LAX DISTRIBUTIVE LAWS FOR TOPOLOGY, II

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Abstract. For a small quantaloid $\mathcal{Q}$ we consider four fundamental 2-monads $\mathbb{T}$ on $\mathcal{Q}$-**Cat**, given by the presheaf 2-monad $\mathbb{P}$ and the copresheaf 2-monad $\mathbb{P}^{\dagger}$, as well as their two composite 2-monads, and establish that they all laxly distribute over $\mathbb{P}$. These four 2-monads therefore admit lax extensions to the category $\mathcal{Q}$-**Dist** of $\mathcal{Q}$-categories and their distributors. We characterize the corresponding $(\mathbb{T},\mathcal{Q})$-categories in each of the four cases, leading us to both known and novel categorical structures.

1. Introduction

The syntax used in *Monoidal Topology* [6] is given by a quantale V, a **Set**-monad $\mathbb{T}$ and, most importantly, by a lax extension of $\mathbb{T}$ to the 2-category V-**Rel** of sets and V-valued relations, or, equivalently, by a lax distributive law of $\mathbb{T}$ over the *discrete* V-presheaf monad $\mathbb{P}_{\mathsf{V}}$, the Kleisli category of which is exactly V-**Rel**. Once equipped with such a lax extension or lax distributive law, the monad $\mathbb{T}$ may then be naturally extended to become a 2-monad on the 2-category V-**Cat**. This lax monad extension from **Set** to V-**Cat** facilitates the study of greatly enriched structures. For example, for V the two-element chain and $\mathbb{T}$ the ultrafilter monad, while the Eilenberg-Moore category over **Set** is **CompHaus**, over V-**Cat** one obtains *ordered* compact Hausdorff spaces, and when V is Lawvere's [12] extended half-line $[0,\infty]$, *metric* compact Hausdorff spaces; see [14, 29, 6]. Moreover, the functorial interaction between the Eilenberg-Moore category $(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$ and the category $(\mathbb{T},\mathsf{V})$-**Cat** of $(\mathbb{T},\mathsf{V})$-categories is a pivotal step for a serious study of *representability*, a powerful property which, in the basic example of the two-chain and the ultrafilter monad, entails core-compactness, or exponentiability, of topological spaces; see [3] and [6, Section III.5].

While this mechanism for generating a 2-monad on V-**Cat** from a **Set**-monad provides an indispensable tool in monoidal topology, the question arises whether it is possible to make a given 2-monad $\mathbb{T}$ on V-**Cat** the starting point of a satisfactory theory, preferably even in the more general context of a small quantaloid $\mathcal{Q}$, (*i.e.*, a **Sup**-enriched category), rather than just a quantale V (*i.e.*, a **Sup**-enriched monoid), a context that has been

Partial financial assistance by National Natural Science Foundation of China (11101297), International Visiting Program for Excellent Young Scholars of Sichuan University and the Natural Sciences and Engineering Research Council (NSERC) of Canada is gratefully acknowledged. This work was completed while the first author held a Visiting Professorship at York University.

2010 Mathematics Subject Classification: 18C15, 18C20, 18D99.

Key words and phrases: quantaloid, monad, presheaf monad, copresheaf monad, double presheaf monad, double copresheaf monad, lax distributive law, lax λ-algebra, lax monad extension, $\mathcal{Q}$-closure space, $\mathcal{Q}$-interior space.

propagated in this paper's predecessor [30]. Such theory should, as a first step, entail the study of lax extensions of $\mathbb{T}$ to the 2-category $\mathcal{Q}$-**Dist** of $\mathcal{Q}$-categories and their distributors (or (bi)modules), rather than just to $\mathcal{Q}$-**Rel**, or, equivalently, the study of lax distributive laws of $\mathbb{T}$ over the *non-discrete* presheaf monad $\mathbb{P}_{\mathcal{Q}}$, rather than over its discrete counterpart. The fact that the non-discrete presheaf monad is, other than its discrete version, lax idempotent (*i.e.*, of Kock-Zöberlein type [32, 10]), serves as a first indicator that this approach should in fact lead to a categorically more satisfactory theory.

This paper makes the case for an affirmative answer to the question raised, even in the extended context of a given small quantaloid $\mathcal{Q}$. It is centred around four naturally arising monads $\mathbb{T}$ on $\mathcal{Q}$-**Cat** which do not come about via the mechanism described above, but should nevertheless be of considerable general interest. They all distribute laxly over $\mathbb{P} = \mathbb{P}_{\mathcal{Q}}$ and, hence, are laxly extendable to $\mathcal{Q}$-**Dist**, and we give a detailed description of the respective lax algebras, or $(\mathbb{T}, \mathcal{Q})$-categories, arising. These monads are

- the presheaf 2-monad $\mathbb{P}$ itself (Section 4);
- the copresheaf 2-monad $\mathbb{P}^{\dagger}$ (Section 5);
- the double presheaf 2-monad $\mathbb{P}\mathbb{P}^{\dagger}$ (Section 6);
- the double copresheaf 2-monad $\mathbb{P}^{\dagger}\mathbb{P}$ (Section 7).

In each of the four cases, the establishment of the needed lax distributive law over $\mathbb{P}$ and the characterization of the corresponding lax algebras, or, equivalently, $(\mathbb{T}, \mathcal{Q})$-categories, takes considerable "technical" effort, especially in the absence of any noticeable formal resemblance between the four cases. However, the lax algebras pertaining to both, $\mathbb{P}$ and $\mathbb{P}^{\dagger}\mathbb{P}$, are identified as $\mathcal{Q}$-*closure spaces*, as considered in [21, 23]. Most challenging has been the identification of the lax algebras pertaining to $\mathbb{P}\mathbb{P}^{\dagger}$, which we describe as $\mathcal{Q}$-*interior spaces*, a structure considered here for the first time. Also the lax algebras pertaining to $\mathbb{P}^{\dagger}$ are of a novel flavour; they are monoid objects in $\mathcal{Q}$-**Dist**. Given that their discrete cousins, *i.e.*, the monoid objects in $\mathcal{Q}$-**Rel**, are $\mathcal{Q}$-categories, they surely deserve further study.

We believe that we have given sufficiently many details in order to make the proofs verifiable for the reader, also since all needed basic tools are comprehensively listed in Section 2. In contrast to Sections 4–7, the introduction of lax distributive laws of a 2-monad over the non-discrete presheaf monad and of their lax algebras (as given in Section 3), as well as the proof of the fact that they correspond bijectively to lax extensions of $\mathbb{T}$ to $\mathcal{Q}$-**Dist**, with lax algebras corresponding to $(\mathbb{T}, \mathcal{Q})$-categories (as given in Section 8), are straightforward extensions of their "discrete" treatment in [30] and should therefore constitute a relatively easy read. We have nevertheless given complete proofs, so that prior reading of [30] is not required for the purpose of understanding this paper.

2. Quantaloid-enriched categories and their distributors

A *quantaloid* [18] is a category enriched in the monoidal-closed category **Sup** [9] of complete lattices and sup-preserving maps. Explicitly, a quantaloid $\mathcal{Q}$ is a 2-category with its 2-cells given by an order "$\preceq$", such that each hom-set $\mathcal{Q}(r,s)$ is a complete lattice and the composition of morphisms from either side preserves arbitrary suprema. Hence, $\mathcal{Q}$ has "internal homs", denoted by $\swarrow$ and $\searrow$, as the right adjoints of the composition functors:

$$-\circ u \dashv - \swarrow u : \mathcal{Q}(r,t) \longrightarrow \mathcal{Q}(s,t) \quad \text{and} \quad v \circ - \dashv v \searrow - : \mathcal{Q}(r,t) \longrightarrow \mathcal{Q}(r,s);$$

explicitly,

$$u \preceq v \searrow w \iff v \circ u \preceq w \iff v \preceq w \swarrow u$$

for all morphisms $u : r \longrightarrow s$, $v : s \longrightarrow t$, $w : r \longrightarrow t$ in $\mathcal{Q}$.

Throughout this paper, we let $\mathcal{Q}$ be a *small* quantaloid. From $\mathcal{Q}$ one forms a new (large) quantaloid $\mathcal{Q}$-**Rel** of $\mathcal{Q}$-*relations* with the following data: its objects are those of $\mathbf{Set}/\mathcal{Q}_0$ (with $\mathcal{Q}_0 := \operatorname{ob}\mathcal{Q}$), *i.e.*, sets X equipped with an *array* (or *type*) map $|\text{-}| : X \longrightarrow \mathcal{Q}_0$, and a morphism $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathcal{Q}$-**Rel** is a map that assigns to every pair $x \in X$, $y \in Y$ a morphism $\varphi(x,y) : |x| \longrightarrow |y|$ in $\mathcal{Q}$; its composite with $\psi : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is defined by

$$(\psi \circ \varphi)(x,z) = \bigvee_{y \in Y} \psi(y,z) \circ \varphi(x,y),$$

and $1_X^{\circ} : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ with

$$1_X^{\circ}(x,y) = \begin{cases} 1_{|x|} & \text{if } x = y, \\ \bot & \text{else} \end{cases}$$

serves as the identity morphism on X. As $\mathcal{Q}$-relations are equipped with the pointwise order inherited from $\mathcal{Q}$, internal homs in $\mathcal{Q}$-**Rel** are computed pointwise as

$$(\theta \swarrow \varphi)(y,z) = \bigwedge_{x \in X} \theta(x,z) \swarrow \varphi(x,y) \quad \text{and} \quad (\psi \searrow \theta)(x,y) = \bigwedge_{z \in Z} \psi(y,z) \searrow \theta(x,z)$$

for all $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $\psi : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $\theta : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$.

A (small) $\mathcal{Q}$-*category* is precisely an (internal) monad in the 2-category $\mathcal{Q}$-**Rel**; or equivalently, a monoid in the monoidal-closed category $(\mathcal{Q}\text{-}\mathbf{Rel}(X,X), \circ)$, for some X over $\mathcal{Q}_0$. Explicitly, a $\mathcal{Q}$-category consists of an object X in $\mathbf{Set}/\mathcal{Q}_0$ and a $\mathcal{Q}$-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ (its "hom"), such that $1_X^{\circ} \preceq a$ and $a \circ a \preceq a$. For every $\mathcal{Q}$-category (X,a), the underlying (pre)order on X is given by

$$x \leqslant x' \iff |x| = |x'| \text{ and } 1_{|x|} \preceq a(x,x'),$$

and we write $x \cong x'$ if $x \leqslant x'$ and $x' \leqslant x$.

A map $f : (X,a) \longrightarrow (Y,b)$ between $\mathcal{Q}$-categories is a $\mathcal{Q}$-*functor* (resp. *fully faithful $\mathcal{Q}$-functor*) if it lives in $\mathbf{Set}/\mathcal{Q}_0$ and satisfies $a(x,x') \preceq b(fx,fx')$ (resp. $a(x,x') = b(fx,fx')$) for all $x, x' \in X$. With the pointwise order of $\mathcal{Q}$-functors inherited from Y, *i.e.*,

$$f \leqslant g : (X,a) \longrightarrow (Y,b) \iff \forall x \in X : \ fx \leqslant gx \iff \forall x \in X : \ 1_{|x|} \preceq b(fx,gx),$$

$\mathcal{Q}$-categories and $\mathcal{Q}$-functors are organized into a 2-category $\mathcal{Q}$-**Cat**.

The one-object quantaloids are the (unital) *quantales* (see [17]); equivalently, a quantale is a complete lattice V with a monoid structure whose binary operation $\otimes$ preserves suprema in each variable. The $\otimes$-neutral element is generally denoted by k; so $\mathsf{k} = 1_*$ if we denote by $*$ the only object of the monoid V, considered as a category.

2.1. Example.

(1) The initial quantale is the two-element chain $\mathsf{2} = \{\bot < \top\}$, with $\otimes = \wedge$, $\mathsf{k} = \top$, and **2-Cat** is the category **Ord** of preordered sets and monotone maps.

(2) The extended real line $[0,\infty]$, ordered by the natural $\geq$, is a quantale with $\otimes = +$, naturally extended to ∞ (see [12]). We write $\mathbf{Met} = [0,\infty]$-**Cat** for the resulting category of (generalized) metric spaces and non-expansive maps.

(3) Every *frame* may be considered as a quantale. In fact, these are precisely the commutative quantales in which every element is idempotent. For example, $([0,\infty],\geq)$ may be considered as a quantale $[0,\infty]_{\max}$ with $\alpha\otimes\beta = \max\{\alpha,\beta\}$. The resulting category $[0,\infty]_{\max}$-**Cat** is the category **UMet** of (generalized) *ultrametric* spaces (see [19]).

(4) From a small site $(\mathcal{C},\mathcal{F})$ one can construct a small quantaloid $\mathcal{R} := \mathcal{R}(\mathcal{C},\mathcal{F})$ (see [31]), defined by the following data:

- objects: the objects of $\mathcal{C}$;
- morphisms: for objects u, v in $\mathcal{C}$, an arrow from u to v is a closed subfunctor $\alpha \subseteq \hat{u}\times\hat{v}$ with respect to the coverage $\mathcal{F}$ in $[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]$, where $\hat{u}$ and $\hat{v}$ are the representable presheaves $\mathcal{C}(-,u)$ and $\mathcal{C}(-,v)$, respectively;
- composition: $\beta\bullet\alpha = \overline{\beta\circ\alpha}$, with $\circ$ denoting the composition of relations in the topos $[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]$ and $\overline{(-)}$ the closure with respect to the coverage $\mathcal{F}$.

It is shown in [4, 5] that Cauchy complete $\mathcal{R}$-categories are equivalent to internal ordered objects in the category $\mathbf{Sh}(\mathcal{C},\mathcal{F})$ of sheaves on the site $(\mathcal{C},\mathcal{F})$.

Recall that a quantale V is called *divisible* (see [7]) if for all $u \preceq v$ in V, there are $w, w' \in \mathsf{V}$ such that $u = v\otimes w = w'\otimes v$, or equivalently, $u = v\otimes(v\searrow u) = (u\swarrow v)\otimes v$. A divisible quantale V, since $\mathsf{k}\preceq\top$ guarantees the existence of some $w\in\mathsf{V}$ with $\top = \mathsf{k}\otimes\top = w\otimes\top\otimes\top = w\otimes\top = \mathsf{k}$, must be *integral*, *i.e.*, $\mathsf{k} = \top$.

Every quantaloid $\mathcal{Q}$ gives rise to the quantaloid $\mathsf{D}\mathcal{Q}$ of "diagonals of $\mathcal{Q}$" (see [26]), which has an easy description when the quantaloid is a divisible quantale V (see [8, 15]): the objects of the quantaloid DV are the elements of V, and a morphism d from u to v is an element in V with $d \preceq u\wedge v$, we write $d : u \rightsquigarrow v$ in this case. The composition of d with $e : v \rightsquigarrow w$ in DV is defined by $e\circ d = e\otimes(v\searrow d) = (e\swarrow v)\otimes d$ in V. The order of the hom-sets of DV is inherited from V.

Given a DV-category (X, a), since in the quantaloid DV one has $1_{|x|} = |x| = a(x, x)$, the conditions on the DV-category structure a, given by a map $X \times X \longrightarrow \mathsf{V}$, may be reformulated as

$$a(x, y) \preceq a(x, x) \wedge a(y, y) \quad \text{and} \quad a(y, z) \otimes (a(y, y) \searrow a(x, y)) \preceq a(x, z),$$

for all $x, y, z \in X$.

There are lax homomorphisms, called *forward* and *backward globalization* functors (see [15, 28]),

$$\gamma : \mathsf{DV} \longrightarrow \mathsf{V}, (d : u \rightsquigarrow v) \mapsto d \swarrow u,$$
$$\delta : \mathsf{DV} \longrightarrow \mathsf{V}, (d : u \rightsquigarrow v) \mapsto v \searrow d,$$

which induce two functors from DV-$\mathbf{Cat}$ to V-$\mathbf{Cat}$.

