summarize: the maximal duality obtainable from the dual adjunction of Proposition 1.1 is the duality between the categories \mathbf{SpFrm} and \mathbf{Sob} .

3 Starting From Nice Categories

So far our exposition hardly made any special assumptions on the concrete categories one starts with. For particular types of concrete categories, some assumptions of our approach may be weakened or the results be strengthened.

3-A Simplifying Conditions SO 1 and SO 2

The crucial assumptions of our approach are the conditions SO 1 and SO 2 of Definition 1.6. We will discuss them briefly under the assumption that our concrete categories are monotopological or essentially algebraic in the sense of [1].

3.1 Lemma *Let $U: \mathbf{A} \rightarrow \mathbf{Set}$ and $V: \mathbf{B} \rightarrow \mathbf{Set}$ be a pair of concrete categories and $(\tilde{A}, \tilde{B}) \in \mathbf{A} \times \mathbf{B}$ a pair of objects equipped with a bijection $[\tilde{A}] \xrightarrow{\tau} [\tilde{B}]$. Then the following hold:*

1. *If V is monotopological, the source $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{A}])_{x \in [A]}$ admits a V -initial lifting.*
2. *If U is essentially algebraic, any lifting of the source $(\sigma\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{B}])_{y \in [B]}$ along U will be a U -initial lifting. In particular, one has that*

(a) $\mathbf{A}_1 = \{A \in \mathbf{A} \mid [\epsilon_A] \text{ is injective}\}$.

(b) *The Conditions SO 1 and WSO 1 are equivalent and, hence, the functors S and S' of Proposition 1.9 coincide.*

Proof Since the families $(\tau\varphi_{A,x}: \mathbf{A}(A, \tilde{A}) \rightarrow [\tilde{A}])_{x \in [A]}$ and $(\sigma\psi_{B,y}: \mathbf{B}(B, \tilde{B}) \rightarrow [\tilde{B}])_{y \in [B]}$ to be lifted are point-separating sources, the first statement is immediate from the definition of a monotopological functor (see [1, 21.40]), while the second statement follows from the fact that, for an essentially algebraic category $U: \mathbf{A} \rightarrow \mathbf{Set}$, every monosource in \mathbf{A} is U -initial (see [1, 23.2]). \diamond

3.2 Corollary *Any dual adjunction in a basic situation, with U and V essentially algebraic, is a natural dual adjunction.* \diamond

3.3 Remark If \mathbf{A} is a category of algebras (in the sense of Universal Algebra) which has concrete products, in view of Remark 1.8, there is only one way to get the desired lifting: one has to define the operations on $\mathbf{B}(B, \tilde{B})$ pointwise because this is the only way to make the induced map Ψ an algebra homomorphism $\Psi: SB \rightarrow \tilde{A}^{[B]}$. Observe that commutativity of the operations is a crucial ingredient for doing this: e.g. $\mathbf{Ab}(A, B)$ becomes an (Abelian) group by pointwise operations for every pair (A, B) of Abelian groups, but $\mathbf{Grp}(G, H)$ does not if H is not Abelian.

3-B Categories with Internal Hom-Functors

Sometimes a concrete category (\mathbf{A}, U) is equipped with an *internal hom-functor*, i.e., there is a functor $H: \mathbf{A}^{\text{op}} \times \mathbf{A} \rightarrow \mathbf{A}$ such that $UH = \mathbf{A}(-, -)$ and, moreover,

$$\begin{aligned} \text{all evaluation maps } \phi_{A, A', x}: \mathbf{A}(A, A') \rightarrow [A'] \text{ with } h \mapsto [h](x) & \quad (14) \\ \text{lift to } \mathbf{A}\text{-morphisms } p_{A, A', x}: H(A, A') \rightarrow A' \text{ for all } A, A' \in \mathbf{A}, x \in [A]. \end{aligned}$$

Typical examples are the category \mathbf{Ab} of abelian groups (here (14) is satisfied due to the definition of pointwise operations) or any cartesian closed (concrete) category *admitting function spaces* in the sense of [1, 27.17], like the category \mathbf{Conv} of convergence spaces (see 4-B). Here the lifted morphism of (14) is the composition

$$H(A, A') \cong T \times H(A, A') \xrightarrow{\hat{x} \times 1} A \times H(A, A') \xrightarrow{\text{ev}_{A, A'}} A'$$

with $T \xrightarrow{\hat{x}} A$ the constant map with value $x \in [A]$ from the terminal object T .

Contravariant Hom's Induce Dual Adjunctions

Given a concrete category (\mathbf{A}, U) equipped with an internal hom-functor $H(-, -)$, for any object $\tilde{A} \in \mathbf{A}$ the triple $(\tilde{A}, 1_{\tilde{A}}, \tilde{A})$ obviously satisfies conditions WSO 1 and WSO 2 with $S = T = H(-, \tilde{A})$. Observe that, when U is essentially algebraic as e.g. for $\mathbf{A} = \mathbf{Ab}$, the adjunction (S, T) will be natural with schizophrenic object $(\tilde{A}, 1_{\tilde{A}}, \tilde{A})$. For a cartesian closed topological category like \mathbf{Conv} , however, this adjunction will hardly ever be natural. Indeed, if it were, it follows from (14) that for every A the source $(T \xrightarrow{\hat{x}} A)_{x \in [A]}$ would be initial, hence every \mathbf{Set} -map would lift to an \mathbf{A} -morphism. In any case, for every $A \in \mathbf{A}$ the contravariant internal hom-functor $\mathbf{A}(-, A)$ is adjoint to itself (see also [1, 27.7] for a related result).

