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FACTORIZATIONS, FIBRES AND CONNECTEDNESS

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It is well known that the Galois correspondence given by

$$\mathfrak{A} \mapsto r(\mathfrak{A}) = \{B \mid \text{all maps } A \to B \text{ with } A \in \mathfrak{A} \text{ are constant}\}$$
$$\mathfrak{B} \mapsto \ell(\mathfrak{B}) = \{A \mid \text{all maps } A \to B \text{ with } B \in \mathfrak{B} \text{ are constant}\}$$

yields a good basis for the study of notions of connectedness and disconnectedness in topology or, more generally, in topological categories over Set (cf. Preuss [6,7,8,9], Herrlich [3], Arhangel'skii and Wiegandt [1]); this concept is closely related to the radical - semi-simple theory of rings and to the torsion - torsion free theory of abelian categories.

Another approach to a general notion of connectedness was provided by Herrlich [3] and Strecker [4], [13] through the notion of a component subcategory of the category Top of topological spaces; this concept was generalized in [5] for topological categories over Set.

Considerable progress was made through the papers [11] by Salicrup and Vázquez and [14] by Tiller. Both papers give, for abstract categories, definitions for "fibre" and "component"; in [11], connectedness is treated via "connection subcategories", and in [14] "component subcategories" are used. Both concepts are more general than that of a left constant subcategory defined by the above correspondence.

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The present paper may be considered as an extension of the two papers [11], [14] but there are two major new aspects, one is technical and the other one is conceptual. Technically we avoid giving absolute definitions for notions like "constant morphism" or "fibre" but let everything depend on a given class $\mathcal{E}$ of morphisms which, in most cases, is assumed to be part of an $(\mathcal{E},\mathcal{M})$-factorization system. The more important difference, however, is our aim to use more constructible methods. The notions of a constant morphism, of a fibre, of a connected object and of a totally disconnected object as used in this paper depend all in a constructible way on a given natural transformation. This idea is, with respect to connectedness, present in Börger's thesis [2]; but his approach, however, is Set-based and not comparable with ours as far as generality is concerned.

For natural transformations $\gamma : Id_{\mathcal{K}} \to C$ (with an endofunctor C of the category $\mathcal{K}$) which belong pointwise to $\mathcal{E}$, the assignments

$$\gamma \mapsto \{B \mid \gamma B : B \to CB \text{ is an isomorphism}\}$$
$$\gamma \mapsto \{A \mid \gamma A : A \to CA \text{ is constant}\}$$

will give us all $\mathcal{E}$-reflective subcategories and all $\mathcal{E}$-component subcategories of $\mathcal{K}$. For every γ one can find an $\mathcal{E}$-reflection $\bar{\gamma}$ which induces the same $\mathcal{E}$-reflective subcategory and an $\mathcal{E}$-connection $\mathring{\gamma}$ which induces the same $\mathcal{E}$-component subcategory. Right constant subcategories appear as those $\mathcal{E}$-reflective subcategories which are induced by an $\mathcal{E}$-connection, and the left constant subcategories are those $\mathcal{E}$-component subcategories which are induced by an $\mathcal{E}$-reflection. This means that the Galois correspondence mentioned first fits perfectly into our more general setting.

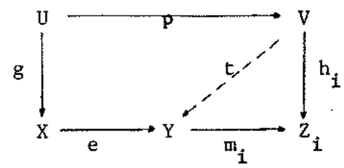
Those readers who get annoyed from our setting of fibres (see Section 3) which seems quite complicated at the first glance, should think of the category Set $\times$ Set: everybody would like a morphism $(f,id_\emptyset) : (X,\emptyset) \to (Y,\emptyset)$ with $X \neq \emptyset$ to have fibres $(f^{-1}y,\emptyset) \to (X,\emptyset)$ for $y \in Y$. But you do not get this by just considering pullbacks of morphisms from the terminal object $(1,1)$ into $(Y,\emptyset)$ - since there aren't any!

1. Factorizations and localizations

A source in a category $\mathcal{K}$ is a family $(X,f_i)_I$ of $\mathcal{K}$-morphisms f_i, $i \in I$, with common domain X; I might be a proper class or void; in the latter case the source consists just of the object X. $(e,m_i)_I$ is called a factorization (of $(X,f_i)_I$) if $e : X \to Y$ is a morphism and $(Y,m_i)_I$ is a source in $\mathcal{K}$ (with $f_i = m_i e$, $i \in I$). In particular, (e,Y) and $(1_X,e)$ are factorizations of the object (= empty source) X and the morphism (= singleton source) e respectively. A morphism p is orthogonal to a factorization $(e,m_i)_I$ if for all g and h_i with $h_i p = m_i eg$, $i \in I$, there is a unique t with $tp = eg$ and $m_i t = h_i$, $i \in I$; we write

$$p \perp (e,m_i)_I$$

in this case and visualize the situation by

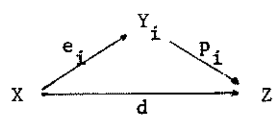


In the following, let $\mathcal{E}$ be a class of morphisms in $\mathcal{K}$ which contains all isomorphisms and which is closed under composition with isomorphisms. A *locally orthogonal* $\mathcal{E}$-*factorization* (*loco* $\mathcal{E}$-*fact*, for short) is a factorization $(e,m_i)_I$ such that $e \in \mathcal{E}$ and $p \perp (e,m_i)_I$ for all $p \in \mathcal{E}$. This last relation is especially fulfilled if $p \perp (1_Y,m_i)_I$ for all $p \in \mathcal{E}$; then $(e,m_i)_I$ is an *orthogonal* $\mathcal{E}$-*factorization* (*ortho* $\mathcal{E}$-*fact*). The category $\mathcal{K}$ is called $\mathcal{E}$-*cocomplete* if

(a) for every $\mathcal{K}$-morphism f and every $e \in \mathcal{E}$ with common domain there is a pushout

with $e' \in \mathcal{E}$,

(b) for every source $(X,e_i)_I$ of $\mathcal{E}$-morphisms there is a multiple pushout



with codiagonal $d \in \mathcal{E}$.

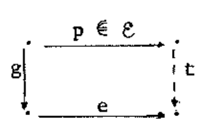
The following theorem was proved in [15]:

THEOREM 1. (1) $\mathcal{K}$ *is* $\mathcal{E}$-*cocomplete if and only if every source in* $\mathcal{K}$ *has a locally orthogonal* $\mathcal{E}$-*factorization*.

(2) $\mathcal{K}$ *is* $\mathcal{E}$-*cocomplete and* $\mathcal{E}$ *is closed under composition if and only if every source in* $\mathcal{K}$ *has an orthogonal* $\mathcal{E}$-*factorization*.

(3) *If* (b) *holds then* $\mathcal{E}$ *consists of epimorphisms only*. □

We can particularly consider (locally) orthogonal $\mathcal{E}$-factorizations of empty sources: an $\mathcal{E}$-morphism $e : X \to Y$ with $p \perp (e,Y)$ for all $p \in \mathcal{E}$ is called an $\mathcal{E}$-*localization* of X; it is uniquely determined by X (up to isomorphisms) according to the property that every solid diagram



has a unique dotted fill-in. Putting $TX = Y$ and $\eta X = e$ we get an endofunctor T of $\mathcal{X}$ and a natural transformation $\eta : Id_{\mathcal{X}} \to T$ provided every X has an $\mathcal{E}$-localization. The latter is certainly true if $\mathcal{X}$ is $\mathcal{E}$-cocomplete; then ηX is the codiagonal of the multiple pushout of all $\mathcal{E}$-morphisms with domain X. If $\mathcal{X}$ has a terminal object 1 it is easier to think of ηX as of the $\mathcal{E}$-part of a loco $\mathcal{E}$-fact of $X \to 1$ provided the latter exists.