When one considers V as a V-category (V, h) with $h(u, v) = v \swarrow u$, there is a full reflective embedding

$$\mathsf{E}_\gamma : \mathsf{DV}\text{-}\mathbf{Cat} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat}/\mathsf{V}.$$

Indeed, given a DV-category (X, a), the V-relation d, defined with the forward globalization functor by

$$\forall x, y \in X : \; d(x, y) = a(x, y) \swarrow a(x, x),$$

makes X a V-category over V, via the V-functor $t : (X, d) \longrightarrow (\mathsf{V}, h)$ with $tx = a(x, x)$ for all $x \in X$. Conversely, for a V-category (X, d) equipped with a V-functor $t : (X, d) \longrightarrow (\mathsf{V}, h)$, define

$$\forall x, y \in X : \; a(x, y) = d(x, y) \otimes tx.$$

To see that (X, a) is indeed a DV-category with array map $t : X \longrightarrow \mathsf{V}$, let us first note that, since $t : (X, d) \longrightarrow (\mathsf{V}, h)$ is a V-functor, $d(x, y) \preceq ty \swarrow tx$, so that $d(x, y) \otimes tx \preceq ty$ and then $a(x, y) = d(x, y) \otimes tx \preceq tx \wedge ty$ follows, for all $x, y \in X$. Secondly, for all $x, y, z \in X$,

$$\begin{aligned} a(y, z) \circ a(x, y) &= d(y, z) \otimes ty \otimes (ty \searrow (d(x, y) \otimes tx)) \\ &= d(y, z) \otimes d(x, y) \otimes tx \preceq d(x, z) \otimes tx = a(x, z). \end{aligned}$$

Thus, (X, a) is a DV-category, as desired.

Let $\mathsf{V}^\star$ be the V-category with underlying set V and V-category structure

$$h^\star(u, v) = v \searrow u.$$

Of course, when V is commutative, $\mathsf{V}^\star$ is the dual of V. One obtains another full reflective embedding

$$\mathsf{E}_\delta : \mathsf{DV}\text{-}\mathbf{Cat} \longrightarrow \mathsf{V}\text{-}\mathbf{Cat}/\mathsf{V}^\star,$$

as follows. Given a DV-category (X, a), the V-relation d, defined by the backward globalization functor,

$$\forall x, y \in X, \; d(x, y) = a(y, y) \searrow a(x, y),$$

makes X a V-category over $\mathsf{V}^\star$, via the V-functor $t : (X, d) \longrightarrow (\mathsf{V}, h^\star)$ with $ty = a(y, y)$ for all $y \in X$.

2.2. Example. A $\mathsf{D}[0,\infty]$-category (X,a) is exactly a (generalized) partial metric space (see [13, 2, 8, 15]). The category structure a is a map $a : X\times X\longrightarrow[0,\infty]$ that must satisfy

(1) $\max\{a(x,x),a(y,y)\}\leq a(x,y)$ for all $x,y\in X$,

(2) $a(x,z)\leq a(x,y)-a(y,y)+a(y,z)$ for all $x,y,z\in X$.

A $\mathsf{D}[0,\infty]$-functor $f:(X,a)\longrightarrow(Y,b)$ is a map $f:X\longrightarrow Y$ such that

(3) $b(fx,fy)\leq a(x,y)$ for all $x,y\in X$,

(4) $b(fx,fx)=a(x,x)$ for all $x\in X$.

We write **ParMet** for the category of partial metric spaces. For $\mathsf{V}=[0,\infty]$, both

$$\mathsf{E}_\gamma:\mathbf{ParMet}\longrightarrow\mathbf{Met}/[0,\infty]\quad\text{and}\quad\mathsf{E}_\delta:\mathbf{ParMet}\longrightarrow\mathbf{Met}/[0,\infty]^\star$$

give isomorphisms of categories.

A $\mathcal{Q}$-relation $\varphi:X\longrightarrow\!\!\!\!\!\!\mapsto Y$ becomes a $\mathcal{Q}$-*distributor* $\varphi:(X,a){\circ\!\!\!\!\longrightarrow}(Y,b)$ if it is compatible with the $\mathcal{Q}$-categorical structures a and b; that is,

$$b\circ\varphi\circ a\preceq\varphi.$$

$\mathcal{Q}$-categories and $\mathcal{Q}$-distributors constitute a quantaloid $\mathcal{Q}$-**Dist** that contains $\mathcal{Q}$-**Rel** as a full subquantaloid, in which the composition and internal homs are calculated in the same way as those of $\mathcal{Q}$-relations; the identity $\mathcal{Q}$-distributor on (X,a) is given by its hom $a:(X,a){\circ\!\!\!\!\longrightarrow}(X,a)$.

Each $\mathcal{Q}$-functor $f:(X,a)\longrightarrow(Y,b)$ induces an adjunction $f_*\dashv f^*$ in $\mathcal{Q}$-**Dist**, given by

$$\begin{array}{ll} f_*:(X,a){\circ\!\!\!\!\longrightarrow}(Y,b), & f_*(x,y)=b(fx,y)\quad\text{and}\\ f^*:(Y,b){\circ\!\!\!\!\longrightarrow}(X,a), & f^*(y,x)=b(y,fx),\end{array}\tag{2.i}$$

and called the *graph* and *cograph* of f, respectively. Obviously, $a=(1_X)_*=1_X^*$ for any $\mathcal{Q}$-category (X,a); hence, $a=1_X^*$ will be our standard notation for identity morphisms in $\mathcal{Q}$-**Dist**.

2.3. Lemma. [20, 23] *Let $f:X\longrightarrow Y$ be a $\mathcal{Q}$-functor.*

(1) *f is fully faithful if, and only if, $f^*\circ f_*=1_X^*$.*

(2) *If f is essentially surjective, in the sense that, for any $y\in Y$, there exists $x\in X$ with $y\cong fx$, then $f_*\circ f^*=1_Y^*$.*

For an object s in $\mathcal{Q}$, and with $\{s\}$ denoting the singleton $\mathcal{Q}$-category, the only object of which has array s and hom 1_s, $\mathcal{Q}$-distributors of the form $\sigma:X{\circ\!\!\!\!\longrightarrow}\{s\}$ are called *presheaves* on X and constitute a $\mathcal{Q}$-category $\mathsf{P}X$, with $1_{\mathsf{P}X}^*(\sigma,\sigma')=\sigma'\swarrow\sigma$. Dually, the *copresheaf* $\mathcal{Q}$-category $\mathsf{P}^\dagger X$ consists of $\mathcal{Q}$-distributors $\tau:\{s\}{\circ\!\!\!\!\longrightarrow}X$ with $1_{\mathsf{P}^\dagger X}^*(\tau,\tau')=\tau'\searrow\tau$.

2.4. Remark. For any $\mathcal{Q}$-category X, it follows from the definition that the underlying order on $\mathsf{P}^\dagger X$ is the *reverse* local order in $\mathcal{Q}$-**Dist**, *i.e.*,

$$\tau \leqslant \tau' \text{ in } \mathsf{P}^\dagger X \iff \tau' \preceq \tau \text{ in } \mathcal{Q}\text{-}\mathbf{Dist}.$$

That is why we use a different symbol "$\leqslant$" for the underlying order of $\mathcal{Q}$-categories and the 2-cells in $\mathcal{Q}$-**Cat**, while the 2-cells in $\mathcal{Q}$ and $\mathcal{Q}$-**Dist** are denoted by "$\preceq$".

A $\mathcal{Q}$-category X is *complete* if the *Yoneda embedding*

$$\mathsf{y}_X : X \longrightarrow \mathsf{P}X, \quad x \mapsto 1_X^*(-, x)$$

has a left adjoint $\sup_X : \mathsf{P}X \longrightarrow X$ in $\mathcal{Q}$-**Cat**; that is,

$$1_X^*(\sup_X \sigma, -) = 1_{\mathsf{P}X}^*(\sigma, \mathsf{y}_X -) = 1_X^* \swarrow \sigma$$

for all $\sigma \in \mathsf{P}X$. It is well known that X is a complete $\mathcal{Q}$-category if, and only if, $X^{\mathrm{op}} := (X, (1_X^*)^{\mathrm{op}})$ with $(1_X^*)^{\mathrm{op}}(x, x') = 1_X^*(x', x)$ is a complete $\mathcal{Q}^{\mathrm{op}}$-category (see [24]), where the completeness of X^{op} may be translated as the *co-Yoneda embedding*

$$\mathsf{y}_X^\dagger : X \longrightarrow \mathsf{P}^\dagger X, \quad x \mapsto 1_X^*(x, -)$$

admitting a right adjoint $\inf_X : \mathsf{P}^\dagger X \longrightarrow X$ in $\mathcal{Q}$-**Cat**.

2.5. Lemma. [23, 24] *Let X be a $\mathcal{Q}$-category.*

(1) *(Yoneda Lemma) For all $\sigma \in \mathsf{P}X$, $\tau \in \mathsf{P}^\dagger X$,*

$$\sigma = (\mathsf{y}_X)_*(-, \sigma) = 1_{\mathsf{P}X}^*(\mathsf{y}_X -, \sigma) \quad \text{and} \quad \tau = (\mathsf{y}_X^\dagger)^*(\tau, -) = 1_{\mathsf{P}^\dagger X}^*(\tau, \mathsf{y}_X^\dagger -).$$

In particular, both $\mathsf{y}_X : X \longrightarrow \mathsf{P}X$ and $\mathsf{y}_X^\dagger : X \longrightarrow \mathsf{P}^\dagger X$ are fully faithful.

(2) $\sup_X \cdot \mathsf{y}_X \cong 1_X$, $\inf_X \cdot \mathsf{y}_X^\dagger \cong 1_X$.

(3) *Both $\mathsf{P}X$ and $\mathsf{P}^\dagger X$ are separated*[1] *and complete, with*

$$\sup_{\mathsf{P}X} \sigma = \sigma \circ (\mathsf{y}_X)_* \quad \text{and} \quad \inf_{\mathsf{P}^\dagger X} \tau = (\mathsf{y}_X^\dagger)^* \circ \tau,$$

for all $\sigma \in \mathsf{PP}X$, $\tau \in \mathsf{P}^\dagger\mathsf{P}^\dagger X$.

Each $\mathcal{Q}$-distributor $\varphi : X \xrightarrow{\circ} Y$ induces *Kan adjunctions* [23] in $\mathcal{Q}$-**Cat** given by

$$\begin{array}{c} \mathsf{P}Y \underset{\varphi_\odot}{\overset{\varphi^\odot}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{P}X \\ \varphi^\odot \tau = \tau \circ \varphi, \quad \varphi_\odot \sigma = \sigma \swarrow \varphi \end{array} \quad \text{and} \quad \begin{array}{c} \mathsf{P}^\dagger Y \underset{\varphi^\oplus}{\overset{\varphi_\oplus}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{P}^\dagger X \\ \varphi_\oplus \tau = \varphi \searrow \tau, \quad \varphi^\oplus \sigma = \varphi \circ \sigma. \end{array} \tag{2.ii}$$

[1] A $\mathcal{Q}$-category X is *separated* if $x \cong x'$ implies $x = x'$ for all $x, x' \in X$.

Moreover, all the assignments in (2.i) and (2.ii) are 2-functorial, and one has two pairs of adjoint 2-functors [4] described by

$$\frac{X \overset{\varphi}{-\!\circ\!\!\rightarrow} Y}{Y \xrightarrow{\overleftarrow{\varphi}} \mathsf{P}X} \qquad \overleftarrow{\varphi} y = \varphi(-,y) \qquad \mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathsf{P}}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}}, \quad (\varphi^{\odot} : \mathsf{P}Y \longrightarrow \mathsf{P}X) \leftarrow\!\shortmid (\varphi : X \to\!\!\!\!\!\circ\; Y)$$

$$\frac{X \overset{\varphi}{-\!\circ\!\!\rightarrow} Y}{X \xrightarrow{\overrightarrow{\varphi}} \mathsf{P}^{\dagger}Y} \qquad \overrightarrow{\varphi} x = \varphi(x,-) \qquad \mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathsf{P}^{\dagger}}{\overset{(-)_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{co}}, \quad (\varphi^{\oplus} : \mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}^{\dagger}Y) \leftarrow\!\shortmid (\varphi : X \to\!\!\!\!\!\circ\; Y) \tag{2.iii}$$

where "co" refers to the dualization of 2-cells. The unit y and the counit ε of the adjunction $(-)^* \dashv \mathsf{P}$ are respectively given by the Yoneda embeddings and their graphs:

$$\varepsilon_X := (\mathsf{y}_X)_* : X \to\!\!\!\!\!\circ\; \mathsf{P}X.$$

The *presheaf 2-monad* $\mathbb{P} = (\mathsf{P}, \mathsf{s}, \mathsf{y})$ on $\mathcal{Q}$-**Cat** induced by $(-)^* \dashv \mathsf{P}$ sends each $\mathcal{Q}$-functor $f : X \longrightarrow Y$ to

$$f_! := (f^*)^{\odot} : \mathsf{P}X \longrightarrow \mathsf{P}Y,$$

which admits a right adjoint $f^! := (f^*)_{\odot} = (f_*)^{\odot} : \mathsf{P}Y \longrightarrow \mathsf{P}X$ in $\mathcal{Q}$-**Cat**; the monad multiplication s is given by

$$\mathsf{s}_X = \varepsilon_X^{\odot} = \sup\nolimits_{\mathsf{P}X} = \mathsf{y}_X^! : \mathsf{PP}X \longrightarrow \mathsf{P}X, \tag{2.iv}$$

where $\sup_{\mathsf{P}X} = \mathsf{y}_X^!$ is an immediate consequence of Lemma 2.5. Similarly, the unit $\mathsf{y}^{\dagger}$ is given by the co-Yoneda embeddings, and $\varepsilon^{\dagger} := (\mathsf{y}_{\Box}^{\dagger})^*$ is the counit of the adjunction $(-)_* \dashv \mathsf{P}^{\dagger}$. The induced *copresheaf 2-monad* $\mathbb{P}^{\dagger} = (\mathsf{P}^{\dagger}, \mathsf{s}^{\dagger}, \mathsf{y}^{\dagger})$ on $\mathcal{Q}$-**Cat** sends f to

$$f_{¡} := (f_*)^{\oplus} : \mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}^{\dagger}Y,$$

which admits a left adjoint $f^{¡} := (f^*)^{\oplus} = (f_*)_{\oplus} : \mathsf{P}^{\dagger}Y \longrightarrow \mathsf{P}^{\dagger}X$ in $\mathcal{Q}$-**Cat**, and the monad multiplication is given by

$$\mathsf{s}_X^{\dagger} = (\varepsilon_X^{\dagger})^{\oplus} = \inf\nolimits_{\mathsf{P}^{\dagger}X} = (\mathsf{y}_X^{\dagger})^{¡} : \mathsf{P}^{\dagger}\mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}^{\dagger}X. \tag{2.v}$$

2.6. Lemma. *Let $f : X \longrightarrow Y$ be a $\mathcal{Q}$-functor.*

(1) *f is fully faithful $\iff$ $f^! \cdot f_! = 1_{\mathsf{P}X}$ $\iff$ $f^{¡} \cdot f_{¡} = 1_{\mathsf{P}^{\dagger}X}$ $\iff$ $f_! : \mathsf{P}X \longrightarrow \mathsf{P}Y$ is fully faithful $\iff$ $f_{¡} : \mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}^{\dagger}Y$ is fully faithful.*

(2) *If f is essentially surjective, then $f_! \cdot f^! = 1_{\mathsf{P}Y}$, $f_{¡} \cdot f^{¡} = 1_{\mathsf{P}^{\dagger}Y}$ and both $f_! : \mathsf{P}X \longrightarrow \mathsf{P}Y$, $f_{¡} : \mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}^{\dagger}Y$ are surjective.*

Proof. Straightforward, with Lemma 2.3 and the definitions of $f_!$, $f^!$, $f_{¡}$, $f^{¡}$. ■

2.7. Lemma. [16, 24] *For all $\mathcal{Q}$-functors $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$,*

$$f \dashv g \iff f_* = g^* \iff f^! = g_! \iff f_¡ = g^¡ \\ \iff f_! \dashv g_! \iff f^! \dashv g^! \iff f_¡ \dashv g_¡ \iff f^¡ \dashv g^¡.$$

2.8. Lemma. *For all $\mathcal{Q}$-functors $f, g : X \longrightarrow Y$ and $\mathcal{Q}$-distributors $\varphi, \psi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,*

(1) $f \leqslant g \iff f_* \succeq g_* \iff f^* \preceq g^* \iff f_! \leqslant g_! \iff f_¡ \leqslant g_¡ \iff f^! \geqslant g^! \iff f^¡ \geqslant g^¡$.