Lifting Contravariant Hom's Along Concrete Functors

Of importance is also the following somewhat more general situation. Let (\mathbf{A}, U) admit an internal hom-functor H ; moreover let there be given a concrete category (\mathbf{B}, V) and a concrete functor $|-|: \mathbf{B} \rightarrow \mathbf{A}$ such that the inclusions $\mathbf{B}(B, C) \hookrightarrow \mathbf{A}(|B|, |C|)$ lift to \mathbf{A} -morphisms $\gamma_{B, C}: \mathbf{B}_{\mathbf{A}}(B, C) \rightarrow H(|B|, |C|)$ (in a monotopological category this can always be done by lifting initially). Now for any $\tilde{B} \in \mathbf{B}$ and $\tilde{A} := |\tilde{B}|$, $\tau = 1_{[\tilde{A}]}$, one easily checks that Condition WSO 2 is fulfilled with $d_{B, \tilde{B}} = p_{|B|, \tilde{A}, \tau} \circ \gamma_{B, \tilde{B}}$, which means that we obtain a contravariant functor $S: \mathbf{B} \rightarrow \mathbf{A}$ with $S(B) = \mathbf{B}_{\mathbf{A}}(B, \tilde{B})$.

The remaining question is whether, for every $A \in \mathbf{A}$, it is possible to lift the \mathbf{A} -source $(p_{A, \tilde{A}, x}: H(A, \tilde{A}) \rightarrow \tilde{A} = |\tilde{B}|)_{x \in [A]}$ along $|-|$ functorially. We will not deal with this question here in full generality; typical examples of this situation however are discussed in 4-B, 4-C, and

4-D. Obviously, if a lifting can be found, Condition WSO 1 is fulfilled, too, and we obtain a contravariant functor $T: \mathbf{A} \rightarrow \mathbf{B}$ with $|TA| = H(A, \tilde{A})^8$.

Putting things together we have got a dual adjunction as in the basic situation where, however, not only $[\tilde{A}] = [\tilde{B}]$ but moreover $\tilde{A} = |\tilde{B}|$. One might describe this situation by saying that, in a basic situation, we have replaced the category **Set** with its (internal) hom-functor $\mathbf{Set}(-, -)$ by the category **A** with its internal hom-functor $H(-, -)$ and consider **A** (by means of $1_{\mathbf{A}}$) and **B** (by means of $|-|$) as concrete categories over **A**. More formally (and correctly) this can be described (see [27, 19]) using the language of *enriched category theory*.

3.4 Remark Clearly, the dual adjunctions arising here fail to be natural by the remarks in 3-B. However replacing the category **Set** by the category **A** as a base category as indicated above, one can develop the notion of an *enriched schizophrenic object* and, correspondingly, of an *enriched natural dual adjunction*, generalizing our definitions in the sense of enriched category theory. The typical examples for this, namely examples 4-B and 4-C, — non-natural dual adjunctions in the **Set**-based setting — are then natural in the enriched sense.

3-C Starting With a Regular Cogenerator

As observed before, the object \tilde{A} must necessarily have certain cogenerator properties. Particular additional properties can be of importance in two ways: they might facilitate the identification of \mathbf{A}_1 and hence, of $\mathbf{Fix}\epsilon$ (see Section 2-B), or, they might give additional information on the structure of the dual category $\mathbf{Fix}\eta$ as we shall see next.

3.5 Proposition Denote by $V': \mathbf{Fix}\eta \rightarrow \mathbf{Set}$ be the restriction of V . If $\mathbf{Fix}\epsilon$ is complete and contains \tilde{A} as a regular cogenerator, the following hold:

1. V' has a left adjoint and reflects isomorphisms; hence $(\mathbf{Fix}\eta, V')$ is an essentially algebraic category.
2. $(\mathbf{Fix}\eta, V')$ is even a regularly algebraic category, provided \tilde{A} is regular injective⁹ in $\mathbf{Fix}\epsilon$.

Proof Since $\mathbf{Fix}\eta$ has coproducts (being dually equivalent to $\mathbf{Fix}\epsilon$) the faithful functor $V': \mathbf{Fix}\eta \rightarrow \mathbf{Set}$, being representable by $B_0 = T\tilde{A}$, has an adjoint. Hence (see [1, 17.B, 18.3]) it remains to prove that V' reflects isomorphisms. Let $f: B \rightarrow B'$ be a morphism in $\mathbf{Fix}\eta$ such that $[f]$ is bijective. Then $[f]$ induces an isomorphism f' of the copowers $\mathbf{B}(B_0, B) \cdot B_0$ and $\mathbf{B}(B_0, B') \cdot B_0$ which makes the following diagram commutative, where q_B and $q_{B'}$ denote the canonical maps from the copowers of the generator B_0 , which, by hypothesis, are regular epimorphisms.

$$\begin{array}{ccc}
 \mathbf{B}(B_0, B) \cdot B_0 & \xrightarrow{f'} & \mathbf{B}(B_0, B') \cdot B_0 \\
 q_B \downarrow & & \downarrow q_{B'} \\
 B & \xrightarrow{f} & B'
 \end{array}$$

Now simply use the fact that the regular epimorphism $q_{B'}$ is an extremal epimorphism. For the last statement, first observe that the definition of a regular projective object just means that the functor V' , represented by the regular projective object $T\tilde{A}$, preserves regular epimorphisms. Now apply [1, 23.E]. \diamond

⁸Observe that, in this situation, any $\tilde{B} \in \mathbf{B}$ yields a dual adjunction of **B** with itself, strictly represented by (\tilde{B}, \tilde{B}) .