In Set with $\mathcal{E}$ = {surjective mappings} or in Top with {quotient maps} $\subset \mathcal{E} \subset$ {surjective maps}, TX will just determine whether X is empty or not:

$$TX = \begin{cases} \emptyset & \text{for } X = \emptyset , \\ 1 & \text{for } X \neq \emptyset . \end{cases}$$

In Set $\times$ Set one has 4 different types of images under T.

It should be noted that even for concrete categories and for $\mathcal{E}$ the class of morphisms with underlying surjective morphisms it might be that $\mathcal{E}$-localizations exist without having $\mathcal{E}$-cocompleteness, not even ortho $\mathcal{E}$-facts of morphisms or a terminal object. For instance, in the category of rings A with 1 which come equipped with a fixed maximal ideal $\mathfrak{m}_A$, and of ring homomorphisms $f : A \to B$ with $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$, the projection $p : (A,\mathfrak{m}_A) \to (A/\mathfrak{m}_A,(0))$ is easily seen to be an $\mathcal{E}$-localization.

2. Constant morphisms and subfibres

The following notions depend on the given class $\mathcal{E}$. For every X, the existence of an $\mathcal{E}$-localization $\eta X : X \to TX$ is assumed. A morphism $c : X \to Y$ is called constant if there exists a d with $c = d \cdot \eta X$. A morphism $v : V \to X$ is called a subfibre of a source $(X,f_i)_I$ if all $f_i v$ are constant. Denoting by $\mathfrak{C}$ the class of constant morphisms and by $fib(X,f_i)_I$ the class of all subfibres of $(X,f_i)_I$ one has the following easy properties:

(0) For all X, $\eta X \in \mathfrak{C}$.

(A) For all f and g, $c \in \mathfrak{C}$ implies $gcf \in \mathfrak{C}$ (if defined).

(B) For every loco $\mathcal{E}$-fact $(e,m_i)_I$ of $(X,f_i)_I$ and all $v : V \to X$, $f_i v \in \mathfrak{C}$ for all $i \in I$ implies $ev \in \mathfrak{C}$;

in particular:

(B') For every ortho $\mathcal{E}$-fact $(e,m_i)_I$ and all $u : U \to \text{codomain}(e)$, $m_i u \in \mathfrak{C}$ for all $i \in I$ implies $u \in \mathfrak{C}$.

If $\mathcal{E}$ is closed under composition one also has:

(C) For every epimorphism $e \in \mathcal{E}$, $fe \in \mathfrak{C}$ implies $f \in \mathfrak{C}$.

For the proof of (C) one uses the easily estiblished

LEMMA 1: If $\mathcal{E}$ is closed under composition then $T(\mathcal{E}) \subset Iso\, \mathcal{X}$. □

Using the terminology of subfibres we can write the conclusion of (B) as

$\mathrm{fib}(X,f_i)_I \subset \mathrm{fib}(X,e)$. In the following proposition, the converse implication of (B) is investigated:

PROPOSITION 1. *Let all finite sources have loco* $\mathcal{E}$-*facts and consider the following assertions*:

(D) *Every* $\mathcal{E}$-*factorization* $(e,m_i)_I$ *of a source* $(X,f_i)_I$ *with* $\mathrm{fib}(X,f_i)_I \subset \mathrm{fib}(X,e)$ *is a loco* $\mathcal{E}$-*fact*.

(D') *Whenever* $p = ge$ *with* $p,e \in \mathcal{E}$ *and* $\mathrm{fib}(X,p) \subset \mathrm{fib}(X,e)$, *then* g *is an isomorphism*.

(D'') *For* $p : X \to Y$ *in* $\mathcal{E}$ *and every* $f : X \to Z$ *with* $\mathrm{fib}(X,p) \subset \mathrm{fib}(X,f)$ *there is* h *with* $hp = f$.

Then (D) $\Rightarrow$ (D') $\Rightarrow$ (D'') *whereas* (D'') $\Rightarrow$ (D) *in case* $\mathcal{E} \subset \mathrm{Epi}\,\mathcal{K}$.

Proof: (D') $\Rightarrow$ (D'') was proved in [16], Lemma 5.1; the implications (D'') $\Rightarrow$ (D) and (D) $\Rightarrow$ (D') are easy. □

Definition. $\mathcal{E}$ is called *fibre determined* if condition (D'') holds.

Remarks. (1) If $\mathcal{K}$ has coequalizers and if they belong to $\mathcal{E}$ then for every constant morphism $c : X \to Y$ one has $cx = cy$ for all parallel x,y with codomain X . Vice versa, if c satisfies this property then c is constant provided all $\mathcal{E}$-localizations are regular epimorphisms. The class $\mathcal{E}$ of regular epimorphisms is often fibre determined, for instance in the category Top of topological spaces; also in the categories of groups, of rings, or of R-modules, but not in the category of commutative semigroups: consider the additive homomorphism $p : \mathbb{N} \to \mathbb{Z}_2$ which sends even numbers ≥ 1 to 0 and odd ones to 1 , and the additive homomorphism $f : \mathbb{N} \to S = \{0,1,e\}$ which does the same except that $f(1) = e$; here S is the commutative semigroup which contains Z_2 as a subsemigroup, and $0 + e = 1$, $1 + e = e + e = 0$.

(2) Under the assumption of Proposition 1, one easily proves that an arbitrary class η of morphisms which satisfies properties (0), (A), and (D'') (in which $\mathfrak{C}$ has to be replaced by η) must necessarily coincide with the class $\mathfrak{C}$ of constant morphisms (cf. [16] and [17], Theorem 2).

3. Fibres

In this section, $\mathcal{E}$ is assumed to be closed under composition and to admit the formation of all $\mathcal{E}$-localizations which are assumed to be epimorphic. Then, every subfibre v of a source $(X,f_i)_I$ gives rise to commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\eta V} & TV \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \bar{v}_i} \\ X & \xrightarrow{f_i} & Y_i \end{array} \qquad (1)$$

with uniquely determined morphisms $\overline{v}_i, i \in I$. Note that $T\eta V = \eta TV$ is an isomorphism by Lemma 1.

Definition. (1) A subfibre $v \in fib(X,f_i)_I$ is a fibre of $(X,f_i)_I$ if the diagrams (1) form a modified pullback in the sense that, given $w : W \to X$ and $h : W \to TV$ with $f_i w = \overline{v}_i h$ for all $i \in I$ and Th an isomorphism, then there is a unique $g : W \to V$ with $vg = w$ and $\eta V \cdot g = h$.

(2) The source $(X,f_i)_I$ is said to have fibres if every subfibre w of $(X,f_i)_I$ factors through a fibre v of $(X,f_i)_I$ by a unique morphism g with Tg an isomorphism.