(2) $\varphi \preceq \psi \iff \varphi^{\odot} \leqslant \psi^{\odot} \iff \varphi^{\oplus} \geqslant \psi^{\oplus} \iff \overleftarrow{\varphi} \leqslant \overleftarrow{\psi} \iff \overrightarrow{\varphi} \geqslant \overrightarrow{\psi}$.

2.9. Lemma. [21, 24] *Let $f : X \longrightarrow Y$ be a $\mathcal{Q}$-functor between complete $\mathcal{Q}$-categories. Then*

$$\sup_Y \cdot f_! \leqslant f \cdot \sup_X \quad \text{and} \quad f \cdot \inf_X \leqslant \inf_Y \cdot f_¡.$$

*Furthermore, f is a left (resp. right) adjoint in $\mathcal{Q}$-**Cat** if, and only if, $\sup_Y \cdot f_! = f \cdot \sup_X$ (resp. $f \cdot \inf_X = \inf_Y \cdot f_¡$).*

The above lemma shows that left (resp. right) adjoint $\mathcal{Q}$-functors between complete $\mathcal{Q}$-categories are exactly sup-preserving (resp. inf-preserving) $\mathcal{Q}$-functors. Thus we denote the 2-subcategory of $\mathcal{Q}$-**Cat** consisting of separated complete $\mathcal{Q}$-categories and sup-preserving (resp. inf-preserving) $\mathcal{Q}$-functors by $\mathcal{Q}$-**Sup** (resp. $\mathcal{Q}$-**Inf**).

2.10. Lemma. *The following identities hold for all $\mathcal{Q}$-distributors $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.*

(1) $\mathsf{y}_X = \overleftarrow{1_X^*}$, $\mathsf{y}_X^{\dagger} = \overrightarrow{1_X^*}$.

(2) $1_{\mathsf{P}X} = \overleftarrow{(\mathsf{y}_X)_*}$, $1_{\mathsf{P}^{\dagger}X} = \overrightarrow{(\mathsf{y}_X^{\dagger})^*}$.

(3) $\overleftarrow{\varphi} = \varphi^{\odot} \cdot \mathsf{y}_Y$, $\overrightarrow{\varphi} = \varphi^{\oplus} \cdot \mathsf{y}_X^{\dagger}$.

(4) $\varphi = \overleftarrow{\varphi}^* \circ (\mathsf{y}_X)_* = (\mathsf{y}_Y^{\dagger})^* \circ \overrightarrow{\varphi}_*$.

(5) $(\mathsf{y}_Y)_* \circ \varphi = \varphi^{\odot *} \circ (\mathsf{y}_X)_*$, $\varphi \circ (\mathsf{y}_X^{\dagger})^* = (\mathsf{y}_Y^{\dagger})^* \circ (\varphi^{\oplus})_*$.

Proof. (1), (3) are trivial, and (2), (4) are immediate consequences of the Yoneda lemma. For (5), note that the 2-functor

$$\mathsf{P} : (\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}, \quad (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y) \mapsto (\varphi^{\odot} : \mathsf{P}Y \longrightarrow \mathsf{P}X)$$

is faithful, and

$$((\mathsf{y}_Y)_* \circ \varphi)^{\odot} = \varphi^{\odot} \cdot \mathsf{y}_Y^! = \varphi^{\odot} \cdot \sup_{\mathsf{P}Y} = \sup_{\mathsf{P}X} \cdot (\varphi^{\odot})_! = \mathsf{y}_X^! \cdot \varphi^{\odot * \odot} = (\varphi^{\odot *} \circ (\mathsf{y}_X)_*)^{\odot}$$

follows by applying Lemma 2.9 to the left adjoint $\mathcal{Q}$-functor $\varphi^{\odot} : \mathsf{P}Y \longrightarrow \mathsf{P}X$. The other identity can be verified analogously. ■

2.11. Lemma. *The following identities hold for all $\mathcal{Q}$-functors $f : X \longrightarrow Y$.*

(1) $f_{¡!} = f^{¡!}$, $\quad f_{!¡} = f^{!¡}$, $\quad (f_!)^! = (f^!)_!$, $\quad (f_¡)^¡ = (f^¡)_¡$.

(2) $\overleftarrow{f_*} = f^! \cdot \mathsf{y}_Y$, $\quad \overrightarrow{f_*} = \mathsf{y}_Y^{\dagger} \cdot f = f_¡ \cdot \mathsf{y}_X^{\dagger}$.

(3) $\overrightarrow{f^*} = f^¡ \cdot \mathsf{y}_Y^{\dagger}$, $\quad \overleftarrow{f^*} = \mathsf{y}_Y \cdot f = f_! \cdot \mathsf{y}_X$.

(4) $(\mathsf{y}_X)_* \circ f^* = (f_!)^* \circ (\mathsf{y}_Y)_*$, $\quad f_! \cdot \mathsf{y}_X^! = \mathsf{y}_Y^! \cdot f_{!!}$, $\quad (\mathsf{y}_X)_¡ \cdot f^¡ = (f_!)^¡ \cdot (\mathsf{y}_Y)_¡$.

(5) $f_* \circ (\mathsf{y}_X^{\dagger})^* = (\mathsf{y}_Y^{\dagger})^* \circ (f_¡)_*$, $\quad f_¡ \cdot (\mathsf{y}_X^{\dagger})^¡ = (\mathsf{y}_Y^{\dagger})^¡ \cdot f_{¡¡}$, $\quad (\mathsf{y}_X^{\dagger})_! \cdot f^! = (f_¡)^! \cdot (\mathsf{y}_Y^{\dagger})_!$.

Proof. For (1), $f_{¡!} = f^{¡!}$ since $(f^¡)_! \dashv f^{¡!}$ and $(f^¡)_! \dashv f_{¡!}$, and the other identities can be checked similarly. The non-trivial identities in (2) and (3) follow respectively from the naturality of $\mathsf{y}^{\dagger}$ and y, while (4) and (5) are immediate consequences of Lemma 2.10(5). ∎

2.12. Lemma. *The following identities hold for all $\mathcal{Q}$-distributors $\varphi : X \to\!\!\!\!\circ\; Y$, $\psi : Y \to\!\!\!\!\circ\; Z$ and $\mathcal{Q}$-functors f whenever the operations make sense:*

(1) $\overleftarrow{\psi \circ \varphi} = \varphi^{\odot} \cdot \overleftarrow{\psi} = \mathsf{y}_X^! \cdot \overleftarrow{\varphi}_! \cdot \overleftarrow{\psi}$, $\quad \overleftarrow{\psi \circ f^*} = f_! \cdot \overleftarrow{\psi}$, $\quad \overleftarrow{f^* \circ \varphi} = \overleftarrow{\varphi} \cdot f$.

(2) $\overrightarrow{\psi \circ \varphi} = \psi^{\oplus} \cdot \overrightarrow{\varphi} = (\mathsf{y}_Z^{\dagger})^¡ \cdot \overrightarrow{\psi}_¡ \cdot \overrightarrow{\varphi}$, $\quad \overrightarrow{\psi \circ f_*} = \overrightarrow{\psi} \cdot f$, $\quad \overrightarrow{f_* \circ \varphi} = f_¡ \cdot \overrightarrow{\varphi}$.

Proof. Straightforward calculations with the help of Lemmas 2.10 and 2.11. ∎

2.13. Lemma. *For $\mathcal{Q}$-functors $f, g : \mathsf{P}X \longrightarrow Y$ (resp. $f, g : \mathsf{P}^{\dagger}X \longrightarrow Y$), if f (resp. g) is a left (resp. right) adjoint in $\mathcal{Q}$-**Cat**, then*

$$f\mathsf{y}_X \leqslant g\mathsf{y}_X \ (\textit{resp. } f\mathsf{y}_X^{\dagger} \leqslant g\mathsf{y}_X^{\dagger}) \iff f \leqslant g.$$

Proof. For the non-trivial direction, suppose that $f \dashv h : Y \longrightarrow \mathsf{P}X$, then $f\mathsf{y}_X \leqslant g\mathsf{y}_X$ implies $\mathsf{y}_X \leqslant hg\mathsf{y}_X$. Consequently, the Yoneda lemma and the $\mathcal{Q}$-functoriality of $hg : \mathsf{P}X \longrightarrow \mathsf{P}X$ imply

$$\sigma = (\mathsf{y}_X)_*(-,\sigma) = 1^*_{\mathsf{P}X}(\mathsf{y}_X-,\sigma) \preceq 1^*_{\mathsf{P}X}(hg\mathsf{y}_X-, hg\sigma) \preceq 1^*_{\mathsf{P}X}(\mathsf{y}_X-, hg\sigma) = hg\sigma$$

and thus $\sigma \leqslant hg\sigma$, hence $f\sigma \leqslant g\sigma$ for all $\sigma \in \mathsf{P}X$. ∎

As one already has the isomorphisms of ordered hom-sets

$$\begin{array}{ccccc} \mathcal{Q}\text{-}\mathbf{Dist}(X,Y) & \cong & \mathcal{Q}\text{-}\mathbf{Cat}(Y,\mathsf{P}X) & \cong & (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{co}}(X,\mathsf{P}^{\dagger}Y) \\ \varphi & \overset{\sim}{\longleftrightarrow} & \overleftarrow{\varphi} & \overset{\sim}{\longleftrightarrow} & \overrightarrow{\varphi} \end{array}$$

from the adjunctions (2.iii), more isomorphisms can be formed in $\mathcal{Q}$-**Sup** and $\mathcal{Q}$-**Inf**:

2.14. Lemma. [23] *For all $\mathcal{Q}$-categories X, Y, one has the natural isomorphisms of ordered hom-sets*

$$\begin{aligned}\mathcal{Q}\text{-}\mathbf{Dist}(X,Y) &\cong (\mathcal{Q}\text{-}\mathbf{Sup})^{\mathrm{co}}(\mathsf{P}X,\mathsf{P}^{\dagger}Y) \cong \mathcal{Q}\text{-}\mathbf{Inf}(\mathsf{P}^{\dagger}Y,\mathsf{P}X)\\ &\cong \mathcal{Q}\text{-}\mathbf{Sup}(\mathsf{P}Y,\mathsf{P}X) \cong (\mathcal{Q}\text{-}\mathbf{Inf})^{\mathrm{co}}(\mathsf{P}X,\mathsf{P}Y).\end{aligned}$$

Proof. Each $\mathcal{Q}$-distributor $\varphi : X \to\!\!\!\!\circ\; Y$ induces the Isbell adjunction $\varphi_{\uparrow} \dashv \varphi^{\downarrow} : \mathsf{P}^{\dagger}Y \longrightarrow \mathsf{P}X$ [23] with

$$\varphi_{\uparrow}\sigma = \varphi \swarrow \sigma \quad \text{and} \quad \varphi^{\downarrow}\tau = \tau \searrow \varphi$$

for all $\sigma \in \mathsf{P}X$, $\tau \in \mathsf{P}^{\dagger}Y$. It is straightforward to check that

$$\begin{array}{ccccc}\mathcal{Q}\text{-}\mathbf{Dist}(X,Y) & \cong & (\mathcal{Q}\text{-}\mathbf{Sup})^{\mathrm{co}}(\mathsf{P}X,\mathsf{P}^{\dagger}Y) & \cong & \mathcal{Q}\text{-}\mathbf{Inf}(\mathsf{P}^{\dagger}Y,\mathsf{P}X)\\ \varphi & \overset{\sim}{\longleftrightarrow} & \varphi_{\uparrow} & \overset{\sim}{\longleftrightarrow} & \varphi^{\downarrow}\\ & \cong & \mathcal{Q}\text{-}\mathbf{Sup}(\mathsf{P}Y,\mathsf{P}X) & \cong & (\mathcal{Q}\text{-}\mathbf{Inf})^{\mathrm{co}}(\mathsf{P}X,\mathsf{P}Y)\\ & \overset{\sim}{\longleftrightarrow} & \varphi^{\odot} & \overset{\sim}{\longleftrightarrow} & \varphi_{\odot}\end{array}$$

gives the required isomorphisms. The readers may refer to [23, Theorems 4.4 & 5.7] for details. ∎

3. The non-discrete version of lax distributive laws and their lax algebras

In this section we establish the non-discrete version of the lax distributive laws considered in [30]. For a 2-monad $\mathbb{T} = (T, m, e)$ on $\mathcal{Q}$-$\mathbf{Cat}$, a *lax distributive law* $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$ is given by a family

$$\{\lambda_X : T\mathsf{P}X \longrightarrow \mathsf{P}TX\}_{X\in \mathrm{ob}(\mathcal{Q}\text{-}\mathbf{Cat})}$$

of $\mathcal{Q}$-functors satisfying the following inequalities for all $\mathcal{Q}$-functors $f : X \longrightarrow Y$:

(a) $$\begin{array}{ccc} T\mathsf{P}X & \xrightarrow{T(f_!)} & T\mathsf{P}Y\\ {\scriptstyle \lambda_X}\downarrow & \leqslant & \downarrow{\scriptstyle \lambda_Y}\\ \mathsf{P}TX & \xrightarrow[(Tf)_!]{} & \mathsf{P}TY\end{array} \qquad (Tf)_! \cdot \lambda_X \leqslant \lambda_Y \cdot T(f_!) \qquad \text{(lax naturality of } \lambda\text{);}$$

(b) $$\begin{array}{ccc} & TX & \\ {\scriptstyle T\mathsf{y}_X}\swarrow & \geqslant & \searrow{\scriptstyle \mathsf{y}_{TX}}\\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX\end{array} \qquad \mathsf{y}_{TX} \leqslant \lambda_X \cdot T\mathsf{y}_X \qquad \text{(lax } \mathbb{P}\text{-unit law);}$$

(c) $$\begin{array}{ccccc} T\mathsf{P}\mathsf{P}X & \xrightarrow{\lambda_{\mathsf{P}X}} & \mathsf{P}T\mathsf{P}X & \xrightarrow{(\lambda_X)_!} & \mathsf{P}\mathsf{P}TX\\ {\scriptstyle T\mathsf{s}_X}\downarrow & & \geqslant & & \downarrow{\scriptstyle \mathsf{s}_{TX}}\\ T\mathsf{P}X & & \xrightarrow[\lambda_X]{} & & \mathsf{P}TX\end{array} \qquad \mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X} \leqslant \lambda_X \cdot T\mathsf{s}_X \qquad \text{(lax } \mathbb{P}\text{-mult. law);}$$

(d)
$$\begin{array}{ccc} & \mathsf{P}X & \\ {}^{e_{\mathsf{P}X}}\swarrow & \geqslant & \searrow^{(e_X)_!} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$
$(e_X)_! \leqslant \lambda_X \cdot e_{\mathsf{P}X}$ (lax $\mathbb{T}$-unit law);

(e)
$$\begin{array}{ccc} TT\mathsf{P}X \xrightarrow{T\lambda_X} T\mathsf{P}TX & \xrightarrow{\lambda_{TX}} & \mathsf{P}TTX \\ {\scriptstyle m_{\mathsf{P}X}}\downarrow & \geqslant & \downarrow{\scriptstyle (m_X)_!} \\ T\mathsf{P}X & \xrightarrow[\lambda_X]{} & \mathsf{P}TX \end{array}$$
$(m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \leqslant \lambda_X \cdot m_{\mathsf{P}X}$ (lax $\mathbb{T}$-mult. law).

Each of these laws is said to hold *strictly* (at f or X) if the respective inequality sign may be replaced by an equality sign; for a *strict distributive law*, all lax laws must hold strictly everywhere. For simplicity, in what follows, we refer to a lax distributive law $\lambda : T\mathsf{P} \longrightarrow T\mathsf{P}$ just as a *distributive law*, which indirectly emphasizes the fact that the ambient 2-cell structure is given by order; we also say that $\mathbb{T}$ *distributes* over $\mathbb{P}$ by λ in this case, adding *strictly* when λ is strict.

