Left-determined model categories and universal homotopy theories

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Abstract

We say that a model category is left-determined if the weak equivalences are generated (in a sense specified below) by the cofibrations. While the model category of simplicial sets is not left-determined, we show that its non-oriented variant, the category of symmetric simplicial sets (in the sense of Lawvere and Grandis) carries a natural left-determined model category structure. This is used to give another and, as we believe simpler, proof of a recent result of D. Dugger about universal homotopy theories.

1 Introduction

Recall that a *model category* $\mathcal{K}$ is a complete and cocomplete category $\mathcal{K}$ equipped with three classes of morphisms $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$, called *cofibrations*, *weak equivalences* and *fibrations*, such that

(1) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorizations systems and

(2) $\mathcal{W}$ is closed under retracts (in the category $\mathcal{K}^{\to}$ of morphisms of $\mathcal{K}$) and has the 2-out-of-3 property

(see [Q], [H], [Ho] or [AHRT2]). Model categories were introduced by D. Quillen to provide a foundation of homotopy theory. Here a weak factorization system is a pair $(\mathcal{L}, \mathcal{R})$ of morphisms such that every morphism has a factorization as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism, and $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$ where $\mathcal{L}^{\square}$ (${}^{\square}\mathcal{R}$) consists of all morphisms having the right (left) lifting property w.r.t. $\mathcal{L}$ ($\mathcal{R}$, respectively). The morphism l has a *left lifting property* with respect to a morphism r (or r has a *right lifting property* w.r.t. l) if in every commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow[v]{} & D \end{array}$$

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there exists a diagonal $d : B \to C$.

A model category is determined by any two of the three classes above. Clearly, $\mathcal{C}$ and $\mathcal{W}$ determine $\mathcal{F}$ because $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\square}$, and from $\mathcal{C}$ and $\mathcal{F}$ one obtains the morphisms of $\mathcal{W}$ as composites $g \cdot f$ with $f \in {}^{\square}\mathcal{F}$ and $g \in \mathcal{C}^{\square}$. In this paper we are interested in the model categories whose model structure is determined by its cofibrations only, and we therefore call them *left-determined*. For example, the model category **SComp** of simplicial complexes is left-determined while the model category **Simp** of simplicial sets is not left-determined.

Simp is, of course, the presheaf category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ where $\mathbf{\Delta}$ is the category of non-zero finite ordinals and order-preserving maps. F. W. Lawvere [L] and M. Grandis [G] introduced *symmetric simplicial sets* as functors $\mathbf{F}^{\mathrm{op}} \to \mathbf{Set}$ where $\mathbf{F}$ is the category of non-zero finite cardinals (= finite sets) and arbitrary maps. We will show that the category $\mathbf{SSimp} = \mathbf{Set}^{\mathbf{F}^{\mathrm{op}}}$ of symmetric simplicial sets is a left-determined model category. Moreover, the model categories **SSimp** and **Simp** are Quillen equivalent, i.e., they have equivalent homotopy categories.

D. Dugger [D] has recently shown that, for a small category $\mathcal{X}$, $\mathbf{Simp}^{\mathcal{X}^{\mathrm{op}}}$ is a universal model category over $\mathcal{X}$. In particular, **Simp** is a universal model category over the (one-morphism) category **1**. We will give another proof of his result, by showing that also $\mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$ serves as a universal model category over $\mathcal{X}$. Since **SSimp** is left-determined, our proof is simpler.

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After having completed this work we learned that the concept of a left-determined model category was independently developed by J. H. Smith [S] who used the term minimal model category instead. He also observed that the usual model structure on simplicial sets fails to be left-determined.

2 Left-determined model categories

Definition 2.1. A model category $\mathcal{K}$ is *left-determined* if $\mathcal{W}$ is the smallest class of morphisms satisfying the following conditions:

(i) $\mathcal{C}^{\square} \subseteq \mathcal{W}$,

(ii) $\mathcal{W}$ is closed under retracts and satisfies the 2-out-of-3 property,

(iii) $\mathcal{C} \cap \mathcal{W}$ is stable under pushout and closed under transfinite composition.

We will denote the smallest class of morphisms satisfying (i)-(iii) by $\mathcal{W}_{\mathcal{C}}$. It has the property that $\mathcal{W}_{\mathcal{C}} \subseteq \mathcal{W}$ for each model category $\mathcal{K}$ having $\mathcal{C}$ as the class of cofibrations. Left-determined model categories are those for which $\mathcal{W} = \mathcal{W}_{\mathcal{C}}$. Recall that $\mathcal{C}^{\square}$ denotes the class of morphisms having the right lifting property w.r.t. $\mathcal{C}$. Of course, $\mathcal{C}^{\square} = \mathcal{F} \cap \mathcal{W}$ is the class of trivial fibrations.

In general, given $\mathcal{C}$, the first principal problem is whether $\mathcal{C}$ and $\mathcal{W}_\mathcal{C}$ yield a model category. The next theorem gives an affirmative answer under an additional set-theoretic hypothesis, the Vopěnka's Principle. (Subsequently J. H. Smith informed us that he has been able to prove the theorem even absolutely, i.e. without any additional set-theoretic hypothesis.) Recall that Vopěnka's Principle is a set-theoretic axiom implying the existence of very large cardinals (see [AR]). We denote by $\operatorname{cof}(\mathcal{I})$ the smallest class of morphisms containing $\mathcal{I}$, closed under retracts in comma-categories $A\backslash\mathcal{K}$ and satisfying (iii). The smallest class containing $\mathcal{I}$ and satisfying (iii) is denoted by $\operatorname{cell}(\mathcal{I})$ (see [AHRT1]).