In (2), necessarily v is, up to an isomorphism, uniquely determined by w . This follows from Lemma 2 below which presents fibres in a different way: let $Fib(X,f_i)_I$ be the category whose class of objects is $fib(X,f_i)_I$ and whose morphisms $f : w \to v$ are those $\mathcal{K}$-morphisms with $vf = w$ and Tf an isomorphism. Like every other category $Fib(X,f_i)_I$ is the disjoint union of its components considered as its subcategories; here two objects belong to the same component if they can be connected by a finite string of arrows (with alternating directions) .

LEMMA 2. v is a fibre of $(X,f_i)_I$ if and only if v is a terminal object in its component in $Fib(X,f_i)_I$. □

For the following theorem only, let $\mathcal{M}$ be any class of morphisms such that every morphism whose domain is of the form TX, $X \in Ob\,\mathcal{K}$, belongs to $\mathcal{M}$.

THEOREM 2. If $\mathcal{K}$ is $\mathcal{M}$-complete[1] then every non-empty source has fibres, and these belong to $\mathcal{M}$.

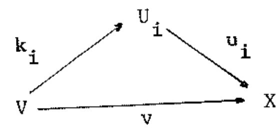
Proof: First we consider a subfibre $w : W \to X$ of a single morphism $f : X \to Y$ which gives us $\overline{w}$ with $fw = \overline{w} \cdot \eta W$. Since $\overline{w} \in \mathcal{M}$ one has the pullback

$$\begin{array}{ccc} V & \xrightarrow{k} & TW \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \overline{w}} \\ X & \xrightarrow{f} & Y \end{array} \qquad (2)$$

and a unique g with $vg = w$ and $kg = \eta W$; here the second equation is redundant since $\overline{w}$ is monomorphic by the dual of Theorem 1(3). So we just have to show that v is a fibre and that Tg is an isomorphism. The morphism $\overline{v} := \overline{w}(\eta TW)^{-1}\,Tk$ satisfies the equation $\overline{v} \cdot \eta V = fv$, hence $v \in fib(X,f)$. Since $\overline{v} \in \mathcal{M}$ is a monomorphism also Tk is one; therefore, since $Tk \cdot Tg = T\eta W$ is an isomorphism also Tg is one. Up to this isomorphism, diagram (1) coincides with the pullback (2).

[1] This is the dual notion of $\mathcal{E}$-cocomplete, i.e. pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms and multiple pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$.

Let now w be a subfibre of $(X,f_i)_I$, $I \neq \emptyset$. For each $i \in I$, one has fibres $u_i : U_i \to X$ and morphisms g_i with $u_i g_i = w$ and Tg_i isomorphisms. Since $u_i \in \mathcal{M}$ for all $i \in I$ one can form the multiple pullback



and get g with $vg = w$ and $k_i g = g_i$, $i \in I$. The morphisms $\bar{v}_i := \bar{u}_i \cdot Tk_i$ satisfy $v_i \cdot \eta V = f_i v$, hence $v \in \mathrm{fib}(X,f_i)_I$ and $\bar{v}_i \in \mathcal{M}$ is a monomorphism. Since $I \neq \emptyset$, from $Tk_i \cdot Tg = Tg_i$ one obtains Tg to be an isomorphism. The rest is routine. □

Remarks. (1) If $\mathcal{K}$ is $\mathcal{E}$-cocomplete the natural choice for $\mathcal{M}$ is the class of all morphisms m such that $p \perp (1,m)$ for all $p \in \mathcal{E}$ which, in fact, contains all morphisms $TX \to Y$.

(2) Notice from the above proof: a fibre of a non-empty source is an intersection of fibres of its single morphisms.

(3) If $\mathcal{K}$ allows the formation of direct products of the form $X \times TW$ such that the projection to TW belongs to $\mathcal{E}$ then the empty source X has fibres (which are projections $X \times TW \to X$).

In Top with $\mathcal{E} = \{\text{quotient maps}\}$, say, a complete set of fibres of $f : X \to Y$ is given by the inclusions $\emptyset \to X$ and $f^{-1}y \to X$, $y \in Y$. Even if f is surjective $\emptyset \to X$ has to be included since $\emptyset \to X$ is a subfibre which has to be factored through a fibre $v : F \to X$ by a map $h : \emptyset \to F$ with Th an isomorphism, so $F = \emptyset$. The condition "Th iso" guarantees the essentially unique choice of a fibre. (This is the main difference between Tiller's notion (cf. [14]) and ours.) In this context one should also notice that in concrete categories which satisfy Theorem 2, even in case of a surjective morphism $f : X \to Y$ it might occur that $\emptyset \to X$ is the only fibre of f, for instance in the category of G-sets (where G is a fixed monoid or group): if Y contains no element y with $Gy = \{y\}$, then there is even no subfibre of f except $\emptyset \to X$.

4. γ-connected and totally γ-disconnected objects

Definitions. A natural transformation $\gamma : Id_{\mathcal{K}} \to C$ (with an endofunctor $C: \mathcal{K} \to \mathcal{K}$) is called an $\mathcal{E}$-prereflection[2] if $\gamma X : X \to CX$ is an epimorphic $\mathcal{E}$-morphism for all objects X; it is an $\mathcal{E}$-reflection if, in addition, $\gamma C : C \to CC$ is a natural equivalence (note that always $\gamma C = C\gamma$). γ is called an $\mathcal{E}$-connection if, for every X,

[2] Börger [2] defines prereflections slightly more generally.

(1) fibres of the single morphism γX exist and, after a representative system $(u_i : F_i \to X)_{i \in I}$ of non-isomorphic fibres of X has been chosen,

(2) the diagrams

$$\begin{array}{ccc} F_i & \xrightarrow{u_i} & X \\ {\scriptstyle \gamma F_i}\downarrow & & \downarrow{\scriptstyle \gamma X} \\ CF_i & \xrightarrow{Cu_i} & CX \end{array} \qquad (3)$$

form a generalized pushout in the sense that, given $g : X \to Y$ and $h_i : CF_i \to Y$ with $h_i \cdot \gamma F_i = gu_i$, $i \in I$, then there is a (unique) t with $t \cdot \gamma X = g$ (and $t \cdot Cu_i = h_i$).

Fibres of γX (or, less precisely, just their domains) are called γ-*quasi-components* of X, and in case of an $\mathcal{E}$-connection, γ-*components* of X. Finally, we call X γ-*connected* if γX is constant, and *totally* γ-*disconnected*, if γX is an isomorphism. Let $\mathfrak{C}_\gamma$ ($\mathfrak{D}_\gamma$ resp.) denote the full subcategory of all γ-connected (totally γ-disconnected resp.) objects.

These phrases are motivated by the first three examples below and become more plausible by the following facts.

PROPOSITION 2. *Let* $\gamma : Id_{\mathcal{K}} \to C$ *be an* $\mathcal{E}$-*prereflection.*

(1) *The following assertions are equivalent:*

(i) γ *is an* $\mathcal{E}$-*reflection,*

(ii) *for all* X, CX *is totally* γ-*disconnected,*

(iii) $\mathfrak{D}_\gamma$ *is reflective with reflexion morphisms* γX.