3.1. Remark. Recall that in the discrete case (see [30]), a distributive law λ of a monad $\mathbb{T} = (T, m, e)$ on $\mathbf{Set}/\mathcal{Q}_0$ over the discrete presheaf monad $\mathbb{P}$ on $\mathbf{Set}/\mathcal{Q}_0$ is usually required to be monotone, *i.e.*,
$$f \leqslant g \Longrightarrow \lambda_X \cdot Tf \leqslant \lambda_X \cdot Tg$$
for all $\mathcal{Q}$-functors $f, g : Y \longrightarrow \mathsf{P}X$. As for the non-discrete case, $\mathbb{T} = (T, m, e)$ becomes a 2-monad on the 2-category $\mathcal{Q}$-$\mathbf{Cat}$ and, hence, the monotonicity of a distributive law of $\mathbb{T}$ over the 2-monad $\mathbb{P}$ on $\mathcal{Q}$-$\mathbf{Cat}$ is automatically satisfied through the 2-functoriality of T.

3.2. Definition. For a distributive law $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$, a *lax λ-algebra* (X, p) over $\mathcal{Q}$ is a $\mathcal{Q}$-category X with a $\mathcal{Q}$-functor $p : TX \longrightarrow \mathsf{P}X$ satisfying

(f)
$$\begin{array}{ccc} & X & \\ {}^{e_X}\swarrow & \geqslant & \searrow^{\mathsf{y}_X} \\ TX & \xrightarrow[p]{} & \mathsf{P}X \end{array}$$
$\mathsf{y}_X \leqslant p \cdot e_X$ (lax unit law);

(g)
$$\begin{array}{ccc} TTX \xrightarrow{Tp} T\mathsf{P}X \xrightarrow{\lambda_X} \mathsf{P}TX & \xrightarrow{p_!} & \mathsf{P}\mathsf{P}X \\ {\scriptstyle m_X}\downarrow & \geqslant & \downarrow{\scriptstyle \mathsf{s}_X} \\ TX & \xrightarrow[p]{} & \mathsf{P}X \end{array}$$
$\mathsf{s}_X \cdot p_! \cdot \lambda_X \cdot Tp \leqslant p \cdot m_X$ (lax mult. law).

A *lax λ-homomorphism* $f : (X, p) \longrightarrow (Y, q)$ of lax λ-algebras is a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ which satisfies

(h)
$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle p}\downarrow & \leqslant & \downarrow{\scriptstyle q} \\ \mathsf{P}X & \xrightarrow[f_!]{} & \mathsf{P}Y \end{array}$$
$f_! \cdot p \leqslant q \cdot Tf$ (lax homomorphism law).

The resulting 2-category is denoted by $(\lambda, \mathcal{Q})$-**Alg**, with the local order inherited from $\mathcal{Q}$-**Cat**.

3.3. Proposition. *$(\lambda, \mathcal{Q})$-**Alg** is topological over $\mathcal{Q}$-**Cat** and, hence, totally complete and totally cocomplete.*

Proof. For any family of λ-algebras (Y_j, q_j) and $\mathcal{Q}$-functors $f_j : X \longrightarrow Y_j$ $(j \in J)$,

$$p := \bigwedge_{j \in J} (f_j)^! \cdot q_j \cdot T f_j$$

gives the initial structure on X with respect to the forgetful functor $(\lambda, \mathcal{Q})$-**Alg**$\longrightarrow\mathcal{Q}$-**Cat**, and thus establishes the topologicity of $(\lambda, \mathcal{Q})$-**Alg** over $\mathcal{Q}$-**Cat** (see [1]). The total completeness and total cocompleteness of $(\lambda, \mathcal{Q})$-**Alg** then follow from that of $\mathcal{Q}$-**Cat** (see [22, Theorem 2.7]). ∎

4. The distributive law of the presheaf 2-monad

The presheaf 2-monad $\mathbb{P}$ on $\mathcal{Q}$-**Cat** is *lax idempotent*, or *of Kock-Zöberlein type* [25], in the sense that

$$(\mathsf{y}_X)_! \leqslant \mathsf{y}_{\mathsf{P}X}$$

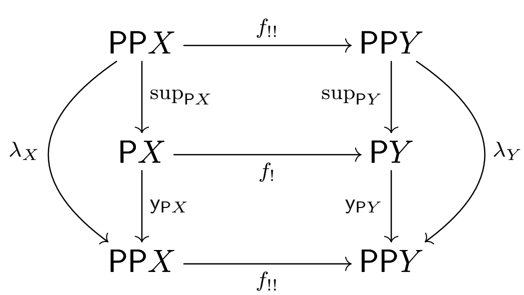
for all $\mathcal{Q}$-categories X. This fact makes it possible to establish the distributivity of $\mathbb{P}$ over itself:

4.1. Theorem. *The presheaf 2-monad $\mathbb{P}$ distributes over itself by λ with*

$$\lambda_X = \mathsf{y}_{\mathsf{P}X} \cdot \mathsf{y}_X^! = \mathsf{y}_{\mathsf{P}X} \cdot \sup_{\mathsf{P}X} : \mathsf{PP}X \longrightarrow \mathsf{PP}X.$$

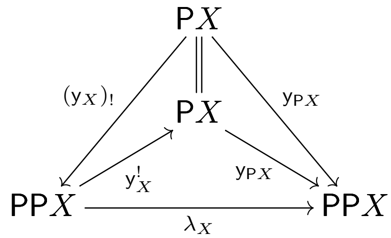
Proof. We show that λ satisfies the laws (a), (b), (c) and (e) strictly and (d) laxly.

(a) $f_{!!} \cdot \lambda_X = \lambda_Y \cdot f_{!!}$ for any $\mathcal{Q}$-functor $f : X \longrightarrow Y$. The commutativity of the upper square and the lower square of the diagram



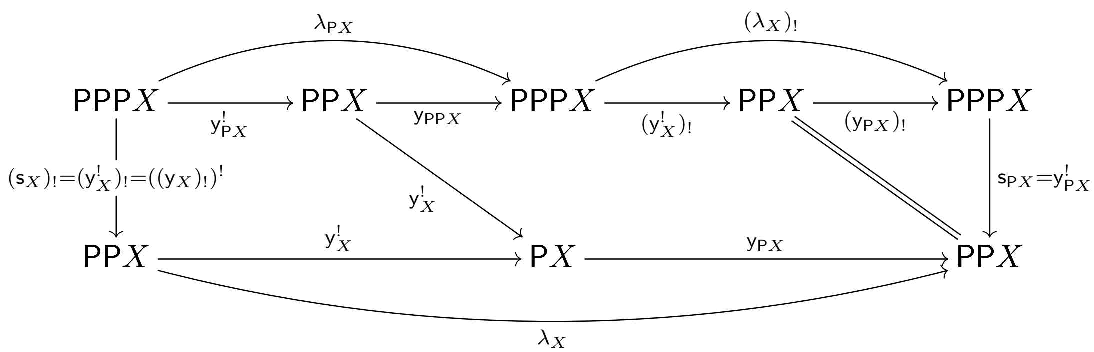
respectively follow from Lemma 2.9 and the naturality of y.

(b) $\mathsf{y}_{\mathsf{P}X} = \lambda_X \cdot (\mathsf{y}_X)_!$. Since y_X is fully faithful, one has $\mathsf{y}_X^! \cdot (\mathsf{y}_X)_! = 1_{\mathsf{P}X}$ and thus the diagram

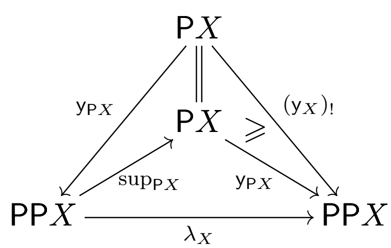


is commutative.

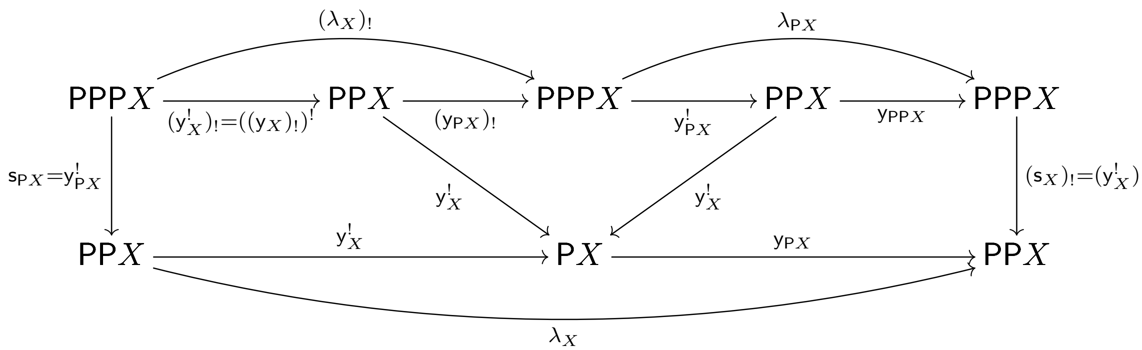
(c) $\mathsf{s}_{\mathsf{P}X} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X} = \lambda_X \cdot (\mathsf{s}_X)_!$. In the following diagram, the commutativity of the left and the middle trapezoids both follow from the naturality of y, and the right triangle commutes since $\mathsf{y}_{\mathsf{P}X}$ is fully faithful.



(d) $(\mathsf{y}_X)_! \leqslant \lambda_X \cdot \mathsf{y}_{\mathsf{P}X}$. In the following diagram, $\sup_{\mathsf{P}X} \cdot \mathsf{y}_{\mathsf{P}X} = 1_{\mathsf{P}X}$ and $\mathbb{P}$ being lax idempotent guarantees $(\mathsf{y}_X)_! \leqslant \mathsf{y}_{\mathsf{P}X}$.



(e) $(\mathsf{s}_X)_! \cdot \lambda_{\mathsf{P}X} \cdot (\lambda_X)_! = \lambda_X \cdot \mathsf{s}_{\mathsf{P}X}$. The naturality of y ensure that the left and the right trapezoids of the diagram



are commutative, and the commutativity of the middle triangle follows from the full faithfulness of $\mathsf{y}_{\mathsf{P}X}$.

■

A $\mathcal{Q}$-*closure space* [21, 23] is a pair (X, c) that consists of a $\mathcal{Q}$-category X and a $\mathcal{Q}$-*closure operation* c on $\mathsf{P}X$; that is, a $\mathcal{Q}$-functor $c : \mathsf{P}X \longrightarrow \mathsf{P}X$ satisfying $1_{\mathsf{P}X} \leqslant c$ and $c \cdot c = c$. A *continuous* $\mathcal{Q}$-*functor* $f : (X, c) \longrightarrow (Y, d)$ between $\mathcal{Q}$-closure spaces is a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ such that

$$f_! \cdot c \leqslant d \cdot f_! : \mathsf{P}X \longrightarrow \mathsf{P}Y.$$

$\mathcal{Q}$-closure spaces and continuous $\mathcal{Q}$-functors constitute a 2-category $\mathcal{Q}$-**Cls** with the local order inherited from $\mathcal{Q}$-**Cat**.

4.2. Theorem. $(\lambda, \mathcal{Q})$-**Alg** $\cong$ $\mathcal{Q}$-**Cls**.

Proof. For any $\mathcal{Q}$-category X, we show that a $\mathcal{Q}$-functor $c : \mathsf{P}X \longrightarrow \mathsf{P}X$ gives a lax λ-algebra structure on X if, and only if, (X, c) is a $\mathcal{Q}$-closure space.

c satisfies (f) $\iff$ $1_{\mathsf{P}X} \leqslant c$: This is an immediate consequence of Lemma 2.13.

c satisfies (g) $\iff$ $c \cdot c \leqslant c$: Note that

$$\begin{aligned} c \cdot c &= \sup_{\mathsf{P}X} \cdot \mathsf{y}_{\mathsf{P}X} \cdot c \cdot \sup_{\mathsf{P}X} \cdot \mathsf{y}_{\mathsf{P}X} \cdot c \\ &= \sup_{\mathsf{P}X} \cdot c_! \cdot \mathsf{y}_{\mathsf{P}X} \cdot \sup_{\mathsf{P}X} \cdot c_! \cdot \mathsf{y}_{\mathsf{P}X} && (\mathsf{y} \text{ is natural}) \\ &= \sup_{\mathsf{P}X} \cdot c_! \cdot \lambda_X \cdot c_! \cdot \mathsf{y}_{\mathsf{P}X}, \end{aligned}$$

and thus

$$\begin{aligned} c \cdot c \leqslant c &\iff \sup_{\mathsf{P}X} \cdot c_! \cdot \lambda_X \cdot c_! \cdot \mathsf{y}_{\mathsf{P}X} \leqslant c \\ &\iff \sup_{\mathsf{P}X} \cdot c_! \cdot \lambda_X \cdot c_! \leqslant c \cdot \sup_{\mathsf{P}X}, && (\sup_{\mathsf{P}X} \dashv \mathsf{y}_{\mathsf{P}X}) \end{aligned}$$

which is precisely the condition (g).

Therefore, the isomorphism between $(\lambda, \mathcal{Q})$-**Alg** and $\mathcal{Q}$-**Cls** follows since a continuous $\mathcal{Q}$-functor $f : (X, c) \longrightarrow (Y, d)$ is exactly a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ satisfying the condition (h). ■

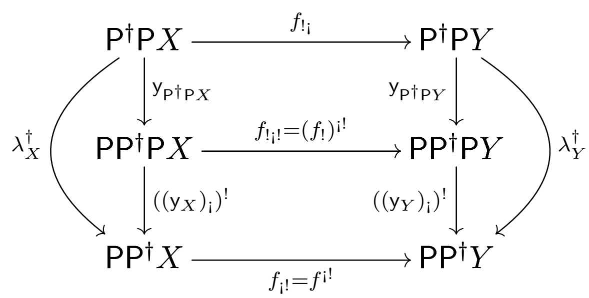
5. The strict distributive law of the copresheaf 2-monad

5.1. Theorem. *The copresheaf 2-monad $\mathbb{P}^\dagger$ distributes strictly over $\mathbb{P}$ by $\lambda^\dagger$ with*

$$\lambda^\dagger_X = ((\mathsf{y}_X)_{\mathsf{i}})^! \cdot \mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X} : \mathsf{P}^\dagger\mathsf{P}X \longrightarrow \mathsf{P}\mathsf{P}^\dagger X.$$

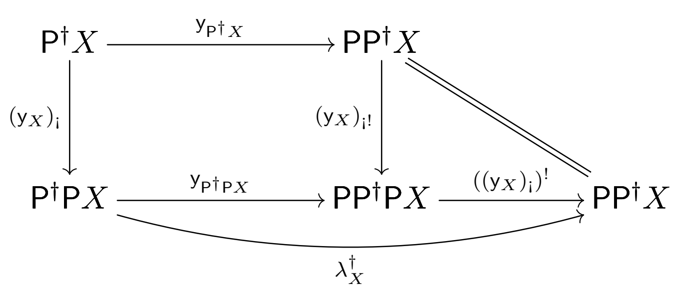
Proof. We show that $\lambda^\dagger$ satisfies the laws (a)-(e) strictly.