Theorem 2.2. *Let $\mathcal{I}$ be a (small) set of morphisms in a locally presentable category $\mathcal{K}$. Under Vopěnka's principle, $\mathcal{C} = \operatorname{cof}(\mathcal{I})$ and $\mathcal{W} = \mathcal{W}_\mathcal{C}$ yield a model category structure on $\mathcal{K}$.*

Proof. According to the theorem of J. H. Smith (see [B] 1.7), it suffices to show that $\mathcal{W}_\mathcal{C}$ satisfies the solution set condition at $\mathcal{I}$. It means that for every $f \in \mathcal{I}$ there is a subset $\mathcal{X}_f$ of $\mathcal{W}_\mathcal{C}$ such that every morphism $f \to g$, $g \in \mathcal{W}_\mathcal{C}$ factorizes through some $h \in \mathcal{X}_f$. Since $f\backslash\mathcal{W}_\mathcal{C}$ is a full subcategory of $f\backslash\mathcal{K}$ and $f\backslash\mathcal{K}$ is locally presentable (see [AR] 1.57), $f\backslash\mathcal{W}_\mathcal{C}$ has a small dense subcategory $\mathcal{X}_f$ provided that we asssume Vopěnka's principle (see [AR] 6.6). Without any loss of generality, we may assume that $\mathcal{X}_f$ contains the initial object of $f\backslash\mathcal{W}_\mathcal{C}$ provided that it exists. A morphism $f \to g$ in $f\backslash\mathcal{W}_\mathcal{C}$ is either initial in $f\backslash\mathcal{W}_\mathcal{C}$ and thus belongs to $\mathcal{X}_f$, or it factorizes through some morphism $f \to h$ from $\mathcal{X}_f$. Hence $\mathcal{X}_f$, $f \in \mathcal{I}$ provide a solution set condition at $\mathcal{I}$. ☐

A model category is called *cofibrantly generated* if $\mathcal{C} = \operatorname{cof}(\mathcal{I})$ and $\mathcal{C} \cap \mathcal{W} = \operatorname{cof}(\mathcal{J})$ for sets $\mathcal{I}$ and $\mathcal{J}$. Following J. H. Smith, a model category $\mathcal{K}$ is called *combinatorial* if it is cofibrantly generated and the category $\mathcal{K}$ is locally presentable. The model categories from Theorem 2.2 are combinatorial.

Left-determined model categories are, in some sense, related to left Bousfield localizations. Recall that, having model categories $\mathcal{K}$ and $\mathcal{L}$, a *left Quillen functor* $H : \mathcal{K} \to \mathcal{L}$ is a left adjoint functor preserving cofibrations and trivial cofibrations (i.e., elements of $\mathcal{C} \cap \mathcal{W}$). Every left Quillen functor preserves weak equivalences between cofibrant objects (see [Ho]). An object A of a model category $\mathcal{K}$ is *cofibrant* if $0 \to A$ is a cofibration.

A model category $\mathcal{K}$ is called *functorial* if both weak factorization systems $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are functorial. This means that, for a weak factorization system $(\mathcal{L}, \mathcal{R})$, there is a functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ and natural transformations $\lambda : \operatorname{dom} \to F$ and $\varrho : F \to \operatorname{cod}$ such that $f = \varrho_f \cdot \lambda_f$ is an $(\mathcal{L}, \mathcal{R})$-factorization of a morphism $f : A \to B$; of course, $\operatorname{dom}(f) = A$ and $\operatorname{cod}(f) = B$. This definition of a functorial weak factorization system is given in [RT] where its relation to functoriality in the sense of Hovey [Ho] is explained. Each combinatorial model category is functorial. In a functorial model category $\mathcal{K}$ we have a *cofibrant replacement functor* $Q : \mathcal{K} \to \mathcal{K}$ where

$$0 \longrightarrow Q(A) \xrightarrow{q_A} A$$

is a functorial weak factorization in $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$. Then $q : Q \to Id_{\mathcal{K}}$ is a natural transformation.

Let $\mathcal{K}$ be a model category and $\mathcal{Z}$ a class of morphism of $\mathcal{K}$. A *left Bousfield localization* of $\mathcal{K}$ w.r.t. $\mathcal{Z}$ is a model category structure $\mathcal{K}\backslash\mathcal{Z}$ on the category $\mathcal{K}$ such that

(a) $\mathcal{K}\backslash\mathcal{Z}$ has the same cofibrations as $\mathcal{K}$,

(b) weak equivalences of $\mathcal{K}\backslash\mathcal{Z}$ contain both the weak equivalences of $\mathcal{K}$ and the morphisms of $\mathcal{Z}$ and

(c) each left Quillen functor $H : \mathcal{K} \to \mathcal{L}$ such that $H \cdot Q$ sends $\mathcal{Z}$-morphisms to weak equivalences is a left Quillen functor $\mathcal{K}\backslash\mathcal{Z} \to \mathcal{L}$

(see [H] 3.3.1). J. H. Smith proved that if $\mathcal{K}$ is a left proper combinatorial model category and $\mathcal{Z}$ a set of morphisms, then a left Bousfield localization $\mathcal{K}\backslash\mathcal{Z}$ exists (see [S]). (The model category is called *left proper* if every pushout of a weak equivalence along a cofibrations is a weak equivalence.) As a consequence, we get the following result

Theorem 2.3. *Let $\mathcal{K}$ be a left proper, combinatorial model category and $\mathcal{Z}$ a class of morphisms of $\mathcal{K}$. Under Vopěnka's principle, a left Bousfield localization $\mathcal{K}\backslash\mathcal{Z}$ exists.*

Proof. We can express $\mathcal{Z}$ as a union of an increasing chain of (small) subsets $\mathcal{Z}_i$ indexed by ordinals. Let $\mathcal{W}_i$ denote the class of weak equivalences in the model category $\mathcal{K}\backslash\mathcal{Z}_i$ (which exists by the result of J. Smith). Then we have $\mathcal{W}_i \subseteq \mathcal{W}_j$ for $i \leq j$; this follows from $Id_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}\backslash\mathcal{Z}_j$ being a left Quillen functor sending $\mathcal{Z}_i$ morphisms to weak equivalences. Hence $Q = Id_{\mathcal{K}} \cdot Q : \mathcal{K}\backslash\mathcal{Z}_i \to \mathcal{K}\backslash\mathcal{Z}_j$ is a left Quillen functor. Let $f : A \to B$ be a weak equivalence from $\mathcal{W}_i$ and consider

$$\begin{array}{ccc} QA & \xrightarrow{Qf} & QB \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle r_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