(2) *If* $\mathcal{E}$ *is fibre determined, for the following assertions one has* (ii) $\Rightarrow$ (i):

(i) γ *is an* $\mathcal{E}$-*connection,*

(ii) *for all* X, γ-*quasi-components of* X *are* γ-*connected.*

(3) *For all* X, *let* γX *have fibres but let only a small set of them be pairwise non-isomorphic. Then* (2)(i) $\Rightarrow$ (ii) *holds if* $\mathcal{K}$ *has coproducts and if there exists a class of morphisms* $\mathcal{M}$ *in* $\mathcal{K}$ *satisfying the following properties:*

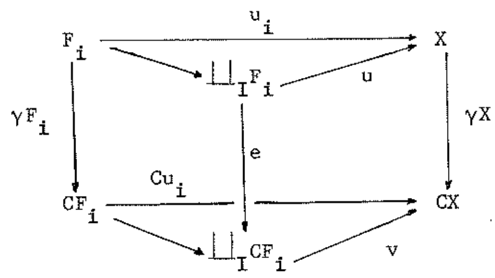
(a) *injections of coproducts belong to* $\mathcal{M}$,

(b) *the induced morphism* $u : \coprod_I F_i \to X$ *belongs to* $\mathcal{M}$,

(c) *for every* $p \in \mathcal{E}$ *and every* $m \in \mathcal{M}$, $p \perp (1,m)$,

(d) *pushouts of* $\mathcal{M}$-*morphisms belong to* $\mathcal{M}$.

Proof: (1) Cf. [2] and [16], Prop. 4.2. (2) Given diagram (3) and g, h_i with $h_i \cdot \gamma F_i = gu_i$ we just have to show $fib(X, \gamma X) \subset fib(X,g)$. But every subfibre v of γX factors through one u_i by a morphism s; since γF_i is constant also $gv = h_i \cdot \gamma F_i \cdot s$ is. (3) The generalized pushout (3) can be constructed as follows:



(4)

Here v is the pushout of u along $e = \coprod_I \gamma F_i$, so $v \in \mathcal{M}$ by (b) and (d). Using (a) and (c) we get $p \perp (1, Cu_i)$ for all $p \in \mathcal{E}$ and $i \in I$. Since $\gamma X \cdot u_i = Cu_i \cdot \gamma F_i$ is constant, from property (B') of Section 2 one derives that γF_i is constant for all $i \in I$. □

Examples. (1) In Top with $\mathcal{E}$ = {quotient maps} , let $\gamma_1 X : X \to C_1 X$ be the map which assigns to every point its component. γ_1-connectedness and totally γ_1-disconnectedness have the usual topological meaning. γ_1 is a reflection and a connection (the latter follows from Prop. 2 with $\mathcal{M}$ = {injective maps}).

(2) Let $\gamma_2 X$ be the projection onto the space $C_2 X$ of quasi-components of X where the quasi-component of a point is the intersection of its open and closed neighborhoods. Then $\mathcal{C}_{\gamma_2} = \mathcal{C}_{\gamma_1}$, but {0-dimensional T_1-spaces} $\subsetneqq \mathcal{D}_{\gamma_2} \subsetneqq \mathcal{D}_{\gamma_1}$. γ_2 is a reflection but not a connection (since $(\gamma_2$-) quasi-components need not be $(\gamma_2$-) connected).

(3) Let $\gamma_3 X$ be the projection onto the space $C_3 X$ of arc-components of X . Then γ_3-connectedness is arcwise connectedness; X is totally γ_3-disconnected iff its arc-components are at most singletons. γ_3 is a connection but not a reflection (for the standard example $X = \{(x , \sin \frac{1}{x}) | x > 0\} \cup \{(0,y) | -1 \leq y \leq 1\}$, $C_3 X$ is the Sierpinski space).

(4) Connections need not arise from partitions directly. For instance, let $\gamma_4 X : X \to \{cl(x) | x \in X\}$ be the T_0-reflexion of X . $\mathcal{C}_{\gamma_4}$ is the subcategory of indiscrete spaces. Somehow surprisingly, γ_4 is also a connection since the γ_4-components of X are its maximal indiscrete subspaces and $\emptyset$. These give an equivalent description of $\gamma_4 X$, and they do form a partition.

(5) If Top is now equipped with $\mathcal{E}$ = {surjective maps} then one may also consider the reflection γ_5 onto the indiscrete spaces. Then $\mathcal{C}_{\gamma_5}$ is the subcategory of trivial spaces X ($|X| \leq 1$) . But even though γ_5-quasi-components are always trivial γ_5 is not a connection; reason: $\mathcal{E}$ is not fibre determined. This shows that Prop. 2(2) is false without this assumption.

(6) In the categroy of R-modules (R an integral domain), let $\mathcal{E}$ be the class of epimorphisms, and let $\gamma_6 M : M \to C_6 M = M/\mathrm{Tor}\, M$ be the projection. Then the γ_6-

connected (totally γ_6-disconnected) objects are the torsion- (torsion free) modules. γ_6 is a reflection and a connection. - Similar examples occur if one considers rings modulo various types of radicals.

(7) In the category Grp of groups with $\mathcal{E}$ the epimorphisms, let $\gamma_7 G : G \to C_7 G = G/[G,G]$ be the projection. Then totally γ_5-disconnected means abelian whereas γ_7-connected means perfect, that is: $G = [G,G]$. γ_7 is a reflection, but not a connection.

5. Characterization of the subcategories $\mathfrak{C}_\gamma$ and $\mathfrak{D}_\gamma$

From now on, let $\mathcal{K}$ be always $\mathcal{E}$-cocomplete and let $\mathcal{E}$ be closed under composition. The following definitions are due to Tiller [14] who also gives the relationships to similar concepts previously used by Herrlich and Strecker for topological categories over Set. But notice again, in our context all notions will depend on $\mathcal{E}$.

A sink $(u_i,X)_I$ is a family of morphisms u_i with common co-domain X ; so it is a source in $\mathcal{K}^{op}$. $(u_i,X)_I$ is called chained (w.r.t. $\mathcal{E}$) if any morphism $g : X \to Y$ is constant whenever all gu_i , $i \in I$, are constant. A full and replete subcategory $\mathfrak{G}$ of $\mathcal{K}$ is an $\mathcal{E}$-component subcategory if for every chained sink $(u_i,X)_I$ one has $X \in \mathrm{Ob}\,\mathfrak{G}$ whenever the domains of all u_i's belong to $\mathfrak{G}$.

The subcategories $\mathfrak{C}_\gamma$ are easily seen to be $\mathcal{E}$-component subcategories. Given an arbitrary $\mathfrak{G}$, one constructs a smallest $\mathcal{E}$-component subcategory containing $\mathfrak{G}$ as follows: let $\pi_{\mathfrak{G}} X : X \to P_{\mathfrak{G}} X$ be the $\mathcal{E}$-part of an ortho $\mathcal{E}$-fact of the source of all morphisms g with domain X such that gu is constant for all $u : A \to X$, $A \in \mathrm{Ob}\,\mathfrak{G}$. In fact, $\pi_{\mathfrak{G}}$ is an $\mathcal{E}$-prereflection (cf. [16], Prop. 5.2), and $\mathfrak{G} \subset \mathfrak{C}_\gamma$ for $\gamma = \pi_{\mathfrak{G}}$. If $\mathfrak{G}$ is an $\mathcal{E}$-component subcategory then also "$\supset$" : to get $X \in \mathrm{Ob}\,\mathfrak{G}$ whenever γX is constant it suffices to show that the sink of all morphisms u with domain in $\mathfrak{G}$ and codomain X is chained; but every g with gu constant for all u factors through $\gamma X = \pi_{\mathfrak{G}} X$, so g must be constant. This proves the first part of

THEOREM 3. (1) The following assertions are equivalent for a full and replete subcategory $\mathfrak{G}$:

(i) $\mathfrak{G}$ is an $\mathcal{E}$-component subcategory,

(ii) $\mathfrak{G} = \mathfrak{C}_\gamma$ for $\gamma = \pi_{\mathfrak{G}}$,

(iii) $\mathfrak{G} = \mathfrak{C}_\gamma$ for some $\mathcal{E}$-prereflection γ .