(a) $f_{\mathsf{i}!} \cdot \lambda^\dagger_X = \lambda^\dagger_Y \cdot f_{!\mathsf{i}}$ for any $\mathcal{Q}$-functor $f : X \longrightarrow Y$. Indeed, both the upper square and the lower square of the diagram



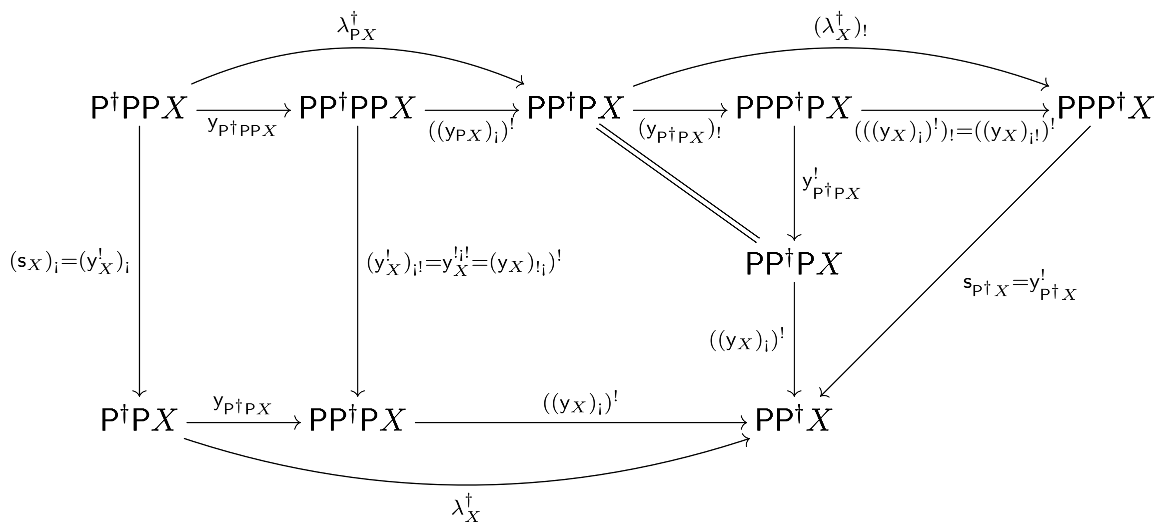
are commutative, respectively by the naturality of y and Lemma 2.11(4).

(b) $\mathsf{y}_{\mathsf{P}^\dagger X} = \lambda^\dagger_X \cdot (\mathsf{y}_X)_{\mathsf{i}}$. For this, note that both, the left square and the right triangle of the diagram

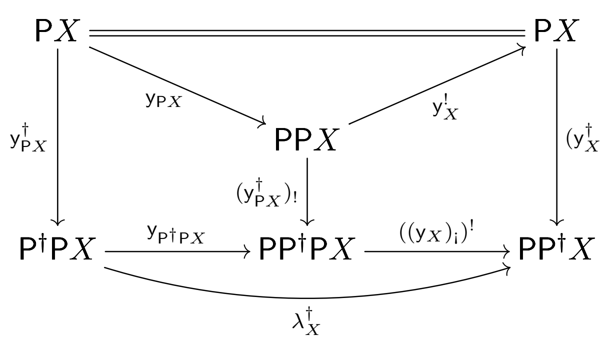


are commutative, by the naturality of y and the full faithfulness of $(\mathsf{y}_X)_{\mathsf{i}}$, respectively.

(c) $\mathsf{s}_{\mathsf{P}^\dagger X} \cdot (\lambda^\dagger_X)_! \cdot \lambda^\dagger_{\mathsf{P}X} = \lambda^\dagger_X \cdot (\mathsf{s}_X)_{\mathsf{i}}$. In the following diagram, the naturality of y guarantees the commutativity of the left square and the right triangle, and together with the full faithfulness of $\mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X}$ it forces the commutativity of the middle square.

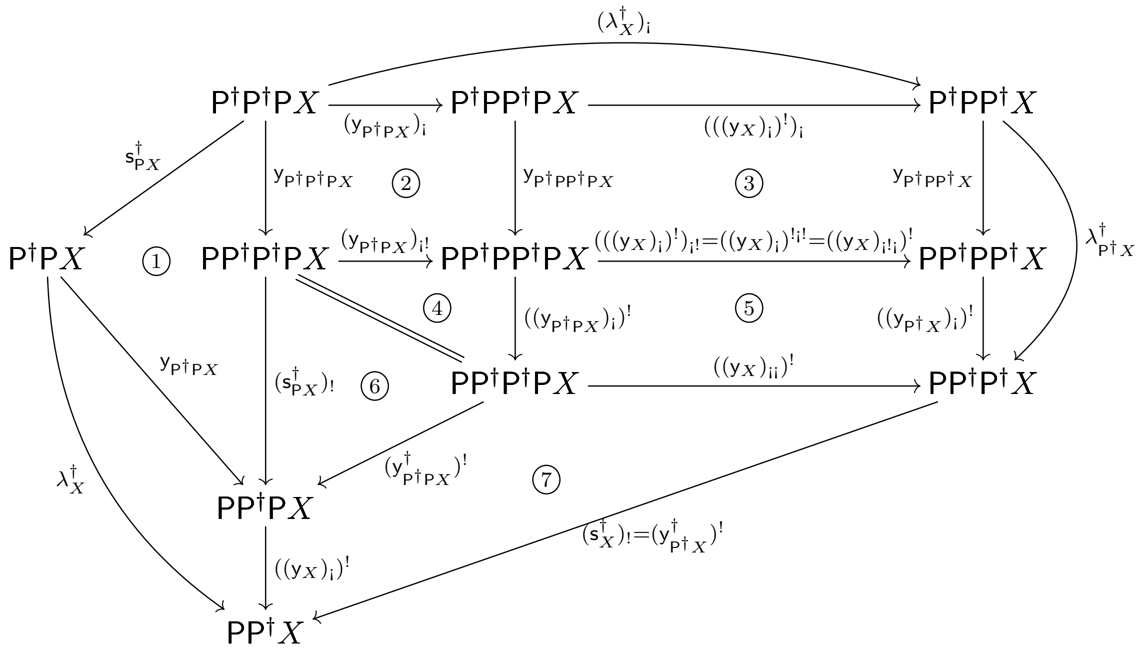


(d) $(\mathsf{y}^\dagger_X)_! = \lambda^\dagger_X \cdot \mathsf{y}^\dagger_{\mathsf{P}X}$. From $\mathsf{y}^!_X = \sup_{\mathsf{P}X}$ one sees that the upper triangle of the diagram



is commutative, and the commutativity of the left and the right trapezoids follow respectively from the naturality of y and Lemma 2.11(5).

(e) $(\mathsf{s}^\dagger_X)_! \cdot \lambda^\dagger_{\mathsf{P}^\dagger X} \cdot (\lambda^\dagger_X)_{\mathsf{i}} = \lambda^\dagger_X \cdot \mathsf{s}^\dagger_{\mathsf{P}X}$. This follows from the commutativity of the following diagram:



Explicitly, the commutativity of ①, ②, ③, ⑤ follow from the naturality of y, ④ follows from the full faithfulness of $(\mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X})_{\mathsf{i}}$, ⑥ follows from Equation (2.v) and Lemma 2.7, and ⑦ follows from the naturality of $\mathsf{y}^\dagger$.

■

5.2. Remark. Stubbe described a strict distributive law of $\mathbb{P}$ over $\mathbb{P}^\dagger$ given by

$$\mathsf{P}\mathsf{P}^\dagger X \xrightarrow{(\mathsf{y}_X)_{\mathsf{i}!}} \mathsf{P}\mathsf{P}^\dagger\mathsf{P}X \xrightarrow{\sup_{\mathsf{P}^\dagger\mathsf{P}X}} \mathsf{P}^\dagger\mathsf{P}X \tag{5.i}$$

in [27]. In fact, the strict distributive law $\lambda^\dagger_X : \mathsf{P}^\dagger\mathsf{P}X \longrightarrow \mathsf{P}\mathsf{P}^\dagger X$ defined in Theorem 5.1 is precisely the right adjoint of (5.i) in $\mathcal{Q}$-**Cat**.

Recall that a $\mathcal{Q}$-category is a monad in $\mathcal{Q}$-**Rel**. Similarly, a monad in $\mathcal{Q}$-**Dist** gives "a $\mathcal{Q}$-category over a base $\mathcal{Q}$-category"; that is, a $\mathcal{Q}$-category X equipped with a $\mathcal{Q}$-distributor $\alpha : X \to\!\!\!\!\!\circ\; X$, such that $1^*_X \preceq \alpha$ and $\alpha \circ \alpha \preceq \alpha$. The latter two inequalities actually force the $\mathcal{Q}$-relation α on X to be a $\mathcal{Q}$-distributor, since with $a = 1^*_X$ one has

$$a \circ (\alpha \circ a) \preceq a \circ (\alpha \circ \alpha) \preceq a \circ \alpha \preceq \alpha \circ \alpha \preceq \alpha.$$

Thus, a monad in $\mathcal{Q}$-**Dist** is given by a set X over $\mathcal{Q}_0$ that comes equipped with two $\mathcal{Q}$-category structures, comparable by "$\preceq$". With morphisms to laxly preserve both structures we obtain the 2-category **Mon**($\mathcal{Q}$-**Dist**); hence, its morphisms $f : (X,\alpha) \longrightarrow (Y,\beta)$ are precisely $\mathcal{Q}$-functors $f : X \longrightarrow Y$ with

$$f_! \cdot \overleftarrow{\alpha} \leqslant \overleftarrow{\beta} \cdot f$$

or, equivalently, $\alpha(x,x') \preceq \beta(fx,fx')$ for all $x,x' \in X$.

We also point out that the copresheaf 2-monad $\mathbb{P}^\dagger$ on $\mathcal{Q}$-**Cat** is *oplax idempotent*, or *of dual Kock-Zöberlein type*, in the sense that

$$\mathsf{y}^\dagger_{\mathsf{P}^\dagger X} \leqslant (\mathsf{y}^\dagger_X)_{\mathsf{i}}$$

for all $\mathcal{Q}$-categories X. We shall use this fact to characterize $(\lambda^\dagger,\mathcal{Q})$-algebras as monads in $\mathcal{Q}$-**Dist**:

5.3. THEOREM. $(\lambda^\dagger,\mathcal{Q})$-**Alg** $\cong$ **Mon**($\mathcal{Q}$-**Dist**).

PROOF. **Step 1.** We show that if (X,p) is a $(\lambda^\dagger,\mathcal{Q})$-algebra, then

$$p = \inf\nolimits_{\mathsf{P}X}\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}}. \tag{5.ii}$$

Indeed, the conditions (f) and (g) for the $(\lambda^\dagger,\mathcal{Q})$-algebra (X,p) read as

(f) $\mathsf{y}_X \leqslant p\cdot\mathsf{y}^\dagger_X$ and

(g) $\mathsf{y}^!_X\cdot p_!\cdot\lambda^\dagger_X\cdot p_{\mathsf{i}} \leqslant p\cdot(\mathsf{y}^\dagger_X)^{\mathsf{i}}$,

and consequently

$$\begin{aligned}
p &= \inf\nolimits_{\mathsf{P}X}\cdot\mathsf{y}^\dagger_{\mathsf{P}X}\cdot p\\
&= \inf\nolimits_{\mathsf{P}X}\cdot p_{\mathsf{i}}\cdot\mathsf{y}^\dagger_{\mathsf{P}^\dagger X} && (\mathsf{y}^\dagger \text{ is natural})\\
&\leqslant \inf\nolimits_{\mathsf{P}X}\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (\mathbb{P}^\dagger \text{ is oplax idempotent})\\
&= (\mathsf{y}^\dagger_X)^!\cdot(\mathsf{y}^\dagger_X)_!\cdot\inf\nolimits_{\mathsf{P}X}\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (\mathsf{y}^\dagger_X \text{ is fully faithful})\\
&= (\mathsf{y}^\dagger_X)^!\cdot\lambda^\dagger_X\cdot\mathsf{y}^\dagger_{\mathsf{P}X}\cdot\inf\nolimits_{\mathsf{P}X}\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (\lambda^\dagger \text{ satisfies (d)})\\
&\leqslant (\mathsf{y}^\dagger_X)^!\cdot\lambda^\dagger_X\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (\mathsf{y}^\dagger_{\mathsf{P}X}\dashv\inf\nolimits_{\mathsf{P}X})\\
&\leqslant (\mathsf{y}^\dagger_X)^!\cdot p^!\cdot p_!\cdot\lambda^\dagger_X\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (p_!\dashv p^!)\\
&\leqslant \mathsf{y}^!_X\cdot p_!\cdot\lambda^\dagger_X\cdot p_{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (p \text{ satisfies (f)})\\
&\leqslant p\cdot(\mathsf{y}^\dagger_X)^{\mathsf{i}}\cdot(\mathsf{y}^\dagger_X)_{\mathsf{i}} && (p \text{ satisfies (g)})\\
&= p. && (\mathsf{y}^\dagger_X \text{ is fully faithful})
\end{aligned}$$

Step 2. As an immediate consequence of (5.ii), p is a right adjoint in $\mathcal{Q}$-**Cat**. For any $\mathcal{Q}$-category X, as one already has

$$\mathcal{Q}\text{-}\mathbf{Dist}(X,X)\cong\mathcal{Q}\text{-}\mathbf{Inf}(\mathsf{P}^\dagger X,\mathsf{P}X)$$

from Lemma 2.14, with the isomorphism given by

$$(\alpha : X \mathrel{\circ\!\!\longrightarrow} X)\mapsto(\alpha^\downarrow : \mathsf{P}^\dagger X\longrightarrow\mathsf{P}X,\quad \alpha^\downarrow\tau=\tau\searrow\alpha),$$

in order for us to establish a bijection between monads on X (in $\mathcal{Q}$-**Dist**) and $(\lambda^\dagger,\mathcal{Q})$-algebra structures on X, it suffices to prove

- $1_X^* \preceq \alpha \iff \alpha^{\downarrow}$ satisfies (f), and
- $\alpha\circ\alpha \preceq \alpha \iff \alpha^{\downarrow}$ satisfies (g)

for all $\mathcal{Q}$-distributors $\alpha : X \to X$.

First, $1_X^* \preceq \alpha \iff \alpha^{\downarrow}$ satisfies (f). Since $\overleftarrow{1_X^*} = \mathsf{y}_X$ and, as one easily sees, $\overleftarrow{\alpha} = \alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}$, the equivalence $1_X^* \preceq \alpha \iff \mathsf{y}_X \leqslant \alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}$ follows immediately.

Second, $\alpha\circ\alpha \preceq \alpha \iff \alpha^{\downarrow}$ satisfies (g), *i.e.*,

$$\mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot(\alpha^{\downarrow})_{\mathsf{i}} \leqslant \alpha^{\downarrow}\cdot(\mathsf{y}_X^{\dagger})^{\mathsf{i}} = \alpha^{\downarrow}\cdot\inf_{\mathsf{P}^{\dagger}X}.$$

Note that

$$\begin{aligned}
\overleftarrow{\alpha\circ\alpha} &= \mathsf{y}_X^{!}\cdot\overleftarrow{\alpha}_!\cdot\overleftarrow{\alpha} && \text{(Lemma 2.12(1))}\\
&= \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot(\mathsf{y}_X^{\dagger})_!\cdot\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger} && (\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}=\overleftarrow{\alpha})\\
&= \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot\mathsf{y}_{\mathsf{P}X}^{\dagger}\cdot\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger} && (\lambda^{\dagger}\text{ satisfies (d)})\\
&= \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot(\alpha^{\downarrow})_{\mathsf{i}}\cdot\mathsf{y}_{\mathsf{P}^{\dagger}X}^{\dagger}\cdot\mathsf{y}_X^{\dagger}, && (\mathsf{y}^{\dagger}\text{ is natural})
\end{aligned}$$

and hence

$$\begin{aligned}
\alpha\circ\alpha\preceq\alpha &\iff \overleftarrow{\alpha\circ\alpha}\leqslant\overleftarrow{\alpha}=\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}\\
&\iff \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot(\alpha^{\downarrow})_{\mathsf{i}}\cdot\mathsf{y}_{\mathsf{P}^{\dagger}X}^{\dagger}\cdot\mathsf{y}_X^{\dagger}\leqslant\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}\\
&\iff \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot(\alpha^{\downarrow})_{\mathsf{i}}\cdot\mathsf{y}_{\mathsf{P}^{\dagger}X}^{\dagger}\leqslant\alpha^{\downarrow}=\alpha^{\downarrow}\cdot\inf_{\mathsf{P}^{\dagger}X}\cdot\mathsf{y}_{\mathsf{P}^{\dagger}X}^{\dagger} && \text{(Lemma 2.13)}\\
&\iff \mathsf{y}_X^{!}\cdot(\alpha^{\downarrow})_!\cdot\lambda_X^{\dagger}\cdot(\alpha^{\downarrow})_{\mathsf{i}}\leqslant\alpha^{\downarrow}\cdot\inf_{\mathsf{P}^{\dagger}X}, && \text{(Lemma 2.13)}
\end{aligned}$$

as desired.