Since r_A, $r_B \in \mathcal{C}^{\square}$, we have r_A, r_B, $Qf \in \mathcal{W}_j$. Hence $f \in \mathcal{W}_j$. Put $\mathcal{W}_* = \bigcup\limits_{i \in Ord} \mathcal{W}_i$. Then $\mathcal{W}_*$ is closed under retracts and satisfies the 2-out-of-3 property. Analogously as in Theorem 2.2, Vopěnka's principle guarantees that $\mathcal{K}, \mathcal{C}$ and $\mathcal{W}_*$ form a model category $\mathcal{K}\backslash\mathcal{Z}$.

Let $H : \mathcal{K} \to \mathcal{L}$ be a left Quillen functor such that $H \cdot Q$ sends $\mathcal{Z}$-morphisms to weak equivalences. Since $H : \mathcal{K}\backslash\mathcal{Z} \to \mathcal{L}$ is a left Quillen functor for each i, $H : \mathcal{K}\backslash\mathcal{Z}_i \to \mathcal{L}$ is a left Quillen functor. Hence $\mathcal{K}\backslash\mathcal{Z}$ is a left Bousfield localization of $\mathcal{K}$ w.r.t. $\mathcal{Z}$. □

Using [AR] as well, C. Casacuberta, D. Sceveneles and J. H. Smith [CSS] proved a related result saying that cohomological localizations of simplicial sets exist under Vopěnka's Principle.

Remark 2.4. In analogy with the definition of a left-determined model category, we define $\mathcal{W}_\mathcal{X}$, where $\mathcal{X}$ is a class of morphisms in a model category $\mathcal{K}$, as the smallest class of morphisms satisfying

(i) $\mathcal{W} \cup \mathcal{X} \subseteq \mathcal{W}_\mathcal{X}$,

(ii) $\mathcal{W}_\mathcal{X}$ is closed under retracts and satisfies the 2-out-of-3 property,

(iii) $\mathcal{C} \cap \mathcal{W}_\mathcal{X}$ is stable under pushout and closed under transfinite composition.

Under Vopěnka's principle, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{W}_\mathcal{X}$ is a model category structure for each combinatorial model category $\mathcal{K}$. This model category structure is evidently $\mathcal{K}\backslash\mathcal{X}$, provided that $\mathcal{K}\backslash\mathcal{X}$ exists (because $\mathcal{W}_\mathcal{X}$ is contained in the class of weak equivalences of $\mathcal{K}\backslash\mathcal{X}$).

Let $\mathcal{K}$ be a cofibrantly generated model category and $\mathcal{X}$ a small category. Then there is a cofibrantly generated model category structure on the functor category $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$ (see [H] 14.2.1); this structure is called the *Bousfield-Kan structure*. To recall it we denote by

$$ev_X : \mathcal{K}^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$$

the evaluation functor given by $ev_X(A) = A(X)$ and by

$$F_X : \mathcal{K} \to \mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$$

its left adjoint given by

$$F_X(K)(Y) = \coprod_{\mathcal{X}^{op}(X,Y)} K\,.$$

If $\mathcal{I}$ ($\mathcal{J}$) is the set of generating (trivial) cofibrations in $\mathcal{K}$ then the Bousfield-Kan model structure has $\overline{\mathcal{I}} = \bigcup\limits_{X\in \mathrm{ob}(\mathcal{X})} F_X(\mathcal{I})$ as generating cofibrations and $\overline{\mathcal{J}} = \bigcup\limits_{X\in \mathrm{ob}(\mathcal{X})} F_X(\mathcal{J})$ as generating trivial cofibrations. Then

(a) $\varphi : A \to B$ is a weak equivalence in $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$ iff $\varphi_X : A(X) \to B(X)$ is a weak equivalence in $\mathcal{K}$ for each X in $\mathcal{X}$,

(b) $\varphi : A \to B$ is a fibration in $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$ iff $\varphi_X : A(X) \to B(X)$ is a fibration in $\mathcal{K}$ for each X in $\mathcal{X}$.

Consequently, trivial fibrations are also morphisms in $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$ which are pointwise trivial fibrations in $\mathcal{K}$.

Proposition 2.5. *Let $\mathcal{K}$ be a cofibrantly generated, left-determined model category and $\mathcal{X}$ a small category. Then $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$ is a left-determined model category.*

Proof. Let $\mathcal{I}(\mathcal{J})$ be the set of (trivial) cofibrations in $\mathcal{K}$, respectively, and let $\overline{\mathcal{W}}$ the set of weak equivalences in $\mathcal{K}^{\mathcal{X}^{\mathrm{op}}}$. We have $\mathcal{W}_{\mathrm{cof}(\overline{\mathcal{I}})} \subseteq \overline{\mathcal{W}}$. Let $w \in \overline{\mathcal{W}}$. Then $w = f \cdot g$ where $f \in \overline{\mathcal{I}}^{\square}$ and $g \in \mathrm{cof}(\overline{\mathcal{J}})$. We have $f \in \mathcal{W}_{\mathrm{cof}(\overline{\mathcal{I}})}$. To prove that $g \in \mathcal{W}_{\mathrm{cof}(\overline{\mathcal{I}})}$, that is $w \in \mathcal{W}_{\mathrm{cof}(\overline{\mathcal{I}})}$, it suffices to show that $\overline{\mathcal{J}} \subseteq \mathcal{W}_{\mathrm{cof}(\overline{\mathcal{I}})}$. But this follows from $\mathcal{J} \subseteq \mathcal{W}_{\mathrm{cof}(\mathcal{I})}$ and the fact that F_X preserve colimits. □

3 Symmetric simplicial sets

A trivial example of a left-determined model category is the category **Set** of sets, with $\mathcal{C}$ the class of all monomorphisms, and with $\mathcal{W}$ the class of the morphisms between non-empty sets and the identity morphism on $\emptyset$. To give a non-trivial example we recall that a *simplicial complex* is a set X equipped with a set $\mathcal{X}$ of non-empty finite subsets of X such that

(a) $\{x\} \in \mathcal{X}$ for each $x \in X$,

(b) $A \in \mathcal{X}, \emptyset \neq B \subseteq A \Rightarrow B \in \mathcal{X}$.