⁹i.e. injective with respect to regular monomorphisms.

Constructing a Duality from a Cogenerator

The observation in the final statement of the previous proposition remains completely formal and — in view of representation problems — without any use, as long as we cannot describe the regularly algebraic category $\mathbf{Fix}\eta$ concretely. This means nothing else but looking for a (known) regularly algebraic category (\mathbf{B}, V) and a duality between $(\mathbf{Fix}\epsilon, U')$ and (\mathbf{B}, V) induced by a schizophrenic object with the given cogenerator C as its first component. There is the following somehow obvious strategy (due to Linton [20]) how to achieve this.

In order to explain this idea, let (\mathbf{A}, U) be a concrete category with U representable and with \mathbf{A} complete and cocomplete; assume furthermore that we are given a regular cogenerator C in \mathbf{A} which is injective with respect to regular monomorphisms. As was made clear in the proof of 3.5 (see also [12, 32.21]), we then know that the category \mathbf{A}^{op} is a regularly algebraic category by means of $\mathbf{A}(-, C): \mathbf{A}^{\text{op}} \rightarrow \mathbf{Set}$.

It now follows from the theory of monads (see e.g. [1, 20.42-44]), that the comparison functor K for the monad \mathbf{T} , defined by $\mathbf{A}(-, C)$ and its left adjoint F (which clearly is given by $F = [C]^{(-)}$), provides a duality between (\mathbf{A}, U) and a full (regular epi-)reflective subcategory \mathbf{B} of $\mathbf{Set}^{\mathbf{T}}$. It is clear from Proposition 2.12 that $\mathbf{B} = \mathbf{Im}K$ is the regular-epireflective hull of $KF1$ in $\mathbf{Set}^{\mathbf{T}}$.

We will apply this method in Section 4-E.

4 Examples

4-A The Duality for Finite - Dimensional Vector Spaces

It is well known that there is a duality between the category $\mathbf{KVec}_{\text{fin}}$ of finite - dimensional vector spaces over some field K and itself. Let us briefly discuss how this fits into our setting. First of all, for the category $\mathbf{A} = \mathbf{B} = \mathbf{KVec}$ of all vector spaces over K the usual underlying functor U is representable (by the one - dimensional space K) and regularly algebraic.

Since all hom-sets $\text{Hom}_K(V, W)$ of K -linear maps between spaces V and W are known to be K -vector spaces by means of pointwise operations, it is clear that $(K, 1_K, K)$ is a schizophrenic object for the category \mathbf{KVec} and itself; it yields the well known dual adjunction which (in both directions) sends a space V to its dual space V^* .

The canonical map $\eta_V: V \rightarrow V^{**}$ is injective for every space V , hence (using the notation introduced in Section 2-A) $\mathbf{KVec}_1 = \mathbf{KVec}$ by Lemma 3.1. Trivially, for V finite-dimensional, η_V is also surjective. In fact, η_V is surjective if and only if V is finite-dimensional (see e.g. [10, 17, ex. 9]). Hence we get $\mathbf{KVec}_2 = \mathbf{KVec}_{\text{fin}}$ and finally $\mathbf{KVec}_1 \cap \mathbf{KVec}_2 = \mathbf{KVec}_{\text{fin}}$, i.e. the familiar duality of the category $\mathbf{KVec}_{\text{fin}}$ of finite-dimensional vector spaces with itself.

4.1 Remark This example clearly proves the previous claim, that one cannot expect the subcategories \mathbf{A}_2 and $\mathbf{Fix}\epsilon$ to be reflective in \mathbf{A} .

4-B The Binz Duality

Let $\mathbf{A} = \mathbf{Conv}$ be the category of convergence spaces, i.e., of sets X where for every $x \in X$ there is given a set $\Lambda(x)$ of filters on X , which are said to converge to x , subject to certain axioms. (For the appropriate definitions and details used in what follows, see [3].) Observe that any topological space X becomes a convergence space in a natural way, putting $\Lambda(X)$ the set of all filters converging to x in the given topology. The underlying functor

$U: \mathbf{Conv} \rightarrow \mathbf{Set}$ is a topological functor (hence representable by a singleton space). Let $\mathbf{B} = \mathbf{ConvAlg}$ be the category of real convergence algebras, i.e. of associative, commutative, unital \mathbb{R} -algebras endowed with a convergence structure where all algebraic operations (and homomorphisms) are supposed to be continuous. The underlying functor $V: \mathbf{ConvAlg} \rightarrow \mathbf{Set}$ factors as $\mathbf{ConvAlg} \xrightarrow{|\cdot|} \mathbf{R-Alg} \xrightarrow{V'} \mathbf{Set}$ over the category $\mathbf{R-Alg}$ of all \mathbb{R} -algebras (of the type described above) and is topologically algebraic and hence representable (by the discrete algebra \mathbb{R}); V' is a (regularly) algebraic and $|\cdot|$ is a topological functor (see [26] for a discussion of situations of this type).