Step 3. $f:(X,\alpha)\longrightarrow(Y,\beta)$ is a morphism in $\mathbf{Mon}(\mathcal{Q}\text{-}\mathbf{Dist})$ if, and only if, $f:(X,\alpha^{\downarrow})\longrightarrow(Y,\psi^{\downarrow})$ satisfies (h). Indeed,

$$\begin{aligned}
f_!\cdot\alpha^{\downarrow}\leqslant\psi^{\downarrow}\cdot f_{\mathsf{i}} &\iff f_!\cdot\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}\leqslant\psi^{\downarrow}\cdot f_{\mathsf{i}}\cdot\mathsf{y}_X^{\dagger} && \text{(Lemma 2.13)}\\
&\iff f_!\cdot\alpha^{\downarrow}\cdot\mathsf{y}_X^{\dagger}\leqslant\psi^{\downarrow}\cdot\mathsf{y}_Y^{\dagger}\cdot f && (\mathsf{y}^{\dagger}\text{ is natural})\\
&\iff f_!\cdot\overleftarrow{\alpha}\leqslant\overleftarrow{\beta}\cdot f,
\end{aligned}$$

which completes the proof. ∎

6. The distributive law of the double presheaf 2-monad

Recall that the adjunctions $(-)^*\dashv\mathsf{P}$ and $(-)_*\dashv\mathsf{P}^{\dagger}$ displayed in (2.iii) give rise to the isomorphisms

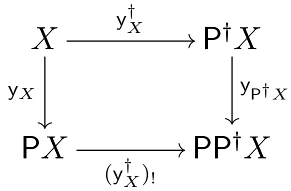
$$\mathcal{Q}\text{-}\mathbf{Cat}(Y,\mathsf{P}X)\cong\mathcal{Q}\text{-}\mathbf{Dist}(X,Y)\cong\mathcal{Q}\text{-}\mathbf{Cat}(X,\mathsf{P}^{\dagger}Y), \tag{6.i}$$

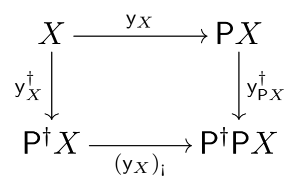
for all $\mathcal{Q}$-categories X, Y. In fact, (6.i) induces another pair of adjoint 2-functors [27]

$$\mathsf{P}_{\mathsf{c}}^{\dagger} \dashv \mathsf{P}_{\mathsf{c}} : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{coop}}, \tag{6.ii}$$

which map objects as $\mathsf{P}^{\dagger}$ and P do, but with $\mathsf{P}_{\mathsf{c}}^{\dagger} f = f^{\mathsf{i}}$ and $\mathsf{P}_{\mathsf{c}} f = f^{!}$ for all $\mathcal{Q}$-functor f. The units and counits of this adjunction are respectively given by

$$\mathsf{y}_{\mathsf{P}^{\dagger}X} \cdot \mathsf{y}_X^{\dagger} = (\mathsf{y}_X^{\dagger})_! \cdot \mathsf{y}_X : X \longrightarrow \mathsf{P}\mathsf{P}^{\dagger}X \quad \text{and} \quad \mathsf{y}_{\mathsf{P}X}^{\dagger} \cdot \mathsf{y}_X = (\mathsf{y}_X)_{\mathsf{i}} \cdot \mathsf{y}_X^{\dagger} : X \longrightarrow \mathsf{P}^{\dagger}\mathsf{P}X$$





for all $\mathcal{Q}$-categories X. This adjunction induces the *double presheaf 2-monad* $(\mathsf{P}_{\mathsf{c}}\mathsf{P}_{\mathsf{c}}^{\dagger}, \mathfrak{y}, \mathfrak{s})$ on $\mathcal{Q}$-**Cat** with the multiplication given by

$$\mathfrak{s}_X = ((\mathsf{y}_{\mathsf{P}^{\dagger}X})_{\mathsf{i}} \cdot \mathsf{y}_{\mathsf{P}^{\dagger}X}^{\dagger})^{!} = (\mathsf{y}_{\mathsf{P}\mathsf{P}^{\dagger}X}^{\dagger} \cdot \mathsf{y}_{\mathsf{P}^{\dagger}X})^{!} = \mathsf{s}_{\mathsf{P}^{\dagger}X} \cdot (\mathsf{y}_{\mathsf{P}\mathsf{P}^{\dagger}X}^{\dagger})^{!} : \mathsf{P}\mathsf{P}^{\dagger}\mathsf{P}\mathsf{P}^{\dagger}X \longrightarrow \mathsf{P}\mathsf{P}^{\dagger}X. \tag{6.iii}$$

As Lemma 2.11(1) implies $\mathsf{P}_{\mathsf{c}}\mathsf{P}_{\mathsf{c}}^{\dagger} = \mathsf{P}\mathsf{P}^{\dagger}$, the double presheaf 2-monad on $\mathcal{Q}$-**Cat** may be alternatively written as

$$\mathbb{P}\mathbb{P}^{\dagger} = (\mathsf{P}\mathsf{P}^{\dagger}, \mathfrak{y}, \mathfrak{s}).$$

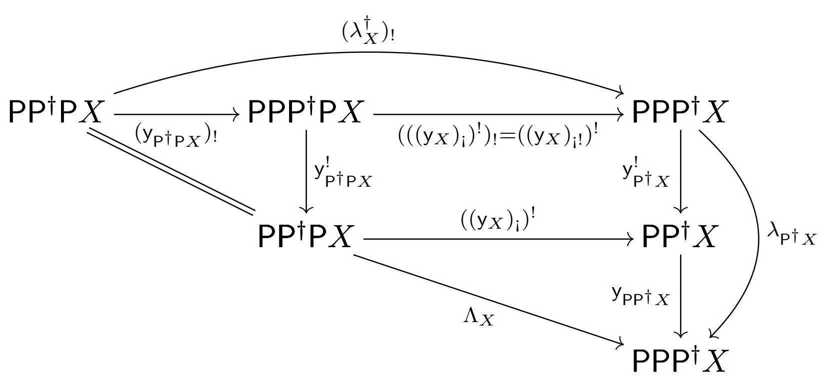
6.1. Theorem. *The double presheaf 2-monad* $\mathbb{P}\mathbb{P}^{\dagger}$ *distributes over* $\mathbb{P}$ *by* Λ *with*

$$\Lambda_X = \mathsf{y}_{\mathsf{P}\mathsf{P}^{\dagger}X} \cdot ((\mathsf{y}_X)_{\mathsf{i}})^{!} : \mathsf{P}\mathsf{P}^{\dagger}\mathsf{P}X \longrightarrow \mathsf{P}\mathsf{P}\mathsf{P}^{\dagger}X.$$

Proof. First note that

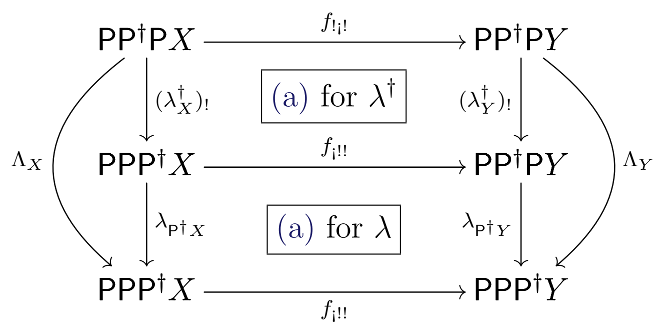
$$\Lambda_X = \lambda_{\mathsf{P}^{\dagger}X} \cdot (\lambda_X^{\dagger})_!. \tag{6.iv}$$

Indeed, from the naturality of y one soon obtains the commutativity of the diagram

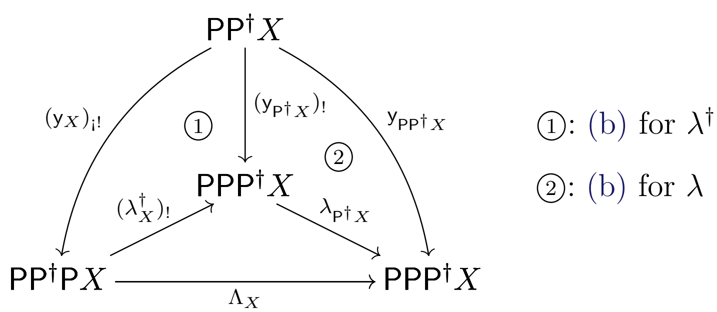


for any $\mathcal{Q}$-category X. Now we check the laws (a)-(e) for Λ:

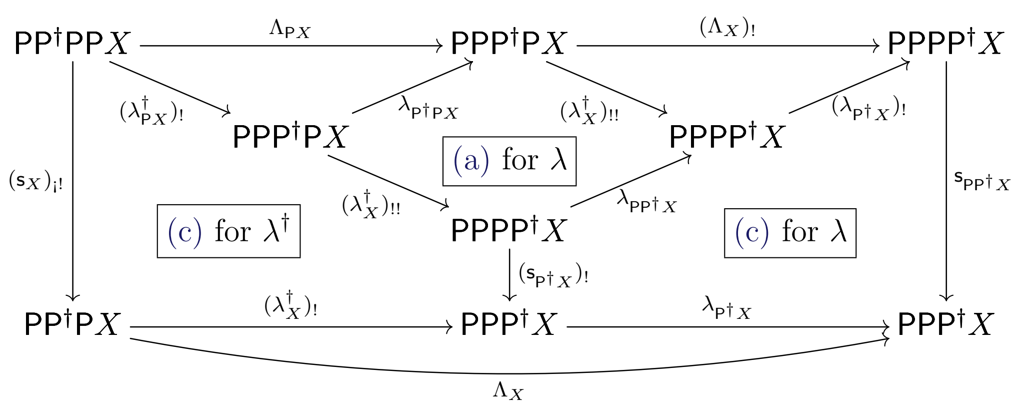
(a) $\Lambda_Y \cdot f_{!¡!} = f_{¡!!} \cdot \Lambda_X$ for every $\mathcal{Q}$-functor $f : X \longrightarrow Y$. This is a direct consequence of the naturality of λ and $\lambda^\dagger$.



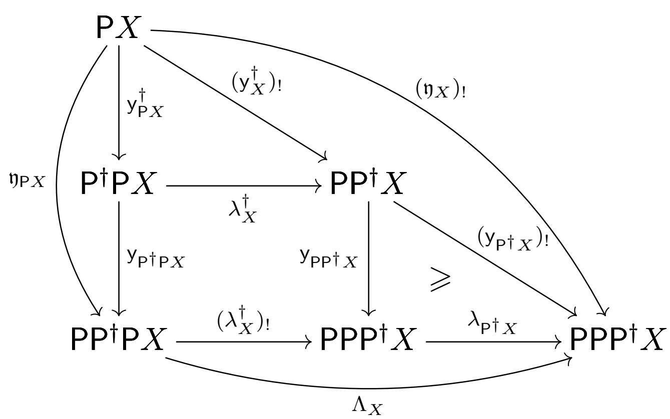
(b) $\mathsf{y}_{\mathsf{PP}^\dagger X} = \Lambda_X \cdot (\mathsf{y}_X)_{¡!}$. This is easy since λ and $\lambda^\dagger$ both satisfy the $\mathbb{P}$-unit law (b) strictly.



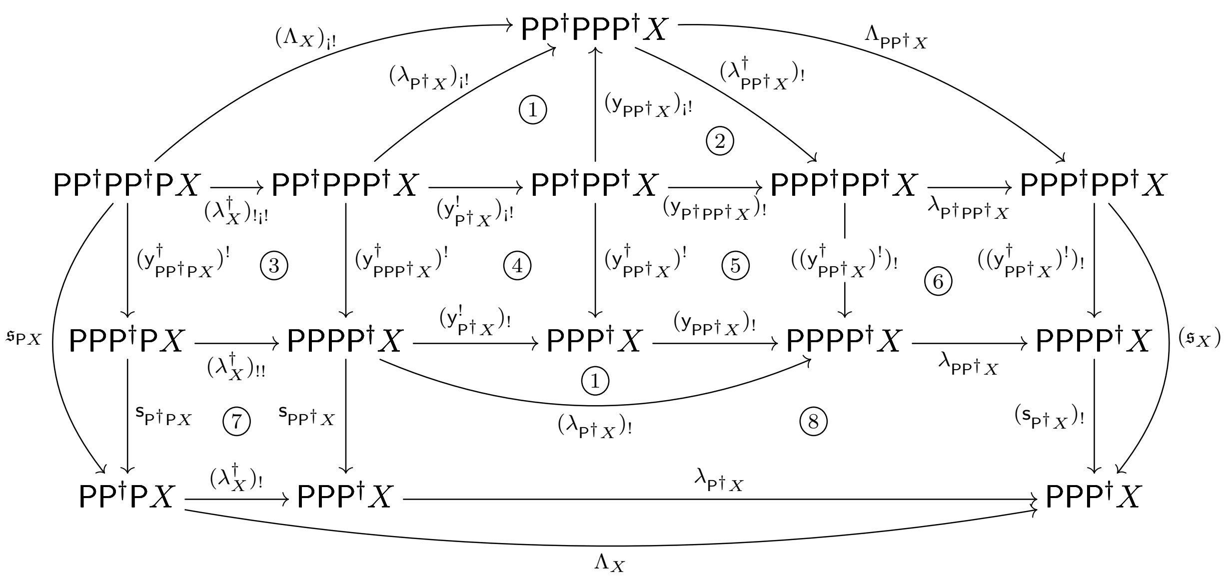
(c) $\mathsf{s}_{\mathsf{PP}^\dagger X} \cdot (\Lambda_X)_! \cdot \Lambda_{\mathsf{P}X} = \Lambda_X \cdot (\mathsf{s}_X)_{¡!}$. This follows from the naturality of λ and the fact that λ and $\lambda^\dagger$ both satisfy the $\mathbb{P}$-multiplication law (c) strictly.



(d) $(\mathfrak{y}_X)_! \leqslant \Lambda_X \cdot \mathfrak{y}_{\mathsf{P}X}$. Since λ satisfies the lax $\mathbb{P}$-unit law (d) and $\lambda^\dagger$ strictly satisfies the $\mathbb{P}^\dagger$-unit law (d), one obtains the upper and the lower right-hand triangles of the following diagram. Moreover, the naturality of y guarantees the commutativity of the lower-left square.



(e) $(\mathfrak{s}_X)_! \cdot \Lambda_{\mathsf{P}\mathsf{P}^\dagger X} \cdot (\Lambda_X)_{¡!} = \Lambda_X \cdot \mathfrak{s}_{\mathsf{P}X}$. We explain the commutativity of the following diagram:



①: The definition of λ.

②: $\lambda^\dagger$ satisfies the $\mathbb{P}$-unit law (b) strictly.