Elements $A \in \mathcal{X}$ with $|A| = n + 1$ are called (non-degenerated) *n-simplices*. For $n = 0, 1$ and 2 we speak about *vertices*, *edges* and *triangles*, respectively. If $|A| \leq 2$ for each $A \in \mathcal{X}$ then $(X, \mathcal{X})$ is called a (non-oriented) *graph* (with loops). Morphisms of complexes $(X, \mathcal{X}) \to (Y, \mathcal{Y})$ are maps $h : X \to Y$ with $h(\mathcal{X}) \subseteq \mathcal{Y}$. We denote the category of simplicial complexes by **SComp**. We will show that $\mathcal{C} = Mono$ yields a left-determined model category structure on **SComp**. But the disadvantage of simplicial complexes is that each simplex is uniquely determined by its vertices, which makes colimits in **SComp** bad. It led S. Eilenberg and B. Zilber [EZ] to introduce complete semisimplicial complexes, which later were renamed as simplicial sets, and which are oriented. Surprisingly, non-oriented simplicial sets were introduced only recently by F. W. Lawvere [L] and M. Grandis [G]; they are called symmetric simplicial sets. Their position to simplicial complexes is the same as the position of multigraphs to graphs in graph theory (one admits multiple edges).

Definition 3.1. Let $\mathbf{F}$ denote the category of non-zero finite cardinals (and all maps). A *symmetric simplicial set* is by definition a functor $\mathbf{F}^{\mathrm{op}} \to \mathbf{Set}$. The category $\mathbf{Set}^{\mathbf{F}^{\mathrm{op}}}$ of symmetric simplicial sets will be denoted by **SSimp**.

We will recall the basic properties of symmetric simplicial sets (see [G]). We have the Yoneda embedding

$$Y : \mathbf{F} \to \mathbf{SSimp}.$$

Its values $\Delta_{n-1} = Y(n)$ are in fact simplicial complexes, which yields the functor

$$\mathbf{F} \to \mathbf{SComp}.$$

The Yoneda embedding Y extends along this functor to the full embedding

$$G : \mathbf{SComp} \to \mathbf{SSimp}$$

sending a simplicial complex $(X, \mathcal{X})$ to the functor $A : \mathbf{F}^{\mathrm{op}} \to \mathbf{Set}$ given by $A(n) = \{S \in \mathcal{X} \,|\, |S| \leq n-1\}$. In what follows we will identify a simplicial complex $(X, \mathcal{X})$ with its image under G. Hence **SComp** will be considered as a full subcategory of **SSimp**.

We have the functor $U : \mathbf{SSimp} \to \mathbf{Set}$ given by precomposition with $\mathbf{1} \to \mathbf{F}^{\mathrm{op}}$ (sending the object of $\mathbf{1}$ to 1). We can view $U(A)$ as the set of vertices of a symmetric simplicial set A and the whole A as the set $U(A)$ equipped with n-simplices corresponding to morphisms $\Delta_n \to A$. We will use the notation $A = (UA, \mathcal{A})$ where $\mathcal{A}$ is the set of simplices of A. For instance, Δ_n has all non-empty subsets of $\{0, 1, \dots, n\}$ as simplices. But note that the functor U is not faithful.

The embedding $\mathbf{\Delta} \to \mathbf{F}$ induces the faithful functor

$$H : \mathbf{SSimp} \to \mathbf{Simp}\,.$$

It has a left adjoint

$$L : \mathbf{Simp} \to \mathbf{SSimp}$$

sending each simplicial set to its symmetrization. There is also a right adjoint

$$R : \mathbf{Simp} \to \mathbf{SSimp}\,.$$

Let $\partial\Delta_n$ be the boundary of Δ_n for $n > 0$, i.e., $U(\partial\Delta_n) = n+1$, and simplices of $\partial\Delta_n$ are all non-empty subsets of $n+1$ distinct from $n+1$. Let $i_n : \partial\Delta_n \to \Delta_n$, $n > 0$, be the embeddings. Let $\mathcal{I} = \{i_n | n \geq 0\}$ where

$$i_0 : 0 \to \Delta_0\,.$$

In what follows, the class of all monomorphisms of **SSimp** is denoted by $Mono$.

Lemma 3.2. $\mathrm{cof}(\mathcal{I}) = Mono$.

The proof is the same as for simplicial sets.

Given $0 \leq k \leq n$, the *k-horn* Δ_n^k is the simplicial complex whose simplices are all subsets $\emptyset \neq S \subsetneqq \{0, 1, \dots, n\}$ distinct from $\{0, \dots, k-1, k+1, \dots, n\}$. Let $\mathcal{J}$ be the set of inclusions

$$j_n : \Delta_n^0 \to \Delta_n\,, \qquad n > 0\,.$$

Lemma 3.3. *$j_n \in \mathcal{W}_{Mono} \cap Mono$ for each $n > 0$.*

Proof. Evidently, each morphism $s_n : \Delta_n \to \Delta_0$, $n \geq 0$ belongs to $Mono^{\square}$. Therefore, by 2.1 (i) and (ii), each morphism $u : \Delta_0 \to \Delta_n$ belongs to $\mathcal{W}_{Mono}$ and, consequently to $\mathcal{W}_{Mono} \cap Mono$. Hence $j_1 \in \mathcal{W}_{Mono} \cap Mono$.