Hence we are in a basic situation. Since, in addition, the category \mathbf{Conv} is cartesian closed (and wellfibre — see [1, 27.20]), with the internal hom-functor given by $C_c(X, Y)$, i.e. by the set of all continuous maps from the space X to the space Y endowed with the *structure of continuous convergence*, Section 3-B applies.

Now let $\tilde{A} = \mathbb{R}_c$ denote the convergence space of the reals and $\tilde{B} = \mathbb{R}_c$ the convergence algebra of the reals (with the convergence structure induced by the natural topology). Since both objects have identical underlying sets one might wonder whether $(\mathbb{R}_c, 1_{\mathbb{R}}, \mathbb{R}_c)$ is a schizophrenic object for \mathbf{Conv} and $\mathbf{ConvAlg}$. According to Section 3-B one obtains a contravariant functor $S: \mathbf{ConvAlg} \rightarrow \mathbf{Conv}$ ¹⁰. Hence we must only show that one can define a (contravariant) functor $C: \mathbf{Conv} \rightarrow \mathbf{ConvAlg}$ ¹¹ such that, for every convergence space X , there are continuous algebra homomorphisms $e_{X,x}: C(X) \rightarrow \mathbb{R}_c$ ($x \in X$) and the following equations hold:

$$|CX| = C_c(X, \mathbb{R}_c) \quad \text{and} \quad V'|e_{X,x}| = \varphi_{X,x}$$

To obtain the necessary internal operations on $C(X, \mathbb{R}_c)$ apply the (product preserving!) functor $C_c(X, -)$ to the corresponding operations of \mathbb{R}_c . The map $s: \mathbb{R}_c \times C_c(X, \mathbb{R}_c) \rightarrow C_c(X, \mathbb{R}_c)$, map corresponding to $m \circ (1_{\mathbb{R}_c} \times ev): \mathbb{R}_c \times C_c(X, \mathbb{R}_c) \times X \rightarrow \mathbb{R}_c$ (m being the multiplication of \mathbb{R}_c) by cartesian closedness, is the continuous scalar multiplication. A straightforward calculation shows that the maps evaluating at $x \in X$ are homomorphisms. Hence we have shown that there is a dual adjunction

$$\begin{array}{ll} C: \mathbf{Conv} \rightarrow \mathbf{ConvAlg} & S: \mathbf{ConvAlg} \rightarrow \mathbf{Conv} \\ X \mapsto C_c(X, \mathbb{R}_c) & A \mapsto \text{Hom}_c(A, \mathbb{R}_c) \end{array}$$

It is now a fundamental result on convergence spaces (see [3, Cor. 18]) that $\mathbf{Conv}_2 = \mathbf{Conv}$ ¹² and, hence, this adjunction is idempotent. The spaces belonging to the subcategory \mathbf{Conv}_1 are called *c-embedded spaces*. It is a difficult task to characterize c-embedded spaces internally (see [3] for a solution). In any case, we know from Theorem 2.12 that the category $\mathbf{c-Emb}$ of c-embedded space is epireflective in \mathbf{Conv} . Idempotency of the adjunction tells us that the category $\mathbf{Fix}\eta$ dually equivalent to $\mathbf{c-Emb}$ is precisely the category $\mathbf{FuncAlg}$ of function algebras (more precisely: of those algebras in $\mathbf{ConvAlg}$ which are isomorphic to some *function algebra* $C_c(X, \mathbb{R}_c)$). This yields the duality between $\mathbf{c-Emb}$ and $\mathbf{FuncAlg}$.

4.2 Remark The dual adjunction (C, S) is not natural¹³. This is clear from 3-B but can also be seen directly: if the adjunction were natural, for a topological space X , $C_c(X, \mathbb{R}_c)$ would be a subspace of the product $\mathbb{R}_c^{[X]}$ and therefore carry the topology of pointwise convergence. It is known, however (see [29]), that the structure of $C_c(X, \mathbb{R}_c)$ is the topology of compact convergence, and that both topologies are different in general.

¹⁰Following traditional terminology we will write $\text{Hom}_c(A, B)$ instead of $\mathbf{B}_A(B, C)$ as in 3-B.

¹¹Again for reasons of traditional terminology we here use C instead of T since $C(X)$ will be the algebra of continuous real-valued maps on X .

¹²Recall the notation introduced in Section 2-A.

¹³See however Remark 3.4.

4-C The Gelfand-Naimark Duality

The Generalized Gelfand-Naimark Adjunction

This example is similar to Example 4-B. Here however we will study function algebras of *topological* spaces only. We will present a complex version of the theory. As we have seen in 4-B, for studying function algebras on a space, one should start with a category of spaces admitting function spaces. A suitable starting point is therefore the category $\mathbf{A} = \mathbf{kSp}$ of *compactly generated Hausdorff spaces* (also called *Kelley spaces* or simply *k-spaces*), i.e., the coreflective hull of the category \mathbf{HComp} of compact Hausdorff spaces in the category \mathbf{Haus} of Hausdorff spaces. The coreflection is often called *Kelley-fication*. \mathbf{kSp} is a monotopological category admitting function spaces (see [8] or [7] for details). Observe in particular, that products in \mathbf{kSp} (which will be denoted by \times_k or \prod^k and called *k-products*) are the Kelley-fications of the topological products, and that the function spaces $C_k(X, Y)$ are given as Kelley-fications of the sets $\mathbf{Top}(X, Y)$ of all continuous maps from X to Y endowed with the compact-open topology. Recall that every locally compact space as well as every metrizable space belongs to \mathbf{kSp} and that, for a locally compact space X , one has $X \times Y = X \times_k Y$ for every space $Y \in \mathbf{kSp}$.