③: Note that $\lambda^\dagger_X = ((\mathsf{y}_X)_¡)^! \cdot \mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X}$ is a right adjoint in $\mathcal{Q}$-**Cat** (with $\sup_{\mathsf{P}^\dagger\mathsf{P}X} \cdot (\mathsf{y}_X)_{¡!}$ as its left adjoint), thus so is $(\lambda^\dagger_X)_!$ by Lemma 2.7. Hence

$$\begin{aligned}(\lambda^\dagger_X)_{!!} \cdot (\mathsf{y}^\dagger_{\mathsf{P}\mathsf{P}^\dagger\mathsf{P}X})^! &= (\lambda^\dagger_X)_{!!} \cdot (\inf_{\mathsf{P}\mathsf{P}^\dagger\mathsf{P}X})_! && \text{(Lemma 2.7)}\\ &= (\inf_{\mathsf{P}\mathsf{P}\mathsf{P}^\dagger X})_! \cdot (\lambda^\dagger_X)_{!¡!} && \text{(Lemma 2.9)}\\ &= (\mathsf{y}^\dagger_{\mathsf{P}\mathsf{P}\mathsf{P}^\dagger X})^! \cdot (\lambda^\dagger_X)_{!¡!}. && \text{(Lemma 2.7)}\end{aligned}$$

④: As $\mathsf{y}^!_{\mathsf{P}^\dagger X}$ is a right adjoint in $\mathcal{Q}$-**Cat**, similar to ③ one deduces

$$\begin{aligned}(\mathsf{y}^!_{\mathsf{P}^\dagger X})_! \cdot (\mathsf{y}^\dagger_{\mathsf{PPP}^\dagger X})^! &= (\mathsf{y}^!_{\mathsf{P}^\dagger X})_! \cdot (\inf_{\mathsf{PPP}^\dagger X})_! && \text{(Lemma 2.7)}\\ &= (\inf_{\mathsf{PP}^\dagger X})_! \cdot (\mathsf{y}^!_{\mathsf{P}^\dagger X})_{¡!} && \text{(Lemma 2.9)}\\ &= (\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^! \cdot (\mathsf{y}^!_{\mathsf{P}^\dagger X})_{¡!}. && \text{(Lemma 2.7)}\end{aligned}$$

⑤: Since $\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X} \dashv \inf_{\mathsf{PP}^\dagger X}$, $\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X}$ is a left adjoint in $\mathcal{Q}$-**Cat**. It follows that

$$\begin{aligned}(\mathsf{y}_{\mathsf{PP}^\dagger X})_! \cdot (\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^! &= \sup^!_{\mathsf{PP}^\dagger X} \cdot (\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^! && \text{(Lemma 2.7)}\\ &= ((\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})_!)^! \cdot \sup^!_{\mathsf{P}^\dagger\mathsf{PP}^\dagger X} && \text{(Lemma 2.9)}\\ &= ((\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!)_! \cdot (\mathsf{y}_{\mathsf{P}^\dagger\mathsf{PP}^\dagger X})_!. && \text{(Lemma 2.11(1))}\end{aligned}$$

⑥: From $\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X} \dashv \inf_{\mathsf{PP}^\dagger X}$ one has

$$\begin{aligned}((\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!)_! \cdot \lambda_{\mathsf{P}^\dagger\mathsf{PP}^\dagger X} &= (\inf_{\mathsf{PP}^\dagger X})_{!!} \cdot \lambda_{\mathsf{P}^\dagger\mathsf{PP}^\dagger X} && \text{(Lemma 2.7)}\\ &= \lambda_{\mathsf{PP}^\dagger X} \cdot (\inf_{\mathsf{PP}^\dagger X})_{!!} && (\lambda \text{ satisfies (a)})\\ &= \lambda_{\mathsf{PP}^\dagger X} \cdot ((\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!)_!. && \text{(Lemma 2.7)}\end{aligned}$$

⑦: Since $(\lambda^\dagger_X)_! \dashv (\lambda^\dagger_X)^!$, Lemma 2.9 implies

$$(\lambda^\dagger_X)_! \cdot \mathsf{s}_{\mathsf{P}^\dagger\mathsf{P}X} = (\lambda^\dagger_X)_! \cdot \sup_{\mathsf{PP}^\dagger\mathsf{P}X} = \sup_{\mathsf{PPP}^\dagger X} \cdot (\lambda^\dagger_X)_{!!} = \mathsf{s}_{\mathsf{PP}^\dagger X} \cdot (\lambda^\dagger_X)_{!!}.$$

⑧: λ satisfies the $\mathbb{P}$-multiplication law (e) strictly.

■

A $\mathcal{Q}$-*interior space* is a pair (X, c) consisting of a $\mathcal{Q}$-category X and a $\mathcal{Q}$-closure operation c on $\mathsf{P}^\dagger X$. A *continuous* $\mathcal{Q}$-*functor* $f : (X, c) \longrightarrow (Y, d)$ between $\mathcal{Q}$-interior spaces is a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ such that

$$c \cdot f^{¡} \leqslant f^{¡} \cdot d : \mathsf{P}^\dagger Y \longrightarrow \mathsf{P}^\dagger X.$$

$\mathcal{Q}$-interior spaces and continuous $\mathcal{Q}$-functors constitute a 2-category $\mathcal{Q}$-**Int**, with the local order inherited from $\mathcal{Q}$-**Cat**.

6.2. Remark. When $\mathcal{Q}$ is a commutative quantale, V, one has $u \swarrow v = v \searrow u$ for all $u, v \in \mathsf{V}$. Considering a set X as a discrete V-category one can display $\mathsf{P}X$ and $\mathsf{P}^\dagger X$ as having the same underlying set V^X, and for all $\varphi, \psi \in \mathsf{V}^X$ one has

$$\mathsf{P}X(\varphi, \psi) = \mathsf{P}^\dagger X(\psi, \varphi),$$

i.e., $\mathsf{P}^\dagger X$ is the dual of $\mathsf{P}X$. Thus, for a closure operation $c : \mathsf{P}^\dagger X \longrightarrow \mathsf{P}^\dagger X$ one has

$$1_{\mathsf{P}^\dagger X} \leqslant c \iff c \leqslant 1_{\mathsf{P}X},$$

that is, c is an *interior operation* on $\mathsf{P}X$ (see [11]). Particularly, when $\mathsf{V} = \mathbf{2}$, $\mathsf{P}X$ is just the powerset of X, and a closure operation c on $\mathsf{P}^\dagger X$ is exactly an interior operation on the powerset of X. So, an interior space (X, c) as defined here coincides with the usual notion.

6.3. THEOREM. $(\Lambda,\mathcal{Q})$-**Alg** $\cong$ $\mathcal{Q}$-**Int**.

PROOF. **Step 1.** We show that if (X,p) is a $(\Lambda,\mathcal{Q})$-algebra, then

$$p = (\inf_{\mathsf{PP}^\dagger X}\cdot p^{\mathsf{i}}\cdot \mathsf{y}^\dagger_{\mathsf{P}X}\cdot \mathsf{y}_X)^! \cdot \mathsf{y}_{\mathsf{PP}^\dagger X} = (\mathsf{y}_X)^!\cdot(\mathsf{y}^\dagger_{\mathsf{P}X})^!\cdot p_{\mathsf{i}!}\cdot \inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X}. \tag{6.v}$$

Indeed, from the definition of the 2-monad $\mathbb{PP}^\dagger$ one may translate the conditions (f) and (g) for (X,p) respectively as

$$\mathsf{y}_X \leqslant p\cdot(\mathsf{y}^\dagger_X)_!\cdot \mathsf{y}_X \quad\text{and}\quad \mathsf{y}^!_X\cdot p_!\cdot\Lambda_X\cdot p_{\mathsf{i}!}\leqslant p\cdot \sup_{\mathsf{PP}^\dagger X}\cdot(\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!.$$

Since from Lemma 2.13 one has

$$\mathsf{y}_X \leqslant p\cdot(\mathsf{y}^\dagger_X)_!\cdot \mathsf{y}_X \iff 1_{\mathsf{P}X}\leqslant p\cdot(\mathsf{y}^\dagger_X)_!$$

and since $\Lambda_X = \mathsf{y}_{\mathsf{PP}^\dagger X}\cdot((\mathsf{y}_X)_{\mathsf{i}})^!$ implies

$$\begin{aligned}
\mathsf{y}^!_X\cdot p_!\cdot\Lambda_X\cdot p_{\mathsf{i}!} &= \mathsf{y}^!_X\cdot p_!\cdot \mathsf{y}_{\mathsf{PP}^\dagger X}\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!}\\
&= \sup_{\mathsf{P}X}\cdot \mathsf{y}_{\mathsf{P}X}\cdot p\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!} && (\mathsf{y}\text{ is natural})\\
&= p\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!},
\end{aligned}$$

the conditions (f) and (g) may be simplified to read as

(f) $1_{\mathsf{P}X}\leqslant p\cdot(\mathsf{y}^\dagger_X)_!$ and

(g) $p\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!}\leqslant p\cdot\sup_{\mathsf{PP}^\dagger X}\cdot(\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!$.

Therefore,

$$\begin{aligned}
p &= \sup_{\mathsf{P}X}\cdot \mathsf{y}_{\mathsf{P}X}\cdot p\\
&= (\mathsf{y}_X)^!\cdot p_!\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (\mathsf{y}\text{ is natural})\\
&= (\mathsf{y}_X)^!\cdot p_!\cdot(\inf_{\mathsf{PP}^\dagger X})_!\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (\inf_{\mathsf{PP}^\dagger X}\text{ is surjective})\\
&\leqslant (\mathsf{y}_X)^!\cdot(\inf_{\mathsf{P}X})_!\cdot p_{\mathsf{i}!}\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (\text{Lemma 2.9})\\
&= (\mathsf{y}_X)^!\cdot(\mathsf{y}^\dagger_{\mathsf{P}X})^!\cdot p_{\mathsf{i}!}\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (\mathsf{y}^\dagger_{\mathsf{P}X}\dashv\inf_{\mathsf{P}X})\\
&= (\mathsf{y}^\dagger_X)^!\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!}\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (\mathsf{y}^\dagger\text{ is natural})\\
&\leqslant p\cdot(\mathsf{y}^\dagger_X)_!\cdot(\mathsf{y}^\dagger_X)^!\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!}\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (p\text{ satisfies (f)})\\
&\leqslant p\cdot((\mathsf{y}_X)_{\mathsf{i}})^!\cdot p_{\mathsf{i}!}\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && ((\mathsf{y}^\dagger_X)_!\dashv(\mathsf{y}^\dagger_X)^!)\\
&\leqslant p\cdot\sup_{\mathsf{PP}^\dagger X}\cdot(\mathsf{y}^\dagger_{\mathsf{PP}^\dagger X})^!\cdot\inf^!_{\mathsf{PP}^\dagger X}\cdot \mathsf{y}_{\mathsf{PP}^\dagger X} && (p\text{ satisfies (g)})\\
&= p.
\end{aligned}$$

Step 2. As an immediate consequence of (6.v), p is a right adjoint in $\mathcal{Q}$-**Cat**. For every $\mathcal{Q}$-category X, as one already has

$$\mathcal{Q}\text{-}\mathbf{Dist}(\mathsf{P}^\dagger X, X)\cong(\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{co}}(\mathsf{P}^\dagger X,\mathsf{P}^\dagger X)\cong(\mathcal{Q}\text{-}\mathbf{Inf})^{\mathrm{co}}(\mathsf{PP}^\dagger X,\mathsf{P}X)$$

from Lemma 2.14, with the isomorphism given by

$$(\varphi : \mathsf{P}^\dagger X \to\!\!\!\!\circ\; X) \mapsto (\overrightarrow{\varphi} : \mathsf{P}^\dagger X \longrightarrow \mathsf{P}^\dagger X) \mapsto (\varphi_\odot : \mathsf{P}\mathsf{P}^\dagger X \longrightarrow \mathsf{P}X),$$

in order to establish a bijection between $\mathcal{Q}$-closure operations on $\mathsf{P}^\dagger X$ and $(\Lambda, \mathcal{Q})$-algebra structures on X, it suffices to prove

- $1_{\mathsf{P}^\dagger X} \leqslant \overrightarrow{\varphi} \iff \varphi_\odot$ satisfies (f), and
- $\overrightarrow{\varphi} \cdot \overrightarrow{\varphi} \leqslant \overrightarrow{\varphi} \iff \varphi_\odot$ satisfies (g)

for all $\mathcal{Q}$-distributors $\varphi : \mathsf{P}^\dagger X \to\!\!\!\!\circ\; X$.

First, $1_{\mathsf{P}^\dagger X} \leqslant \overrightarrow{\varphi} \iff \varphi_\odot$ satisfies (f). Indeed,

$$\begin{aligned} \overrightarrow{(\mathsf{y}^\dagger_X)^*} = 1_{\mathsf{P}^\dagger X} \leqslant \overrightarrow{\varphi} &\iff \varphi^\odot \leqslant (\mathsf{y}^\dagger_X)^{*\odot} = (\mathsf{y}^\dagger_X)_! && \text{(Lemma 2.8(2))} \\ &\iff 1_{\mathsf{P}X} \leqslant \varphi_\odot \cdot (\mathsf{y}^\dagger_X)_!. && (\varphi^\odot \dashv \varphi_\odot) \end{aligned}$$

Second, $\overrightarrow{\varphi} \cdot \overrightarrow{\varphi} \leqslant \overrightarrow{\varphi} \iff \varphi_\odot$ satisfies (g), *i.e.*,

$$\varphi_\odot \cdot ((\mathsf{y}_X)_\mathsf{i})^! \cdot (\varphi_\odot)_{\mathsf{i}!} \leqslant \varphi_\odot \cdot \sup_{\mathsf{P}\mathsf{P}^\dagger X} \cdot (\mathsf{y}^\dagger_{\mathsf{P}\mathsf{P}^\dagger X})^!.$$

Since

$$\begin{aligned} \varphi_\odot \cdot ((\mathsf{y}_X)_\mathsf{i})^! \cdot (\varphi_\odot)_{\mathsf{i}!} &= \varphi_\odot \cdot ((\mathsf{y}_X)_\mathsf{i})^! \cdot (\varphi_\odot)^{\mathsf{i}!} && \text{(Lemma 2.11(1))} \\ &= \varphi_\odot \cdot ((\mathsf{y}_X)_\mathsf{i})^! \cdot ((\varphi^\odot)_\mathsf{i})^! && (\varphi^\odot \dashv \varphi_\odot) \\ &= \varphi_\odot \cdot (\overleftarrow{\varphi}_\mathsf{i})^!, && \text{(Lemma 2.10(3))} \end{aligned}$$

and since from (6.iii) one already knows

$$\mathfrak{s}_X = \sup_{\mathsf{P}\mathsf{P}^\dagger X} \cdot (\mathsf{y}^\dagger_{\mathsf{P}\mathsf{P}^\dagger X})^! = (\mathsf{y}^\dagger_{\mathsf{P}^\dagger X})^! \cdot ((\mathsf{y}_{\mathsf{P}^\dagger X})_\mathsf{i})^!,$$

the condition (g) for $\varphi_\odot$ may be alternatively expressed as

$$\varphi_\odot \cdot (\overleftarrow{\varphi}_\mathsf{i})^! \leqslant \varphi_\odot \cdot (\mathsf{y}^\dagger_{\mathsf{P}^\dagger X})^! \cdot ((\mathsf{y}_{\mathsf{P}^\dagger X})_\mathsf{i})^!.$$

Moreover, from Lemma 2.10(4) one has

$$\varphi^\oplus = (\overleftarrow{\varphi}^* \circ (\mathsf{y}_X)_*)^\oplus = \overleftarrow{\varphi}^\mathsf{i} \cdot (\mathsf{y}_{\mathsf{P}^\dagger X})_\mathsf{i}, \tag{6.vi}$$

and, consequently,

$$\begin{aligned}
&\overrightarrow{\varphi}\cdot\overrightarrow{\varphi}\leqslant\overrightarrow{\varphi} \\
\iff\ & \varphi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X}\cdot\varphi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X}\leqslant \varphi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X} && \text{(Lemma 2.10(3))}\\
\iff\ & \varphi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X}\cdot\varphi^{\oplus}\leqslant\varphi^{\oplus} && \text{(Lemma 2.13)}\\
\iff\ & \varphi^{\oplus}\cdot(\varphi^{\oplus})_{\mathsf{i}}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}\mathsf{P}^{\dagger}X}\leqslant\varphi^{\oplus}=\varphi^{\oplus}\cdot\inf_{\mathsf{P}^{\dagger}\mathsf{P}^{\dagger}X}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}\mathsf{P}^{\dagger}X} && (\mathsf{y}^{\dagger}\text{ is natural})\\
\iff\ & \varphi^{\oplus}\cdot(\varphi^{\oplus})_{\mathsf{i}}\leqslant\varphi^{\oplus}\cdot\inf_{\mathsf{P}^{\dagger}\mathsf{P}^{\dagger}X}=\varphi^{\oplus}\cdot(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})^{\mathsf{i}} && \text{(Lemma 2.13)}\\
\iff\ & (\varphi\circ(\varphi^{\oplus})_{*})^{\oplus}\leqslant(\varphi\circ(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})^{*})^{\oplus}\\
\iff\ & (\varphi\circ(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})^{*})^{\odot}\leqslant(\varphi\circ(\varphi^{\oplus})_{*})^{\odot} && \text{(Lemma 2.8(2))}\\
\iff\ & (\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})_{!}\cdot\varphi^{\odot}\leqslant(\varphi^{\oplus})^{!}\cdot\varphi^{\odot}=((\mathsf{y}_{\mathsf{P}^{\dagger}X})_{\mathsf{i}})^{!}\cdot\overleftarrow{\varphi}^{\mathsf{i}!}\cdot\varphi^{\odot} && \text{(Equation (6.vi))}\\
\iff\ & \varphi_{\odot}\cdot(\overleftarrow{\varphi}_{\mathsf{i}})^{!}\leqslant\varphi_{\odot}\cdot(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})^{!}\cdot((\mathsf{y}_{\mathsf{P}^{\dagger}X})_{\mathsf{i}})^{!}\\
\iff\ & \varphi\text{ satisfies (g);}
\end{aligned}$$

here the penultimate equivalence is an immediate consequence of

$$(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})_{!}\cdot\varphi^{\odot}\dashv\varphi_{\odot}\cdot(\mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X})^{!}\quad\text{and}\quad\overleftarrow{\varphi}^{\mathsf{i}!}\cdot\varphi^{\odot}\dashv\varphi_{\odot}\cdot(\overleftarrow{\varphi}_{\mathsf{i}})^{!}.$$