Assume that $j_1, \dots, j_n \in \mathcal{W}_{Mono} \cap Mono$. Consider the pushout

$$\begin{array}{ccc} \Delta_n + \Delta_0 & \xrightarrow{p_n} & \Delta_n \\ {\scriptstyle \mathrm{id}_{\Delta_n} + u_1^0}\downarrow & & \downarrow{\scriptstyle g_n} \\ \Delta_n + \Delta_1 & \longrightarrow & P_n \end{array}$$

where $u_1^0(0) = 0$ and p_n is induced by id_{Δ_n} and $u_n^0 : \Delta_0 \to \Delta_n$ (again $u_n^0(0) = 0$). Since $\mathrm{id}_{\Delta_n} + u_1^0 \in \mathcal{W}_{Mono} \cap Mono$, we have $g_n \in \mathcal{W}_{Mono} \cap Mono$ (by 2.1 (iii)). P_n is the simplicial complex given by attaching an edge at the vertex 0. By successive use of $j_2, \dots, j_n$, we fill horns to simplices. This is done via pushouts, starting with

$$\begin{array}{ccc} \Delta_2^0 & \xrightarrow{h} & P_n \\ {\scriptstyle j_2}\downarrow & & \downarrow \\ \Delta_2 & \longrightarrow & P_n' \end{array}$$

where h sends one edge of Δ_2^0 to the attached edge and the other edge to an edge of Δ_n. Doing this for all edges of Δ_n containing 0, we start to fill by using j_3, etc. At the end we obtain Δ_{n+1}^0 and a morphism

$$q_n : \Delta_n \xrightarrow{g_n} P_n \longrightarrow P_n' \longrightarrow \cdots \longrightarrow \Delta_{n+1}^0$$

which belongs to $\mathcal{W}_{Mono} \cap Mono$. Since, in the diagram

$$\begin{array}{ccc} \Delta_n & \xrightarrow{q_n} & \Delta_{n+1}^0 \\ & {\scriptstyle s_n}\searrow \quad \swarrow{\scriptstyle t_n} & \\ & \Delta_0 & \end{array}$$

we have $s_n \in \mathcal{W}_{Mono}$, we get $t_n \in \mathcal{W}_{Mono}$. Since, in the diagram

$$\begin{array}{ccc} \Delta_{n+1}^0 & \xrightarrow{j_{n+1}} & \Delta_{n+1} \\ & {\scriptstyle t_n}\searrow \quad \swarrow{\scriptstyle s_{n+1}} & \\ & \Delta_0 & \end{array}$$

we have t_n, $s_{n+1} \in \mathcal{W}_{Mono}$, we get $j_{n+1} \in \mathcal{W}_{Mono}$. Hence $j_{n+1} \in \mathcal{W}_{Mono} \cap Mono$. □

Theorem 3.4. **SSimp** *is a left-determined model category with* $\mathcal{C} = Mono$ *and* $\mathcal{F} = \mathcal{J}^{\square}$.

Proof. $(Mono, Mono^{\square})$ is a weak factorization system (see [B] or [AHRT2]); analogously for $(\mathrm{cof}(\mathcal{J}), \mathcal{J}^{\square})$. To prove the result it suffices to show that

$$\mathcal{W}_{Mono} \cap Mono = \mathrm{cof}(\mathcal{J})$$

(cf. [B]). Following Lemma 3.3, we have

$$\mathrm{cof}(\mathcal{J}) \subseteq \mathcal{W}_{Mono} \cap Mono\,.$$

The opposite inclusion will follow from properties of the adjunction $L \dashv H$ between symmetric simplicial sets and simplicial sets.

Since L preserves monomorphisms, H preserves trivial fibrations, i.e.,

$$H(Mono^{\square}) \subseteq \mathcal{W},$$

where $\mathcal{W}$ denotes the class of weak equivalences of simplicial sets. Since H preserves monomorphisms as well, we have

$$H(\mathcal{W}_{Mono} \cap Mono) \subseteq \mathcal{W} \cap Mono^{*} = \mathrm{cof}(\mathcal{J}^{*})$$

where $Mono^{*}$ denotes the monomorphisms in **Simp** and $\mathcal{J}^{*}$ is the generating set of horns in simplicial sets (cf. [Ho]). Since $L(\mathcal{J}^{*}) = \mathcal{J}$, we have $L(\mathrm{cof}(\mathcal{J}^{*})) \subseteq \mathrm{cof}(\mathcal{J})$. Consequently,

$$LH(\mathcal{W}_{Mono} \cap Mono) \subseteq \mathrm{cof}(\mathcal{J})\,.$$

The functor H sends a symmetric simplicial set $A = (UA, \mathcal{A})$ to the simplicial set HA having as (oriented) simplices all possible orientations of simplices from $\mathcal{A}$. The functor L then produces from each orientation a non-oriented simplex in LHA. Hence LH multiplies each non-degenerated n-simplex in A $n!$-times. By sending each simplex in $\mathcal{A}$ to its standard orientation, we get a natural transformation $\varrho : Id \to LH$ which splits the adjunction counit $\varepsilon : LH \to Id$. Hence each morphism f in **SSimp** is a retract of $LH(f)$. Consequently,

$$\mathcal{W}_{Mono} \cap Mono \subseteq \mathrm{cof}(\mathcal{J})\,.$$

$\square$

Remark 3.5. Both L and H are left Quillen functors. Moreover, $L \dashv H$ is a Quillen equivalence. Following [Ho] 1.3.13, this amounts to showing that

$$X \xrightarrow{\eta_X} HLX \xrightarrow{Hr_{LX}} H(LX)_f$$

is a weak equivalence for each simplicial set X (where η is the adjunction unit and $r_{LX} : LX \to (LX)_f$ is a fibrant replacement) and that

$$\varepsilon_Y : LHY \to Y$$

is a weak equivalence for each fibrant symmetric simplicial set Y. But η_X is a trivial cofibration because it is given by completing horns to simplices, and Hr_{LX} is a trivial cofibration too, because H is a left Quillen functor. That ε_Y is a trivial fibration follows from its description given in the proof above.