(2) The following assertions are equivalent for a full and replete subcategory $\mathfrak{B}$:

(i) $\mathfrak{B}$ is $\mathcal{E}$-reflective,

(ii) $\mathfrak{B} = \mathfrak{D}_\gamma$ for $\gamma = \rho_{\mathfrak{B}}$,

(iii) $\mathfrak{B} = \mathfrak{D}_\gamma$ for some $\mathcal{E}$-prereflection γ .

Here, for every $\mathcal{K}$-object X , $\rho_{\mathfrak{B}} X : X \to R_{\mathfrak{B}} X$ denotes the $\mathcal{E}$-part of an ortho $\mathcal{E}$-fact of the source of all morphisms with domain X and codomain in $\mathfrak{B}$. It is well known that $\rho_{\mathbb{B}}$ is an $\mathcal{E}$-reflection since $\mathcal{D}_\gamma$ for $\gamma = \rho_{\mathbb{B}}$ is the $\mathcal{E}$-reflective hull of $\mathbb{B}$. Therefore one has (2) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) whereas (iii) $\Rightarrow$ (i) follows from the easily justified fact that $X \in \mathrm{Ob}\mathcal{D}_\gamma$ whenever there is a source $(X,m_i)_I$ with the codomains of all m_i's in $\mathcal{D}_\gamma$ and $p \perp (1_X,m_i)_I$ for all $p \in \mathcal{E}$. □

One can introduce a preordering for $\mathcal{E}$-prereflections by writing $\gamma \leq \delta$ if there is a natural transformation σ with $\sigma\gamma = \delta$; obviously $\gamma \cong \delta$ (as objects in the comma category of natural transformations with domain $\mathrm{Id}_{\mathcal{K}}$) iff $\gamma \leq \delta$ and $\delta \leq \gamma$. Notice that $\gamma \leq \delta$ implies $\mathcal{C}_\gamma \subset \mathcal{C}_\delta$ and $\mathcal{D}_\delta \subset \mathcal{D}_\gamma$. For $\pi_{\mathbb{G}}$ and $\rho_{\mathbb{B}}$ as constructed above one has

LEMMA 3. $\pi_{\mathbb{G}} \cong \min\{\delta \mid \mathbb{G} \subset \mathcal{C}_\delta\}$ and $\rho_{\mathbb{B}} \cong \max\{\delta \mid \mathbb{B} \subset \mathcal{D}_\delta\}$. □

We define

$$\mathring{C} := P_{\mathbb{G}} \text{ and } \mathring{\gamma} := \pi_{\mathbb{G}} \text{ with } \mathbb{G} = \mathcal{C}_\gamma ,$$
$$\overline{C} := R_{\mathbb{B}} \text{ and } \overline{\gamma} := \rho_{\mathbb{B}} \text{ with } \mathbb{B} = \mathcal{D}_\gamma$$

for every $\mathcal{E}$-prereflection γ. Then:

THEOREM 4. (1) $\mathring{\gamma} \leq \gamma \leq \overline{\gamma}$,

(2) $\mathcal{C}_{\mathring{\gamma}} = \mathcal{C}_\gamma \subset \mathcal{C}_{\overline{\gamma}}$ and $\mathcal{D}_{\overline{\gamma}} = \mathcal{D}_\gamma \subset \mathcal{D}_{\mathring{\gamma}}$,

(3) $\overline{\gamma} \cong \min\{\delta \mid \gamma \leq \delta$ and δ is an $\mathcal{E}$-reflection$\}$,

(4) $\mathring{\gamma}$ _is an_ $\mathcal{E}$-_connection, if all morphisms_ $\mathring{\gamma}X$ _have fibres; if, in addition,_ $\mathcal{E}$ _is fibre determined and if_ (i) $\Rightarrow$ (ii) _in Prop._ 2(2) _holds true, then_

$\mathring{\gamma} \cong \max\{\delta \mid \delta \leq \gamma$ and δ is an $\mathcal{E}$-connection$\}$.

Proof: (1) and (2) from Lemma 3 and Theorem 3. (3) $\overline{\gamma}$ is an $\mathcal{E}$-reflection, and for any other $\mathcal{E}$-reflection $\delta : \mathrm{Id}_{\mathcal{K}} \to D$ with $\gamma \leq \delta$ one has $DX \in \mathcal{D}_\delta \subset \mathcal{D}_\gamma$ by Prop. 2(1), hence $\sigma X \cdot \overline{\gamma}X = \delta X$ with a unique morphism $\sigma X : \overline{C}X \to DX$. So $\overline{\gamma} \leq \delta$.

(4) Consider diagram (3) and any morphisms g,h_i such that $h_i \cdot \mathring{\gamma}F_i = gu_i$ for all $i \in I$. By construction of $\pi_{\mathbb{G}}$ (with $\mathbb{G} = \mathcal{C}_\gamma$) it suffices to show that gv is constant for every $v : A \to X$ with $A \in \mathrm{Ob}\mathbb{G}$. But since $\pi_{\mathbb{G}}X \cdot v$ is constant v factors through one of the fibres u_i by a morphism $t : A \to F_i$. Since $\pi_{\mathbb{G}}F_i \cdot t$ is constant, also $gv = h_i \cdot \pi_{\mathbb{G}}F_i \cdot t$ is.

Finally suppose $\delta \leq \gamma$ with δ-connected δ-quasi-components. Since $\mathcal{C}_\delta \subset \mathcal{C}_\gamma = \mathcal{C}_{\mathring{\gamma}}$ those are also $\mathring{\gamma}$-connected. Hence $\delta \leq \mathring{\gamma}$ follows if $\mathcal{E}$ is fibre determined. □

Considering the examples of the previous section, we trivially have $\mathring{\gamma}_1 = \gamma_1 = \overline{\gamma}_1$. Since $\gamma_1 \leq \gamma_2$, from $\mathcal{C}_{\gamma_2} = \mathcal{C}_{\gamma_1}$ one gets $\mathring{\gamma}_2 = \gamma_1$. We do not know a good description of $\overline{\gamma}_3$. Although $\mathcal{E}$ is not fibre determined in example (5), there is a greatest $\mathcal{E}$-connection $\mathring{\gamma}_5 \leq \gamma_5$, namely $\mathring{\gamma}_5 = 1$ (which is in fact the least $\mathcal{E}$-prereflection). $\mathring{\gamma}_7G$ is the projection $G \to G/G^\infty$ where $G^\infty = \bigcap_\alpha G^{(\alpha)}$ with $G^{(\alpha+1)} = [G^{(\alpha)},G^{(\alpha)}]$, α an ordinal number (see the remarks after Theorem 5 below).