Since the field \mathbb{C} of complex numbers belongs to \mathbf{kSp} , one can form the category \mathbf{kAlg} of complex *k-algebras*: objects are \mathbb{C} -algebras (associative, commutative, and unital as in Section 4-B) carrying a Kelley-topology, such that all algebra operations are continuous with respect to *k-products*. Morphisms are the continuous algebra homomorphisms. Since the Kelley-fication of a topological space is simply a refinement of its topology, every topological complex algebra belongs to \mathbf{kAlg} , while a complex *k-algebra* might fail to be a topological algebra. Clearly, the complex numbers are a *k-algebra*. As in 4-B we will denote this algebra by \mathbb{C}_a , while \mathbb{C} denotes the space of complex numbers.

We now can argue literally as in 4-B in order to obtain a dual adjunction

$$\begin{array}{ll} C: \mathbf{kSp} \rightarrow \mathbf{kAlg} & , S: \mathbf{kAlg} \rightarrow \mathbf{kSp} \\ X \mapsto C_k(X, \mathbb{C}_a) & A \mapsto \text{Hom}_k(A, \mathbb{C}_a) \end{array}$$

which we will call the *Generalized Gelfand-Naimark Adjunction*. Clearly, this adjunction is not natural.

C^* -Algebras

For every topological space X one traditionally introduces the following structure on the set $C^*(X)$ of all bounded continuous complex-valued functions on X :

- By pointwise operations $C^*(X)$ becomes an associative, commutative, unital \mathbb{C} -algebra.
- By pointwise conjugation one gets an operation $f \mapsto f^*$ such that $C^*(X)$ becomes an involutive algebra.
- By the supremum-norm $\|f\| = \sup_{x \in X} |f(x)|$ then $C^*(X)$ becomes a normed algebra (which, due to the required boundedness, is even a Banach algebra) satisfying the additional axiom $\|f\|^2 = \|f \cdot f^*\|$.

Therefore $C^*(X)$ becomes a *C^* -algebra* (more precisely a commutative unital *C^* -algebra*). Taking as morphisms the involution-preserving unital \mathbb{C} -algebra homomorphisms (which are necessarily continuous) we obtain the category \mathbf{C}^* (see [6] for details). \mathbf{C}^* becomes a concrete category in two ways: first, there is the restriction V of the perfectly "well-behaved" underlying functor of

\mathbf{kAlg} assigning to a C^* -algebra its underlying set. Secondly, when assigning to a C^* -algebra A its unit ball $\{a \in A \mid \|a\| \leq 1\}$ and to a C^* -morphism f the restriction of f to the unit ball of its domain, one obtains a faithful functor $\circlearrowleft: C^* \rightarrow \mathbf{Set}$ which, perhaps somewhat surprisingly, turns out to be monadic (see [23] or [22, 3.1 exercise 16]). In particular, \circlearrowleft is representable while V is not (see [6, 1.3.3] for a hint).

Observe that the category C^* is a full subcategory of the category \mathbf{kAlg} ¹⁴.

Dualities Induced by the Generalized Gelfand-Naimark Adjunction

Obviously, for any compact Hausdorff space X , the C^* -algebra $C^*(X)$ and the function k -algebra $C(X) = C_k(X, \mathbb{C}_s)$ coincide algebraically. In fact they also coincide topologically since, due to compactness of X , the topology of $C^*(X)$ is the compact-open topology (see [8, 4.2.17]). Hence, by restriction, we get a function-algebra functor $C: \mathbf{HComp} \rightarrow C^*$.

Now there arises the question if the functor S can be restricted correspondingly. Indeed, since all evaluations are continuous, it follows from basic results on the topology of function spaces (see e.g. [17, Chap. 7] and [7, p.309]) that, for every C^* -algebra A , the space $S(A) = \text{Hom}_k(A, \mathbb{C}_s)$ is a compact space and that, moreover, its topology is the topology of pointwise convergence. Hence, by restriction of S to C^* , we get the so-called *spectrum*-functor $S: C^* \rightarrow \mathbf{HComp}$. It follows that the Generalized Gelfand-Naimark Adjunction induces by restriction a dual adjunction (which will be called *Gelfand-Naimark Adjunction*)

$$\begin{array}{ccc} C: \mathbf{HComp} \rightarrow C^* & & S: C^* \rightarrow \mathbf{HComp} \\ X \mapsto C_k(X, \mathbb{C}_s) & & A \mapsto \text{Hom}_k(A, \mathbb{C}_s) \end{array}$$

This might appear as a surprise, because, in view of Proposition 1.2, one would expect a compact Hausdorff space instead of \mathbb{C}_s to "represent" the function algebra functor. Recall however that one should consider C^* as a concrete category by means of \circlearrowleft rather than by means of V . Denoting by $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ the unit disc in \mathbb{C} , it is now easy to prove (see [23, 2.2]) that, for any compact Hausdorff space X , one has $\circlearrowleft C(X) = \mathbf{HComp}(X, D)$. It follows that the Gelfand-Naimark Adjunction is a natural adjunction between the concrete categories (\mathbf{HComp}, U) and (C^*, \circlearrowleft) with schizophrenic object $(D, 1_D, \mathbb{C}_s)$.