As a consequence we obtain that **Simp** and **SSimp** have equivalent homotopy categories.

Remark 3.6. The model category **Simp** is not left-determined. To prove this we consider the class $\mathcal{X}$ of morphisms $f : A \to B$ such that one of the following possibilities occur (U^* denotes the underlying functor $\mathbf{Simp} \to \mathbf{Set}$):

(a) there are vertices $b_1 \in U^*B$, $b_2 \in U^*B - (U^*f)(U^*A)$, an edge e in B from b_1 to b_2 but no edge in B from b_2 to b_1;

(b) there are vertices a_1, $a_2 \in U^*A$ and an edge e in A from a_1 to a_2 such that there is no edge in A from a_2 to a_1 but there is an edge in B from $U^*(f)(a_2)$ to $U^*(f)(a_1)$;

(c) there are vertices a_1, $a_2 \in U^*A$ and an edge e in B from $U^*(f)(a_1)$ to $U^*(f)(a_2)$ but there is no edge in B from $U^*(f)(a_2)$ to $U^*(f)(a_1)$ and no edge in A from a_1 to a_2.

Since (the oriented) horn $j_1^* : \Delta_1^0 \to \Delta_1$ belongs to $\mathcal{X}$, it suffices to show that $\mathcal{X} \cap \mathcal{W}_{Mono^*} = \emptyset$. But $\mathcal{X} \cap Mono^\square = \emptyset$ and no element of $\mathcal{X}$ can arise by operations 2.1 (ii) and (iii) from morphisms not belonging to $\mathcal{X}$.

Remark 3.7. **SComp** is a left-determined model category with $\mathcal{C} = Mono$ and $\mathcal{F} = \mathcal{J}^\square$. In fact, both i_n, $n \geq 0$ and j_n, $n > 0$ are morphisms of simplicial complexes. Hence the result follows from Theorem 3.4.

Corollary 3.8. *For each small category $\mathcal{X}$ the functor category $\mathbf{SSimp}^{\mathcal{X}}$ is a left-determined model category (with the Bousfield-Kan model category structure).*

The proof follows from Theorem 3.4 and Proposition 2.5.

4 Universal model categories

We will show that $\mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$ is a universal model category over $\mathcal{X}$ in the sense of D. Dugger [D] for each small category $\mathcal{X}$. In particular, **SSimp** is a universal model category over the one-morphism category. We will denote by

$$Y^* : \mathcal{X} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$$

the composition

$$\mathcal{X} \xrightarrow{Y_{\mathcal{X}}} \mathbf{Set}^{\mathcal{X}^{\mathrm{op}}} \xrightarrow{D_{\mathcal{X}}} (\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}})^{\mathbf{F}^{\mathrm{op}}}$$

where $D_{\mathcal{X}}$ is a left adjoint to the underlying functor

$$U_{\mathcal{X}} : (\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}})^{\mathbf{F}^{\mathrm{op}}} \longrightarrow \mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$$

given by evaluation at 1, i.e., $U_{\mathcal{X}} = ev_1$. Of course, we use the identifications

$$(\mathbf{Set}^{\mathbf{F}^{\mathrm{op}}})^{\mathcal{X}^{\mathrm{op}}} \cong \mathbf{Set}^{(\mathbf{F}\times\mathcal{X})^{\mathrm{op}}} \cong (\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}})^{\mathbf{F}^{\mathrm{op}}}.$$

Objects of $\mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$ may be called symmetric simplicial presheaves; then $D_{\mathcal{X}}(A)$ is the discrete symmetric simplicial presheaf over A. We also have the Yoneda embedding

$$\overline{Y} : \mathbf{F} \times \mathcal{X} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$$

and we will use the notation

$$\Delta_{n,X} = \overline{Y}(n+1, X)$$

for $n \geq 0$ and $X \in \mathcal{X}$. It is easy to see that

$$\Delta_{n,X} = F_X(\Delta_n)$$

where $F_X : \mathbf{SSimp} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$ is a left adjoint to the evaluation functor ev_X. We will also denote

$$\partial\Delta_{n,X} = F_X(\partial\Delta_n)\,.$$

Theorem 4.1. *Let $\mathcal{K}$ be a functorial model category, $\mathcal{X}$ a small category and $H : \mathcal{X} \to \mathcal{K}$ a functor such that all objects HX, $X \in \mathcal{X}$ are cofibrant. Then there is a left Quillen functor $H^* : \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$ such that $H^* \cdot Y^* = H$.*

Proof. Let $\mathbf{F}_n$ be the full subcategory of $\mathbf{F}$ consisting of cardinals $0 < k \leq n+1$. We get the induced inclusions

$$\mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}} \subseteq \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$$

where $\mathbf{SSimp}_n = \mathbf{Set}^{\mathbf{F}_n^{\mathrm{op}}}$ ($\mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}} \hookrightarrow \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}}$ is given by $\mathbf{SSimp}_n \hookrightarrow \mathbf{SSimp}$ which is induced by the functor $\mathbf{F}_n \hookrightarrow \mathbf{F} \xrightarrow{Y} \mathbf{SSimp}$). In particular, $\mathbf{SSimp}_0^{\mathcal{X}^{\mathrm{op}}} \cong \mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$ is the category of discrete symmetric simplicial presheaves. Since $\mathcal{K}$ is cocomplete and $\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$ is a free cocompletion of $\mathcal{X}$ (see [AR] 1.45), H extends to a colimit preserving functor

$$H_0^* : \mathbf{SSimp}_0^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$$

such that $H_0^* Y = H$.