There is a natural transfinite construction of the reflexion $\overline{\gamma}X : X \to \overline{C}X$: let $\gamma_1 X : X \to CX$ be the morphism γX , let $\gamma_{\alpha+1}X : X \to C_{\alpha+1}X$ be the morphism $\gamma C_\alpha X \cdot \gamma_\alpha X$, and let $\gamma_\lambda X : X \to C_\lambda X \cong \varinjlim_{\alpha<\lambda} C_\alpha X$ be the canonical injection in case of a limit ordinal λ (the direct limit exists since $\mathcal{K}$ is $\mathcal{E}$-cocomplete). If $\mathcal{K}$ admits only a small set of non-isomorphic $\mathcal{E}$-morphisms with domain X then, for some α , $\gamma C_\alpha X : C_\alpha X \to C_{\alpha+1}X$ must be an isomorphism, that is: $C_\alpha X \in \mathcal{D}_\gamma$, Hence $\overline{\gamma}X \cong \gamma_\alpha X$.

A similar observation can be made with respect to the construction of $\mathring{\gamma}X : X \to \mathring{C}X$: we shall "know" $\mathring{\gamma}X$ in all examples as soon as we know the $\mathring{\gamma}$-components of X , but these can be (under mild conditions) obtained by a transfinite construction by first forming the γ-quasi-components of X , then the γ-quasi-components of the γ-quasi-components, and so on. The proof of the following theorem will describe this in more detail.

We denote by $\mathcal{K}^*$ the (non-full) subcategory of $\mathcal{K}$ which has the same objects but just those morphisms $f : X \to Y$ with Tf an isomorphism (where T comes from the $\mathcal{E}$-localizations $\eta X : X \to TX)$; let $\mathcal{C}^*_\gamma$ be the full subcategory of $\mathcal{K}^*$ with $\mathrm{Ob}\mathcal{C}^*_\gamma = \mathrm{Ob}\mathcal{C}_\gamma$. For the next theorem only, let us assume that fibres can be constructed as in Theorem 2 with a class $\mathcal{M}$ which is closed under composition and for which $\mathcal{K}$ is $\mathcal{M}$-well powered; also, let T preserve colimits of chains of $\mathcal{M}$-morphisms. Then:

THEOREM 5. <u>For any $\mathcal{E}$-prereflection γ , $\mathcal{C}^*_\gamma$ is a multi-$\mathcal{M}$-coreflective subcategory</u> of $\mathcal{K}^*$ (cf. Salicrup [12] for terminology).

<u>Proof</u>: We show that $\mathcal{C}^*_\gamma$ is even multicoreflective in $\mathcal{K}$; so for every $w : A \to X$ with $A \in \mathrm{Ob}\mathcal{C}_\gamma$ we have to construct morphisms $v : F \to X$ and $f : A \to F$ with $F \in \mathrm{Ob}\mathcal{C}_\gamma$, $vf = w$, and Tf an isomorphism, such that the following universal property holds: whenever one has $zg = w$ with $g : A \to G \in \mathrm{Ob}\mathcal{C}_\gamma$ and Tg isomorphic then there is a unique $h : G \to F$ with $vh = z$.

First of all, since $A \in \mathrm{Ob}\mathcal{C}_\gamma$, w is a subfibre of γX which must factor through an essentially unique fibre $v_1 : F_1 \to X$ by a unique morphism $f_1 : A \to F_1$ with Tf_1 an isomorphism. Inductively we may construct $v_\alpha : F_\alpha \to X$ and $f_\alpha : A \to F_\alpha$ with Tf_α an isomorphism: f_α must factor through an essentially unique fibre $v_{\alpha+1,\alpha} : F_{\alpha+1} \to F_\alpha$ of γF_α by a unique $f_{\alpha+1} : A \to F_{\alpha+1}$ with $Tf_{\alpha+1}$ an isomorphism; since $\mathcal{M}$ is closed under composition, $v_{\alpha+1} := v_\alpha \cdot v_{\alpha+1,\alpha}$ belongs to $\mathcal{M}$, so $\mathcal{M}$-completeness will allow us to form the inverse limit $F_\lambda = \varprojlim_{\alpha<\lambda} F_\alpha$ in case of a limit ordinal. If T preserves this limit we are sure that, for the induced morphism $f_\lambda : A \to F_\lambda$, Tf_λ is an isomorphism. If $\mathcal{K}$ is $\mathcal{M}$-wellpowered this construction will stop for some α , that is : $v_{\alpha+1,\alpha}$ is an isomorphism. But since $v_{\alpha+1,\alpha}$ is a fibre of γF_α the latter must be constant, hence $F := F_\alpha \in \mathcal{C}_\gamma$.

Given z and g as above, it is easily checked that the subfibre z of γX must factor through the fibre v_1 of γX , and then through v_2 , and so on, which gives the desired factorization after a transfinite induction. □

COROLLARY. $\mathcal{C}^*_\gamma$ <u>is closed under connected colimits in</u> $\mathcal{K}^*$, <u>in particular under filtered colimits</u>. □

The needed assumptions for Theorem 5 are satisfied in examples (1) - (7) mentioned in Section 4. However, the transfinite construction is needed only if γ is not an $\mathcal{E}$-connection since, otherwise, the γ-components serve as local coreflections (cf. [17] Prop. 6 for the respective simplified version of Theorem 5). For $\gamma = \gamma_2$, it is not the fact of multicoreflectivity that is interesting (since $\mathcal{C}_{\gamma_2} = \mathcal{C}_{\gamma_1}$) but the way the γ_1-component $K(x)$ of a point $x \in X$ is constructed by Theorem 5: one forms its γ_2-quasi-component $Q_1(x)$, then its γ_2-quasi-component $Q_2(x)$ in the subspace $Q_1(x)$ and so on; for some ordinal α, $Q_\alpha(x) = K(x)$.

Since the category of groups has a zero-object, Theorem 5 gives in case $\gamma = \gamma_7$ that the full subcategory of perfect groups is monocoreflective, and the coreflector is obtained by the transfinitely iterated formation of the commutator subgroup. By the way, here $\mathcal{C}^*_\gamma = \mathcal{C}_\gamma$. However, Theorem 5 becomes false if $\mathcal{C}^*_\gamma$ and $\mathcal{K}^*$ are replaced by $\mathcal{C}_\gamma$ and $\mathcal{K}$ as can be already seen in case $\gamma = \gamma_1$: $\mathcal{C}_\gamma$ is the full subcategory of connected spaces, $\emptyset$ included, which is not multico-reflective in <u>Top</u>; it is if one removes the empty space, or if one just removes the inclusion mappings $\emptyset \to X \neq \emptyset$ which will give precisely the category $\mathcal{C}^*_\gamma$.