In order to identify the fixed subcategories of this adjunction we observe first that, since D cogenerates all of \mathbf{HComp} , we have $\mathbf{HComp}_1 = \mathbf{HComp}$. Now use the argument of e.g. [7, 3.4] to prove that also $\mathbf{HComp}_2 = \mathbf{HComp}$. Hence $\mathbf{Fix}_\epsilon = \mathbf{HComp}$, and the Gelfand-Naimark adjunction is idempotent. Application of the Stone-Weierstraß Theorem gives $\mathbf{Fix}_\eta = C^*$ (see [6, 1.4.1] or [24, 14, Thm 2]). This is the Gelfand-Naimark Duality between \mathbf{HComp} and C^* .

As pointed out in [7] the fixed subcategories of the Generalized Gelfand-Naimark Adjunction are strictly larger.

4-D The Pontrjagin - van Kampen Duality

In complete analogy to the definition of the category \mathbf{kAlg} one can define the category \mathbf{kAb} of Abelian k -groups, i.e. Abelian groups equipped with a compactly generated Hausdorff topology, such that the group operations are continuous with respect to the k -product. We denote the forgetful functor from \mathbf{kAb} to \mathbf{kSp} again by $|-|$. A k -group is a topological group if its topology is locally compact. \mathbf{LcAb} denotes the full subcategory of \mathbf{kAb} of locally compact topological groups.

¹⁴ C^* -algebras are metric spaces topologically, and hence k -spaces.

Let T denote the unit circle in the complex plane, i.e. $T = \{z \in \mathbb{C} \mid |z| = 1\}$ considered as a topological group under complex multiplication. In the same way as we have defined function algebras in 4-B and 4-C, for any Abelian k -group G , we can supply the set $\mathbf{kAb}(G, T) = \{f: G \rightarrow T \mid f \text{ is a continuous homomorphism}\}$ with the structure of an Abelian k -group \widehat{G} by naturally lifting the inclusion $\mathbf{kAb}(G, T) \hookrightarrow C_k(|G|, |T|)$. This yields (see footnote 8) a dual adjunction

$$\begin{array}{ccc} \widehat{\cdot}: \mathbf{kAb} & \rightarrow & \mathbf{kAb} \\ G \mapsto \widehat{G} & & G \mapsto \widehat{G} \end{array}$$

strictly represented by (T, T) , which is called the *Generalized Pontrjagin - van Kampen Adjunction*. \widehat{G} is called the *character group* of G .

Note that, for a locally compact Abelian group G , its character group \widehat{G} usually is defined as $\mathbf{LcAb}(G, T)$ equipped with the compact open topology, which then turns out to be (i) a locally compact group topology and (ii) the initial topology with respect to the family of point evaluations (see [24, 31.2] or [13, 23.15]). Hence, for these groups, our definition coincides with the classical one and, moreover, the Generalized Pontrjagin-van Kampen Adjunction restricts to a natural dual adjunction

$$\begin{array}{ccc} \widehat{\cdot}: \mathbf{LcAb} & \rightarrow & \mathbf{LcAb} \\ G \mapsto \widehat{G} & & G \mapsto \widehat{G} \end{array}$$

with schizophrenic object $(T, 1_T, T)$. One now needs quite a bit of functional analysis, which is beyond the scope of this paper, to prove that this adjunction is in fact a duality, the so-called *Pontrjagin - van Kampen Duality*. It follows from [18] that the fixed subcategory of the Generalized Pontrjagin-van Kampen Adjunction is larger than \mathbf{LcAb} .

The observations (see [13, 23.17]) that, for any discrete (respectively compact) group G , the character group \widehat{G} is compact (discrete respectively) then leads to the so called *Pontrjagin Duality* between \mathbf{Ab} and the category $\mathbf{HCompAb}$ of compact Hausdorff Abelian groups:

$$\begin{array}{ccc} \widehat{\cdot}: \mathbf{Ab} & \rightarrow & \mathbf{HCompAb} \\ G \mapsto \widehat{G} & & G \mapsto \widehat{G} \end{array}$$

Obviously the Pontrjagin Duality is strictly represented by the pair (T_d, T) , where T_d denotes the circle group considered as a discrete group. Since \mathbf{Ab} as well as $\mathbf{HCompAb}$ are regularly algebraic categories, it is in fact a natural dual adjunction with $(T_d, 1_T, T)$ as its schizophrenic object.

For a direct approach to Pontrjagin Duality one certainly could have started checking whether $(T_d, 1_T, T)$ is a schizophrenic object for $\mathbf{A} = \mathbf{Ab}$ and $\mathbf{B} = \mathbf{HCompAb}$. Condition SO 2 is trivially fulfilled; in order to check SO 1 one only has to observe that, for any Abelian group G , $\mathbf{Ab}(G, T_d)$ considered as a subspace of $T^{[G]}$ is (i) a topological subgroup of $T^{[G]}$ and (ii) a closed (hence compact) subspace. It is by far less trivial to determine the fixed subcategories as \mathbf{Ab} and $\mathbf{HCompAb}$ respectively; for a nice account of this we refer to [15, VI 4.8, 4.9].