Assume that we have the functor

$$H_n^* : \mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$$

extending H_{n-1}^*. Since $\partial\Delta_{n+1,X}$, $\Delta_{0,X}$ belong to $\mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}}$ for $X \in \mathcal{X}$, we can define $H_{n+1}^*(\Delta_{n+1,X})$ by the functorial (cofibration, trivial fibration) factorization

$$H_n^*(\partial\Delta_{n+1,X}) \xrightarrow{c_{n+1,X}} H_{n+1}^*(\Delta_{n+1,X}) \xrightarrow{r_{n+1,X}} H_n^*(\Delta_{0,X})$$

of the morphism $H_n^*(F_X(p_n)) : H_n^*(\partial\Delta_{n+1,X}) \to H_n^*(\Delta_{0,X})$ where $p_{n+1} : \partial\Delta_{n+1} \to \Delta_0$. To get an extension H_{n+1}^* of H_n^*, below we will define

(a) $H_{n+1}^*(f)$ for $f = \overline{Y}(f_1, f_2) : \Delta_{n+1,X} \to \Delta_{n+1,Y}$ where $f_1 : n+2 \to n+2$ is a bijection,

(b) $H^*_{n+1}(t)$ for $t = \overline{Y}(t_1, t_2) : \Delta_{m,X} \to \Delta_{n+1,Y}$ where $m \le n$

and

(c) H^*_{n+1} for $u = \overline{Y}(u_1, u_2) : \Delta_{n+1,X} \to \Delta_{m,Y}$ where $m \le n$.

(a) f_1 induces the isomorphisms $\overline{f}_1 : \Delta_{n+1} \to \Delta_{n+1}$, $\partial\overline{f}_1 : \partial\Delta_{n+1} \to \partial\Delta_{n+1}$ and f_2 induces a natural transformation $\varphi_{f_2} : F_X \to F_Y$. Hence f induces the homomorphism $\partial f : \partial\Delta_{n+1,X} \to \partial\Delta_{n+1,Y}$. We define $H^*_{n+1}(f)$ by the functorial filling

$$\begin{array}{ccccc}
H^*_n(\partial\Delta_{n+1,X}) & \xrightarrow{c_{n+1,X}} & H^*_{n+1}(\Delta_{n+1,X}) & \xrightarrow{r_{n+1,X}} & H^*_n(\Delta_{0,X}) \\
\big\downarrow{\scriptstyle H^*_n(\partial f)} & & \big\downarrow{\scriptstyle H^*_{n+1}(f)} & & \big\downarrow{\scriptstyle H^*_n((\varphi_{f_2})_{\Delta_0})} \\
H^*_n(\partial\Delta_{n+1,Y}) & \xrightarrow[c_{n+1,Y}]{} & H^*_{n+1}(\Delta_{n+1,Y}) & \xrightarrow[r_{n+1,Y}]{} & H^*_n(\Delta_{0,Y})
\end{array}$$

(b) Since t factorizes through $i_{n+1,Y}$

$$t : \Delta_{m,X} \xrightarrow{t'} \partial\Delta_{n+1,Y} \xrightarrow{i_{n+1,Y}} \Delta_{n+1,Y}$$

we put $H^*_{n+1}(t) = c_{n+1,Y} \cdot H^*_n(t')$.

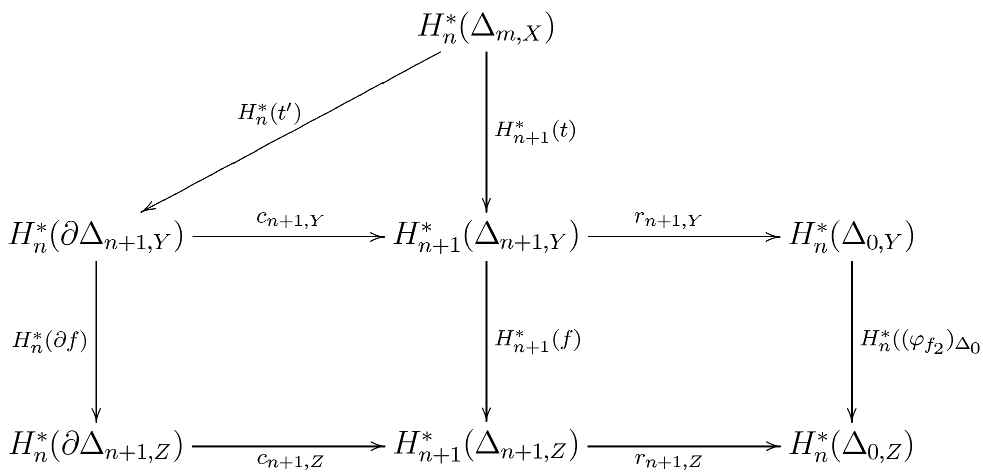
(c) To define $H^*_{n+1}(u)$ for each u, it suffices to do this for the retraction u^0 of $\Delta_{n+1,X}$ to one of its face $\Delta_{n,X}$. In this case, we take $H^*_{n+1}(u^0)$ given by the lifting property

$$\begin{array}{ccc}
H^*_n(\partial\Delta_{n+1,X}) & \xrightarrow{c_{n+1,X}} & H^*_{n+1}(\Delta_{n+1,X}) \\
\big\downarrow{\scriptstyle H^*_n(u^0 \cdot i_{n+1,X})} & \swarrow{\scriptstyle H^*_{n+1}(u^0)} & \big\downarrow{\scriptstyle r_{n+1,X}} \\
H^*_n(\Delta_{n,X}) & \xrightarrow[H^*_n(s_{n,X})]{} & H^*_n(\Delta_{0,X})
\end{array}$$

where $s_{n,X} = \overline{Y}(s, \mathrm{id}_X)$ and $s : n+1 \to 1$.