6. Relationships to left and right constant subcategories

For full subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{K}$ the full subcategories $r(\mathcal{A})$ and $\ell(\mathcal{B})$ are defined by

$$\mathrm{Ob}\, r(\mathcal{A}) = \{B \mid \forall\, A \in \mathrm{Ob}\,\mathcal{A}\ \forall\, f : A \to B : f \text{ constant}\}$$

$$\mathrm{Ob}\, \ell(\mathcal{B}) = \{A \mid \forall\, B \in \mathrm{Ob}\,\mathcal{B}\ \forall\, f : A \to B : f \text{ constant}\} .$$

It follows easily that the <u>right constant subcategory</u> $r(\mathcal{A})$ coincides with the subcategory $\mathcal{D}_\gamma$ for $\gamma = \pi_{\mathcal{A}}$ (cf. Section 5). On the other hand one obtains immediately from the definitions and property (B) of Section 2 that the <u>left constant subcategory</u> $\ell(\mathcal{B})$ is the subcategory $\mathcal{C}_\gamma$ for $\gamma = \rho_{\mathcal{B}}$. So by Theorems 4 and 5 we obtain immediately the known facts that right constant subcategories are $\mathcal{E}$-reflective (but not vice versa: $\mathcal{D}_{\gamma_7}$) and that left constant subcategories are $\mathcal{E}$-component subcategories (but not vice versa: $\mathcal{C}_{\gamma_3}$) and under mild side conditions multicoreflective. However, the more important consequence is stated by the following

THEOREM 6. (1) <u>For a full and replete subcategory</u> $\mathcal{A}$, <u>the following assertions are equivalent</u>:

(i) $\mathcal{A} = \ell(r(\mathcal{A}))$,

(ii) $\mathcal{A} = \ell(\mathcal{B})$ <u>for some subcategory</u> $\mathcal{B}$,

(iii) $\mathcal{A} = \mathcal{C}_\gamma$ <u>for some</u> $\mathcal{E}$<u>-reflection</u> γ .

(2) <u>For a full and replete subcategory</u> $\mathcal{B}$, <u>the following assertions are equiv-</u>

alent, if $\mathcal{K}$ has fibres of morphisms:

(i) $\mathbb{B} = r(\ell(\mathbb{B}))$,

(ii) $\mathbb{B} = r(\mathbb{A})$ for some subcategory $\mathbb{A}$,

(iii) $\mathbb{B} = \mathbb{D}_\gamma$ for some $\mathcal{E}$-connection γ .

Proof: Since ℓ and r form a Galois correspondence, (i) $\Leftrightarrow$ (ii) in (1) and (2) are trivial. (ii) $\Rightarrow$ (iii) follows in both cases from the remarks above. We prove (iii) $\Rightarrow$ (ii): (1) If $\mathbb{A} = \mathbb{C}_\gamma$ for $\gamma \cong \overline{\gamma}$ then $\mathbb{A} = \mathbb{C}_{\overline{\gamma}} = \ell(\mathbb{B})$ with $\mathbb{B} = \mathbb{D}_\gamma$ by Theorem 4 (3). (2) If $\mathbb{B} = \mathbb{D}_\gamma$ for $\gamma \cong \mathring{\gamma}$ then $\mathbb{B} = \mathbb{D}_{\mathring{\gamma}} = r(\mathbb{A})$ with $\mathbb{A} = \mathbb{C}_\gamma$. □

Remarks. (1) In Theorem 6 (2), the existence of fibres is needed only to conclude that $\pi_{\mathbb{A}}$ is an $\mathcal{E}$-connection. Calling any $\gamma \cong \pi_{\mathbb{A}}$ for some $\mathbb{A}$ an $\mathcal{E}$-connection we could have avoided this assumption.

(2) To conclude $\gamma \cong \mathring{\gamma}$ for an $\mathcal{E}$-connection γ one does not need Theorem 4 (4); this follows directly from the definitions without further assumptions.

COROLLARY. If γ is an $\mathcal{E}$-reflection and an $\mathcal{E}$-connection then $\mathbb{D}_\gamma = r(\mathbb{C}_\gamma)$ and $\mathbb{C}_\gamma = \ell(\mathbb{D}_\gamma)$. □

8. Global aspects

Let $\mathbb{P}$ be the ordered conglomerate of all $\mathcal{E}$-prereflections modulo $\cong$, and $\mathbb{P}_r$ ($\mathbb{P}_c$) the subconglomerate of $\mathcal{E}$-reflections ($\mathcal{E}$-connections). The ordered conglomerate $\mathbb{C}$ of all full and replete subcategories of $\mathcal{K}$ contains the subconglomerate $\mathbb{C}_r$ ($\mathbb{C}_c$) of $\mathcal{E}$-reflective ($\mathcal{E}$-component) subcategories which contains the subconglomerates $\mathbb{R}$ ($\mathbb{L}$) of right (left) constant subcategories. Lemma 2 tells us that we have adjoint functors

$$\Delta \dashv \Phi : \mathbb{C}^{op} \to \mathbb{P} \quad \text{and} \quad \Psi \dashv \Gamma : \mathbb{P} \to \mathbb{C}$$

which induce 1 - 1-correspondences

$$\mathbb{C}_r^{op} \underset{\Delta}{\overset{\Phi}{\rightleftarrows}} \mathbb{P}_r \quad \text{and} \quad \mathbb{P}_c \underset{\Psi}{\overset{\Gamma}{\rightleftarrows}} \mathbb{C}_c \ ;$$

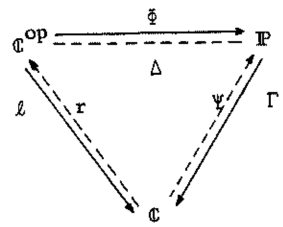
here $\Delta : \gamma \mapsto \mathbb{D}_\gamma$, $\Gamma : \gamma \mapsto \mathbb{C}_\gamma$, $\Phi : \mathbb{B} \mapsto \rho_{\mathbb{B}}$, $\Psi : \mathbb{A} \mapsto \pi_{\mathbb{A}}$.
The assignments $\mathbb{A} \mapsto r(\mathbb{A})$ and $\mathbb{B} \mapsto \ell(\mathbb{B})$ yield the adjuction

$$r \dashv \ell : \mathbb{C}^{op} \to \mathbb{C}$$

which induces a 1 - 1-correspondence

$$\mathbb{R}^{op} \underset{r}{\overset{\ell}{\rightleftarrows}} \mathbb{L} \ .$$

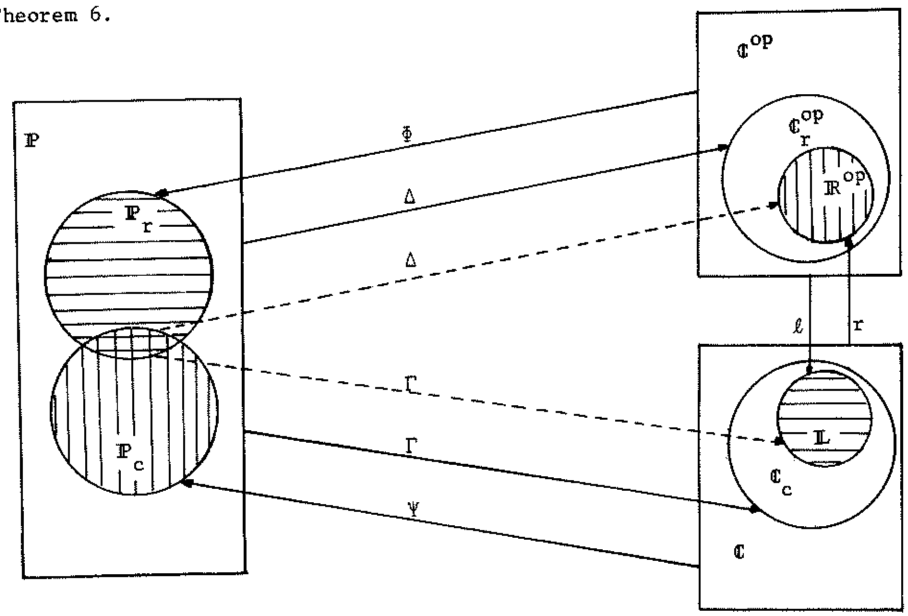
Since $\ell(\mathbb{B}) = \mathbb{C}_\gamma$ for $\gamma = \rho_{\mathbb{B}}$ and $r(\mathbb{A}) = \mathbb{D}_\delta$ for $\delta = \pi_{\mathbb{A}}$ this third adjuction is the composite of the two others, that is:



is commutative for both, the solid and the dotted arrows. Moreover,

$$\Delta(\mathbb{P}_c) = \mathbb{R}^{op} \quad \text{and} \quad \Gamma(\mathbb{P}_r) = \mathbb{L}$$

by Theorem 6.