4-E The Stone Duality

As already mentioned in the introduction, the duality discussed in this section originates from the question whether every Boolean algebra can be represented as a collection of subsets of some set. Certainly, in general one cannot expect a representation as a full powerset of some set — except for finite Boolean algebras (see below) — since there exist non-complete Boolean algebras (e.g. the finite-cofinite algebra over \mathbb{N} (see e.g. [4, 7.7]), whereas powersets are always

complete. Hence this question actually has two different facets: (i) which type of Boolean algebra arises as powerset algebras? and (ii) can every Boolean algebra be represented as a subalgebra of a powerset algebra? **Bool** denotes the category of Boolean algebras and Boolean homomorphisms.

Powerset Algebras

In attacking the first question, the previous discussion suggests considering the contravariant powerset functor \mathcal{P} on **Set**, or rather the naturally isomorphic functor $\mathbf{Set}(-, \mathbf{2})$. Since the two-element set $\mathbf{2}$ is regular injective and a regular cogenerator in the category **Set**, the strategy developed in Section 3-C applies.

Step 1: For every set X , the map $\eta_X: X \rightarrow \mathcal{P}\mathcal{P}X$ with $X \mapsto \{A \subset X \mid x \in A\}$ is easily checked to be \mathcal{P} -universal (see also 3-C). Hence we have identified the monad we are looking for: it is the so-called *Double Power-Set Monad* \mathbf{P} . For a complete description of \mathbf{P} see [22, 3.19].

Step 2: The Eilenberg-Moore category $\mathbf{Set}^{\mathbf{P}}$ can — by a somewhat lengthy calculation (see [22, 5.17-20] or [15, VI 4.3]) — be identified as the category **CABool** of complete atomic Boolean algebras. Recall that a Boolean algebra A is called *complete* if it is complete as a partially ordered set, and that it is called *atomic* provided every $a \in A$ is the supremum of the atoms (i.e. minimal elements in $A \setminus \{0\}$)¹⁵ $\leq a$.

Step 3: Since the comparison functor K acts as $X \mapsto \mathcal{P}X$ one can apply the *Lindenbaum-Tarski Theorem* whereby every complete atomic Boolean algebra B is isomorphic to the power algebra of its set of atoms $\mathcal{A}(B)$ (by $B \ni b \mapsto \{x \in \mathcal{A}(B) \mid x \leq b\}$ and, conversely, $\mathcal{A}(B) \ni A \mapsto \sup A \in B$), and obtain $\mathbf{Im}K = \mathbf{CABool}$. Hence we have established a duality between **Set** and **CABool**. Observe that this duality in so far exceeds the Lindenbaum-Tarski Theorem as it tells us in addition that the representation of a complete atomic Boolean algebra as a powerset algebra can, up to a natural isomorphism, *only* be achieved by means of its set of atoms.

Obviously, the duality between **Set** and **CABool** can be restricted to finite sets and finite complete atomic Boolean algebras. Now any finite Boolean algebra certainly is complete and atomic. Hence this yields a duality between \mathbf{Set}_{fin} and \mathbf{Bool}_{fin} , as mentioned at the beginning of this section.

The Stone Duality

An approach for attacking the second question now would be to establish a duality between $\mathbf{A} = \mathbf{Bool}$ and a “reasonable” concrete category (\mathbf{B}, V) starting with $\tilde{\mathbf{A}} = \mathbf{2}_a$, the two-element chain, as **Bool**-component of the corresponding schizophrenic object (only maps into $\mathbf{2}$ can be interpreted as subsets). We claim that $\mathbf{2}_a$ is a (regular) injective (regular) cogenerator on **Bool**¹⁶. Indeed, given a pair of different **Bool**-morphisms $A \xrightarrow[f]{g} B$, a Zorn’s Lemma argument shows that for $f(a) \neq g(a)$ there exists a prime ideal I in B with $f(a) \notin I \ni g(a)$ (see e.g. [4, 10.4]). Denoting by $x: B \rightarrow \mathbf{2}_a$ the Boolean homomorphism with $b \in I \Leftrightarrow x(b) = 0$ we conclude $x \circ f \neq x \circ g$, hence $\mathbf{2}_a$ is a cogenerator. To prove that $\mathbf{2}_a$ is injective one might either use the hints given in [4, Exercise 10.8] or simply use the fact that complete Boolean algebras are injective (see [2]).

¹⁵In a powerset the atoms obviously are the singletons.

¹⁶Recall that *regular* can be dropped here since in the regularly algebraic category **Bool** all monomorphisms are embeddings.

Now we can apply the strategy of 3-C again. The monad \mathbf{M} induced by $\mathbf{Bool}(-, \mathbf{2}_a)$ is described by $\mathbf{M}(X) = \mathbf{Bool}(\mathbf{2}_a^X, \mathbf{2}_a)$. Since the Boolean homomorphisms into $\mathbf{2}_a$ can be identified with prime ideals, hence with ultrafilters via complementation (see [4, exercise 9.1]) on their domain, \mathbf{M} will be the so called *Ultrafilter monad* on \mathbf{Set} . The Eilenberg-Moore category of \mathbf{M} is well known to be the category \mathbf{HComp} (see e.g. [22, 5.24-33]).

The full subcategory of \mathbf{HComp} cogenerated by the two-element discrete space $\mathbf{2}_s$ is the category \mathbf{Stone} of zero-dimensional compact Hausdorff spaces, also called *Stone spaces* (see e.g. [8, 6.2])¹⁷. Since the continuous maps from a Stone space X into $\mathbf{2}_s$ can be identified with the *clopen*, i.e. closed and open, subsets of X we might summarize our results as follows:

- There exists a natural duality for the concrete categories \mathbf{Bool} and \mathbf{Stone} induced by the schizophrenic object $(\mathbf{2}_a, \mathbf{1}_2, \mathbf{2}_s)$.
- Every Boolean algebra can be represented as the set of clopen subsets of a Boolean space, uniquely determined up to homeomorphism.
- The category of \mathbf{Stone} is regularly algebraic, but not monadic.