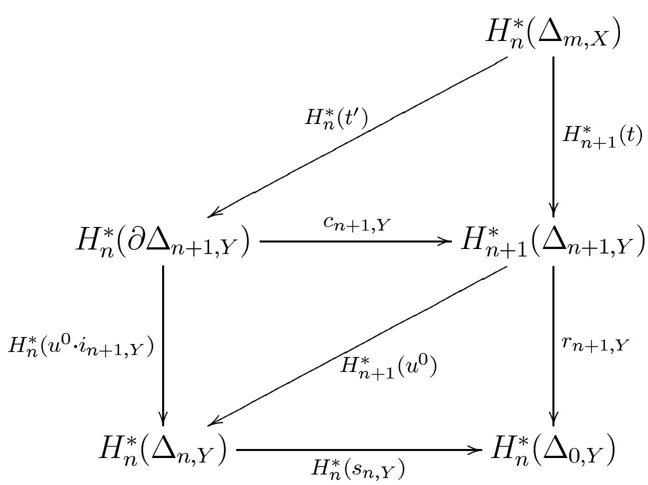
To prove that H^*_{n+1} is a functor, it suffices to consider the following cases:

(1) In



we have $(ft)' = \partial f \cdot t'$ and thus $H_{n+1}^*(f) \cdot H_{n+1}^*(t) = H_{n+1}^*(ft)$.

(2) In



we have $u^0 \cdot t = u^0 \cdot i_{n+1,Y} \cdot t'$ and thus $H_{n+1}^*(u^0) \cdot H_{n+1}^*(t) = H_{n+1}^*(u^0 \cdot t)$.

We have defined H_{n+1}^* on the image of the Yoneda embedding

$$\overline{Y}_n : \mathbf{F}_n \times \mathcal{X} \to \mathbf{Set}^{(\mathbf{F}_n \times \mathcal{X})^{op}} \cong \mathbf{SSimp}_n^{\mathcal{X}^{op}}.$$

Since this is a free cocompletion of $\mathbf{F}_n \times \mathcal{X}$, we obtain a colimit preserving functor $H_{n+1}^* : \mathbf{SSimp}_n^{\mathcal{X}^{op}} \to \mathcal{K}$ extending H_n^*.

We have constructed an increasing chain of colimit preserving functors $H_n^* : \mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$ for $n = 0, 1, \ldots$, which yields a colimit preserving functor

$$H^* : \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$$

with the restriction H_n^* on $\mathbf{SSimp}_n^{\mathcal{X}^{\mathrm{op}}}$. Moreover, H^* is a left adjoint functor (see (a) in the proof of 1.45 in [AR]). In particular, $H^*Y^* = H_0^*Y^* = H$. It remains to be proved that H^* preserves cofibrations and trivial cofibrations. Since, $\partial\Delta_{n+1,X}$

is a colimit of $\partial_{m,X}$, $m \leq n$, by (b) of the construction we get that $H^*(i_{n+1,X}) = c_{n+1,X}$. Hence H^* preserves cofibrations (using Lemma 3.2 and the fact that H^* preserves colimits). Since $s_{n+1,X} = s_{n,X} \cdot u^0$, by (c) of the construction we get that $H^*(s_{n+1,X}) = r_{n+1,X}$. Hence $H^*(s_{n+1,X})$ is a weak equivalence in $\mathcal{K}$ for $n > 0$ and $X \in \mathcal{X}$. Since $s_{m,X} \cdot f = s_{n,Y}$ for each morphism $f : \Delta_{n,X} \to \Delta_{m,Y}$, $H^*(f)$ is a weak equivalence in $\mathcal{K}$ as well. In particular, $H^*F_X(j_1) : H^*F_X(\Delta_1^0) = H^*\Delta_{0,X} \to H^*\Delta_{1,X}$ is a weak equivalence. Since it is also a cofibration, it is a trivial cofibration. Assume that $H^*F_X(j_1), \dots, H^*F_X(j_n)$ are trivial cofibrations in $\mathcal{K}$. In the notation of Lemma 3.3, we get that $H^*F_X(g_n)$ is a trivial cofibration in $\mathcal{K}$ because $u_1^0 = j_1$. Since by assumption $H^*F_X(j_1), \dots, H^*F_X(j_n)$ are trivial cofibrations, $H^*F_X(q_n)$ is a trivial cofibration too. Therefore $H^*F_X(t_n)$ is a weak equivalence in $\mathcal{K}$ and thus $H^*F_X(j_{n+1})$ is a weak equivalence too. Since it is a cofibration, it is a trivial cofibration. □

Remark 4.2. Let $\mathcal{K}$ be a functorial model category, $\mathcal{X}$ a small category and $H : \mathcal{X} \to \mathcal{K}$ an arbitrary functor. Then there is a left Quillen functor $H^* : \mathbf{SSimp}^{\mathcal{X}^{\mathrm{op}}} \to \mathcal{K}$ and a natural transformation $\gamma : H^*Y^* \to H$ which is a pointwise trivial fibration in $\mathcal{K}$.

In fact, let $H_0 : \mathcal{X} \to \mathcal{K}$ be the composition $Q \cdot H$ where $Q : \mathcal{K} \to \mathcal{K}$ is the (functorial) cofibrant replacement functor and $\gamma = qH : H_0 \to H$ the corresponding natural transformation. Then γ is a pointwise trivial fibration in $\mathcal{K}$ (because $q : Q \to Id_{\mathcal{K}}$ is). If we start the construction of the functor H^* with H_0 (i.e., $H_0^* \cdot Y^* = H_0$), we get the result.

Corollary 4.3. *Let $\mathcal{K}$ be a functorial model category and K an object in $\mathcal{K}$. Then there is a left Quillen functor H^* :* **SSimp** $\to \mathcal{K}$ *and a trivial fibration $\gamma : H^*(\Delta_0) \to K$.*

This is a special case of Theorem 4.1 (for $\mathcal{X}$ a one-morphism category).

Remark 4.4. Again if K is cofibrant then $H^*(\Delta_0) = K$.

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