$\mathbb{P}$ carries the structure of a complete lattice: $\gamma \cong \sup\{\gamma_i | i \in I\}$ is obtained as the multiple pushout γX of all morphisms $\gamma_i X$ (for every object X); for δX the $\mathcal{E}$-fact of $(X, \gamma_i X)_I$ one has $\delta \cong \inf\{\gamma_i | i \in I\}$. $\eta : Id_{\mathcal{K}} \to T$ is the top and $1 : Id_{\mathcal{K}} \to Id_{\mathcal{K}}$ the bottom element. Since $\mathbb{P}_r$ is reflective in $\mathbb{P}(\gamma \mapsto \bar{\gamma}$ being the reflector) $\mathbb{P}_r$ is a complete lattice as well and closed under inf's (but not under sup's) in $\mathbb{P}$. Similarly: $\mathbb{P}_c$ is coreflective in $\mathbb{P}(\gamma \mapsto \mathring{\gamma}$ being the coreflector), hence a complete lattice and closed under sup's (but not under inf's) in $\mathbb{P}$. Both η and 1 belong to $\mathbb{P}_r \cap \mathbb{P}_c$.

$\mathbb{C}_r$ and $\mathbb{C}_c$ are both reflective in $\mathbb{C}$ which carries a complete lattice structure with $\cup$ and $\cap$; $\mathcal{K}$ is the top and the full subcategory with object class $\{X | \eta X$ is iso$\}$ is the bottom element. Also $\mathbb{R}$ and $\mathbb{L}$ are both reflective in $\mathbb{C}$. So all four subcategories are complete lattices, but only the inf's are formed like in $\mathbb{C}$. Automatically, $\mathcal{K} \in \mathbb{R} \cap \mathbb{L}$, but it is not difficult to show that also the bottom element of $\mathbb{C}$ belongs to $\mathbb{R} \cap \mathbb{L}$.

From adjointness one derives immediately some rules for the behaviour of the functors Φ, Δ, Ψ, Γ, ℓ, r with respect to the formation of inf's and sup's in the various lattices involved.

8. Appendix: the center of a group

The center $Z(G)$ of a group G is known not to be a functorial construction since a homomorphism $f : G \to H$ need not map $Z(G)$ into $Z(H)$. However, Z is a functor if we restrict ourselves to the category Epi Grp of groups and surjective homomorphisms. This is still a good category for our purposes since it is $\mathcal{E}$-cocomplete with $\mathcal{E}$ the class of all morphisms in Epi Grp. A morphism $f : G \to H$ is constant if and only if $H = 1$; only in that case there are subfibres of f, and then, of course, every $u : K \to G$ is a subfibre of f and 1_G is the fibre through which all these subfibres factor. So every morphism (even every non-empty source) has fibres in the sense of the definition in Section 3. However, $\mathcal{E}$ is obviously not fibre determined. Nevertheless it turns out to be interesting to consider the $\mathcal{E}$-prereflection

$$\gamma G : G \to CG = G/Z(G) .$$

The reflective subcategory $\mathfrak{D}_\gamma$ consists of the groups without center $(Z(G) = 1)$, and the component subcategory $\mathfrak{C}_\gamma$ contains exactly the abelian groups (which are, of course, also reflective in Epi Grp). γ is neither a reflection since CG need not be without center (consider the quaternion group) nor a connection: a representative system of γ-quasi-components has exactly one element (if and only if G is abelian) or it is empty; in the latter case the pushout condition of the definition in Section 4 reduces to saying that every $g : G \to H$ factors through γG which is, of course, not true. However, all existing γ-quasi-components are γ-connected which shows once again the relevance of the condition '$\mathcal{E}$ is fibre determined' in Prop. 2 (2).

The reflection $\bar{\gamma}$ with $\mathfrak{D}_{\bar{\gamma}} = \mathfrak{D}_\gamma$ can be formed as outlined before Theorem 5. This construction gives exactly the quotient series

$$G \to CG \to C^2G \to \dots \to \bar{C}G$$

which comes from the transfinitely continued ascending central series

$$1 < Z_1(G) = Z(G) < Z_2(G) < \dots\dots < Z_\alpha(G) = \bar{Z}(G)$$

of the group G where α is the smallest ordinal such that $Z_\alpha(G) = Z_{\alpha+1}(G)$. (By definition, $Z_{\beta+1}(G)$ is the inverse image of $Z(C^\beta(G))$ under the projection $G \to G/Z_\beta(G) \cong C^\beta(G)$.) The left constant subcategory $\mathfrak{C}_{\bar{\gamma}}$ consists of the groups G with $\bar{Z}(G) = G$; it contains all nilpotent groups (those for which the above α is finite), in particular all abelian groups.

The connections $\mathring{\gamma}$ and $\mathring{\bar{\gamma}}$ belong to a general type of prereflections which are both connections and reflections, so in particular we have $\bar{\mathring{\gamma}} = \mathring{\gamma}$ and $\bar{\mathring{\bar{\gamma}}} = \mathring{\bar{\gamma}}$. Those prereflections are induced by arbitrary nonempty full subcategories $\mathfrak{C}$ which are closed under the formation of quotient structures. Then a prereflection $\delta_{\mathbb{G}}$ can be defined by

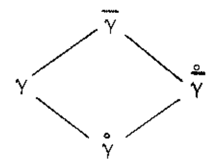
$$\delta_{\mathbb{G}} G = \begin{cases} G \to 1 & \text{if } G \in \mathrm{Ob}\mathbb{G} \\ 1_G : G \to G & \text{otherwise.} \end{cases}$$

$\delta_{\mathbb{G}}$ is a connection and reflection, and

$$\mathbb{C}_{\delta_{\mathbb{G}}} = \mathbb{G} \quad \text{and} \quad \mathbb{D}_{\delta_{\mathbb{G}}} = \mathbb{G}'$$

where $\mathbb{G}'$ is the full subcategory of all objects which are not in $\mathbb{G}$ or trivial. In particular, $\mathbb{G}$ is left constant and $\mathbb{G}'$ is right constant.

Going back to the definition of $\mathring{\gamma}$ it is easy to see that $\mathring{\gamma} = \delta_{\mathbb{G}}$ with (necessarily) $\mathbb{G} = \mathbb{C}_{\gamma}$ the subcategory of abelian groups and $\mathring{\bar{\gamma}} = \delta_{\mathbb{G}}$ with $\mathbb{G} = \mathbb{C}_{\bar{\gamma}}$. In the lattice $\mathbb{P}$ we got



The reflection of the abelian groups is not comparable with any of those.

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