5 Further Topics

We will touch two further interesting questions related to concrete dualities. Due to lack of space we won't give proofs however, referring to the literature instead.

5-A Uniqueness of Dualities

The question of uniqueness of dualities has different facets. Clearly, if there is a duality (S, T) between concrete categories (\mathbf{A}, U) and (\mathbf{B}, V) with U and V representable by A_0 and B_0 respectively, then \mathbf{A} determines \mathbf{B} up to equivalence as an abstract category. Such a duality, if represented by (\tilde{A}, \tilde{B}) , can equivalently be seen as a concrete equivalence between the concrete categories $(\mathbf{A}^{\text{op}}, \mathbf{A}(-, \tilde{A}))$ and (\mathbf{B}, V) . It follows from uniqueness of representations that, given two dualities (S_i, T_i) between (\mathbf{A}, U) and (\mathbf{B}_i, V_i) ($i = 1, 2$) represented by $(\tilde{A}_i, \tilde{B}_i)$, the concrete categories (\mathbf{B}_1, V_1) and (\mathbf{B}_2, V_2) will be concretely equivalent iff \tilde{A}_1 and \tilde{A}_2 are isomorphic (though the categories \mathbf{B}_1 and \mathbf{B}_2 are always equivalent as abstract categories).

The more interesting question is whether different dualities for two concrete categories can be represented by the same pair of objects. It is for an answer to this question that, in our study of schizophrenic objects and natural dual adjunctions, we were always careful not to assume a priori that the bijection $\tau: [\tilde{A}] \rightarrow [\tilde{B}]$ is an identity.

Call two schizophrenic objects $(\tilde{A}, \tau, \tilde{B})$ and $(\tilde{A}, \sigma, \tilde{B})$ for the concrete categories (\mathbf{A}, U) and (\mathbf{B}, V) *equivalent* if there are automorphisms a and b of \tilde{A} and \tilde{B} respectively such that the diagram

$$\begin{array}{ccc} U\tilde{A} & \xrightarrow{\tau} & V\tilde{B} \\ Ua \downarrow & & \downarrow Vb \\ U\tilde{A} & \xrightarrow{\sigma} & V\tilde{B} \end{array}$$

¹⁷See [15, II 4.2] for a set of equivalent descriptions of Stone spaces.

commutes. Call two dual adjunctions (S, T) and (S', T') between (\mathbf{A}, U) and (\mathbf{B}, V) *equivalent* if there are natural equivalences $\gamma: T \rightarrow T'$ and $\delta: S \rightarrow S'$ such that

$$\gamma S \cdot \eta = T' \delta \cdot \eta \quad \text{and} \quad \delta T \cdot \epsilon = S' \gamma \cdot \epsilon' \quad (15)$$

One then can prove the following result for a fixed pair of objects $(\tilde{A}, \tilde{B}) \in \mathbf{A} \times \mathbf{B}$:

5.1 Theorem ([5]) *There is a bijective correspondence between*

- *equivalence classes of schizophrenic objects $(\tilde{A}, \tau, \tilde{B})$ and*
- *equivalence classes of natural dual adjunctions (S, T) strictly represented by (\tilde{A}, \tilde{B}) .*

This can be used for example to prove (see [5]) uniqueness of each of the following dualities: Stone duality, Pontrjagin-van Kampen duality, the localic duality of our leading example.

5-B Duality and Rank

Examples discussed in Section 4 might have led to the impression that, for a duality of concrete categories (\mathbf{A}, U) and (\mathbf{B}, V) with (\mathbf{A}, U) regularly algebraic (e.g. $\mathbf{A} = \mathbf{Bool}, \mathbf{Ab}, \mathbf{C}^*$), the category (\mathbf{B}, V) has to be, in some sense, "topological". However, in each of these cases, (\mathbf{B}, V) is in fact regularly algebraic. So what makes it look "topological"?

The answer to this question requires recalling the notion of *rank* of an algebraic category (see e.g. [22, 5.14] and [11]). Roughly speaking, an algebraic category has rank, if arities of operations needed for defining the objects of the category, are bounded by a certain cardinal. All categories mentioned above have in fact rank (for \mathbf{Bool} and \mathbf{Ab} only finitary operations are needed, \mathbf{C}^* requires in addition to finitary operations another one of arity \aleph_0 (see [22, 3.1 exercise 16]). By contrast, their duals (i.e. the categories \mathbf{Stone} , $\mathbf{HCompAb}$, and \mathbf{HComp}) do not have rank.

The following theorem gives the reason behind this observation.

5.2 Theorem ([9, 25]) *Let (\mathbf{A}, U) be a regularly algebraic category such that also $(\mathbf{A}^{\text{op}}, V)$ is regularly algebraic by means of some \mathbf{Set} -valued functor V . Then not both of these categories can have a rank, except for the trivial cases $\mathbf{A} = \mathbf{1}$ or $\mathbf{A} = \mathbf{2}^{18}$.*

¹⁸Here $\mathbf{1}$ is the category with precisely one morphism and $\mathbf{2}$ is the category $\bullet \rightarrow \bullet$.

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