Step 3. $f:(X,\overrightarrow{\varphi})\longrightarrow(Y,\overrightarrow{\psi})$ is a continuous $\mathcal{Q}$-functor if, and only if, $f:(X,\varphi_{\odot})\longrightarrow(Y,\psi_{\odot})$ satisfies (h), *i.e.*,

$$f_{!}\cdot\varphi_{\odot}\leqslant\psi_{\odot}\cdot f_{\mathsf{i}!}.$$

Indeed,

$$\begin{aligned}
&\overrightarrow{\varphi}\cdot f^{\mathsf{i}}\leqslant f^{\mathsf{i}}\cdot\overrightarrow{\psi}\\
\iff\ & \varphi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}X}\cdot f^{\mathsf{i}}\leqslant f^{\mathsf{i}}\cdot\psi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}Y} && \text{(Lemma 2.10(3))}\\
\iff\ & \varphi^{\oplus}\cdot(f^{\mathsf{i}})_{\mathsf{i}}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}Y}\leqslant f^{\mathsf{i}}\cdot\psi^{\oplus}\cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}Y} && (\mathsf{y}^{\dagger}\text{ is natural})\\
\iff\ & \varphi^{\oplus}\cdot(f^{\mathsf{i}})_{\mathsf{i}}\leqslant f^{\mathsf{i}}\cdot\psi^{\oplus} && \text{(Lemma 2.13)}\\
\iff\ & \varphi^{\oplus}\cdot(f_{\mathsf{i}})^{\mathsf{i}}\leqslant f^{\mathsf{i}}\cdot\psi^{\oplus} && \text{(Lemma 2.11(1))}\\
\iff\ & (\varphi\circ(f_{\mathsf{i}})^{*})^{\oplus}\leqslant(f^{*}\circ\psi)^{\oplus}\\
\iff\ & (f^{*}\circ\psi)^{\odot}\leqslant(\varphi\circ(f_{\mathsf{i}})^{*})^{\odot} && \text{(Lemma 2.8(2))}\\
\iff\ & \psi^{\odot}\cdot f_{!}\leqslant f_{\mathsf{i}!}\cdot\varphi^{\odot}\\
\iff\ & f_{!}\cdot\varphi_{\odot}\leqslant\psi_{\odot}\cdot f_{\mathsf{i}!}; && (\varphi^{\odot}\dashv\varphi_{\odot}\text{ and }\psi^{\odot}\dashv\psi_{\odot})
\end{aligned}$$

here Lemma 2.13 is applicable to the third equivalence because $f^{\mathsf{i}}=(f^{*})^{\oplus}$, as well as $\psi^{\oplus}$, is a right adjoint in $\mathcal{Q}$-**Cat**. This completes the proof. ∎

7. The distributive law of the double copresheaf 2-monad

As the adjunction (6.ii) has its dual

$$\mathsf{P}_{\mathrm{c}}^{\mathrm{coop}}\dashv(\mathsf{P}^{\dagger}_{\mathrm{c}})^{\mathrm{coop}}:\mathcal{Q}\text{-}\mathbf{Cat}\longrightarrow(\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{coop}},\tag{7.i}$$

one naturally constructs the *double copresheaf 2-monad*

$$\mathbb{P}^{\dagger}\mathbb{P} = (\mathsf{P}^{\dagger}\mathsf{P}, \mathfrak{y}^{\dagger}, \mathfrak{s}^{\dagger})$$

on $\mathcal{Q}$-**Cat**, with the units given by

$$\mathfrak{y}^{\dagger}_X = \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \mathsf{y}_X = (\mathsf{y}_X)_{\mathsf{i}} \cdot \mathsf{y}^{\dagger}_X : X \longrightarrow \mathsf{P}^{\dagger}\mathsf{P}X \tag{7.ii}$$

and the multiplication by

$$\mathfrak{s}^{\dagger}_X = (\mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} = ((\mathsf{y}^{\dagger}_{\mathsf{P}X})_{!} \cdot \mathsf{y}_{\mathsf{P}X})^{\mathsf{i}} = \mathsf{s}^{\dagger}_{\mathsf{P}X} \cdot \mathsf{y}^{\mathsf{i}}_{\mathsf{P}^{\dagger}\mathsf{P}X} : \mathsf{P}^{\dagger}\mathsf{P}\mathsf{P}^{\dagger}\mathsf{P}X \longrightarrow \mathsf{P}^{\dagger}\mathsf{P}X. \tag{7.iii}$$

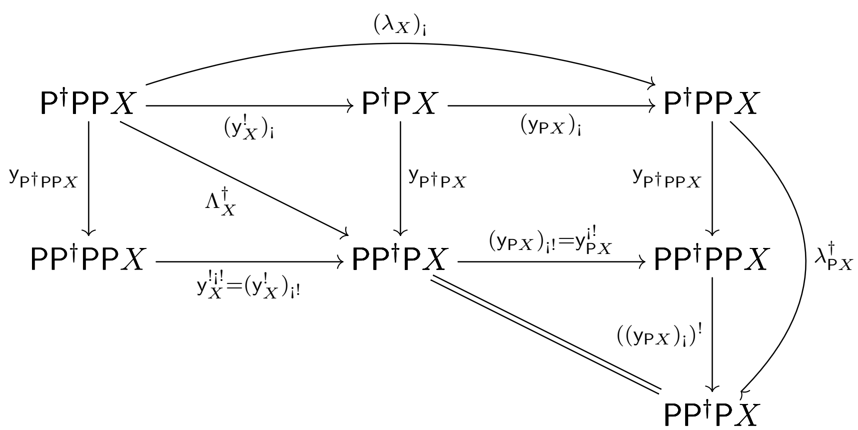
7.1. Theorem. *The double copresheaf 2-monad* $\mathbb{P}^{\dagger}\mathbb{P}$ *distributes over* $\mathbb{P}$ *by* $\Lambda^{\dagger}$ *with*

$$\Lambda^{\dagger}_X = \mathsf{y}^{!\mathsf{i}!}_X \cdot \mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}\mathsf{P}X} : \mathsf{P}^{\dagger}\mathsf{P}\mathsf{P}X \longrightarrow \mathsf{P}\mathsf{P}^{\dagger}\mathsf{P}X.$$

Proof. First note that

$$\Lambda^{\dagger}_X = \lambda^{\dagger}_{\mathsf{P}X} \cdot (\lambda_X)_{\mathsf{i}}. \tag{7.iv}$$

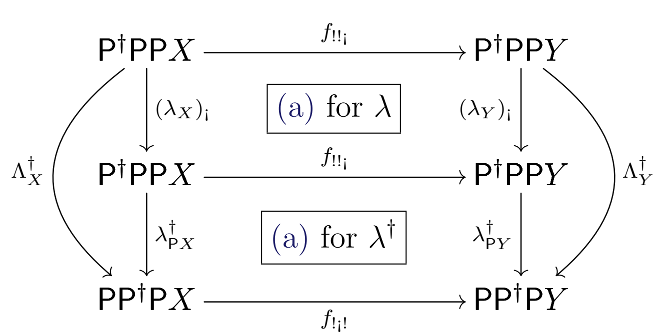
Indeed, with the naturality of y and the full faithfulness of $(\mathsf{y}_{\mathsf{P}X})_{\mathsf{i}}$ one easily sees that the diagram



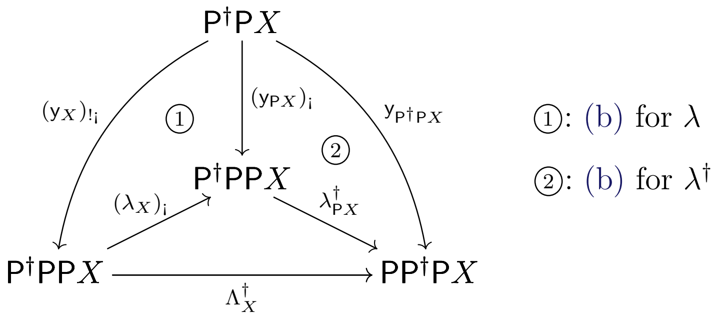
(7.v)

commutes for every $\mathcal{Q}$-category X. Now we check the laws (a)-(e) for $\Lambda^{\dagger}$:

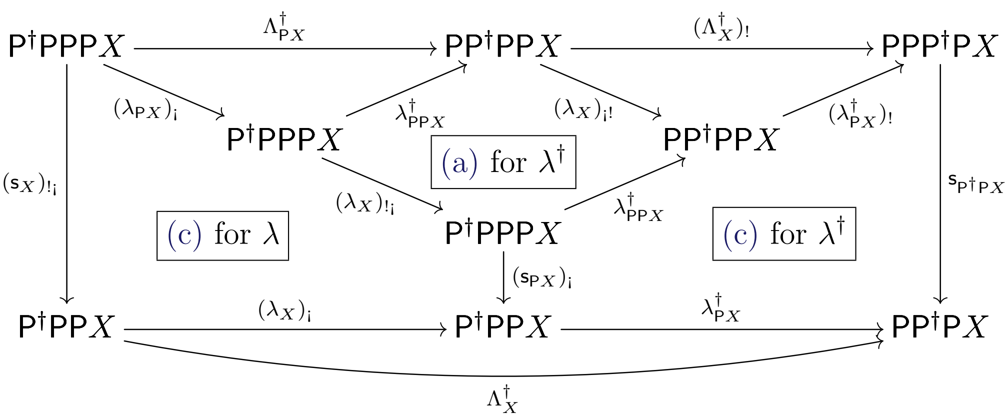
(a) $\Lambda^{\dagger}_Y \cdot f_{!!\mathsf{i}} = f_{!\mathsf{i}!} \cdot \Lambda^{\dagger}_X$ for every $\mathcal{Q}$-functor $f : X \longrightarrow Y$. This is a direct consequence of the naturality of λ and $\lambda^{\dagger}$.



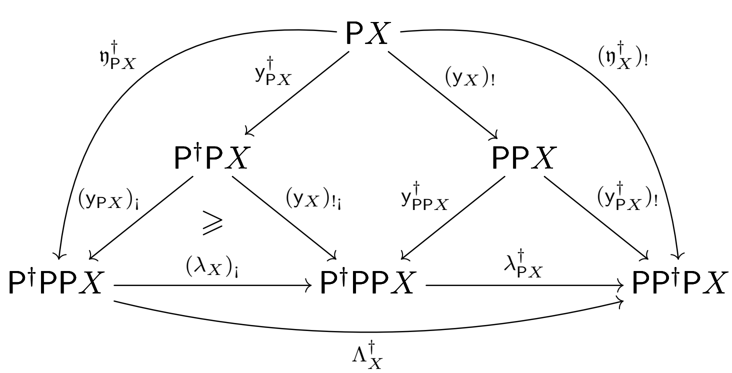
(b) $\mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X} = \Lambda_X^\dagger \cdot (\mathsf{y}_X)_{!_{\mathsf{i}}}$. This is easy since λ and $\lambda^\dagger$ both satisfy the $\mathbb{P}$-unit law (b) strictly.



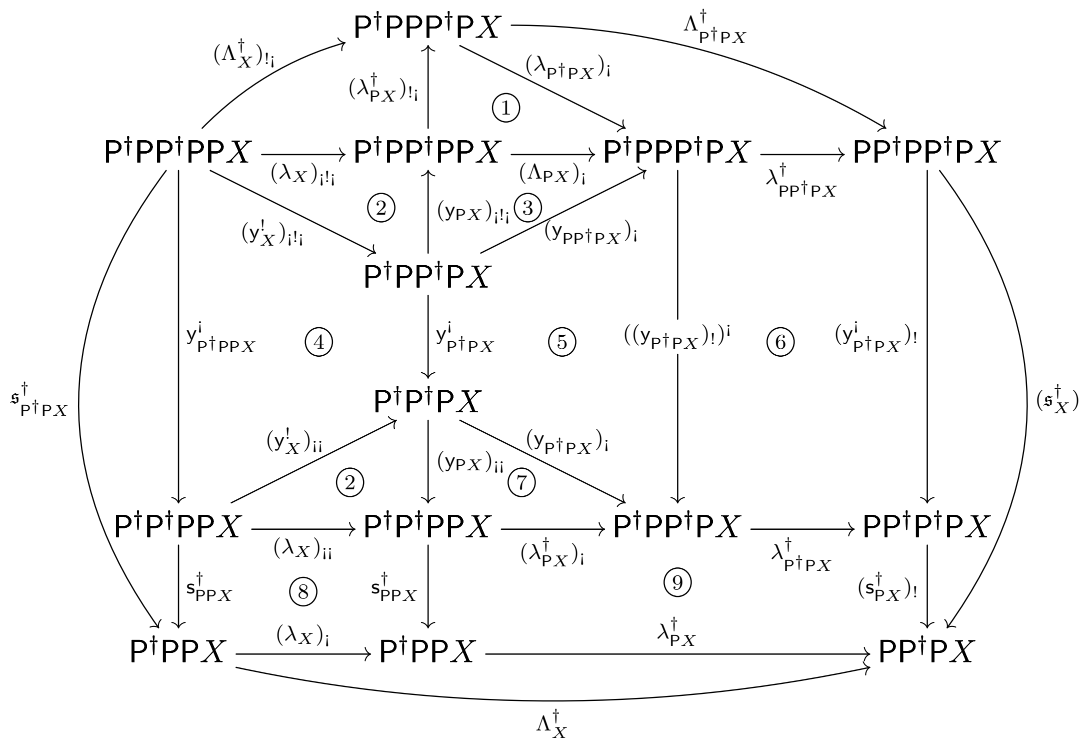
(c) $\mathsf{s}_{\mathsf{P}^\dagger\mathsf{P}X} \cdot (\Lambda_X^\dagger)_! \cdot \Lambda_{\mathsf{P}X}^\dagger = \Lambda_X^\dagger \cdot (\mathsf{s}_X)_{!_{\mathsf{i}}}$. This follows from the naturality of $\lambda^\dagger$ and the fact that λ and $\lambda^\dagger$ both satisfy the $\mathbb{P}$-multiplication law (c) strictly.



(d) $(\mathfrak{y}_X^\dagger)_! \leqslant \Lambda_X^\dagger \cdot \mathfrak{y}_{\mathsf{P}X}^\dagger$. Since λ satisfies the lax $\mathbb{P}$-unit law (d) and $\lambda^\dagger$ strictly satisfies the $\mathbb{P}^\dagger$-unit law (d), one obtains the commutativity of the two lower triangles of the following diagram. Moreover, the naturality of $\mathsf{y}^\dagger$ guarantees the commutativity of the middle rhombus.



(e) $(\mathfrak{s}_X^\dagger)_! \cdot \Lambda^\dagger_{\mathsf{P}^\dagger\mathsf{P}X} \cdot (\Lambda^\dagger_X)_{!_{\mathsf{i}}} = \Lambda^\dagger_X \cdot \mathfrak{s}^\dagger_{\mathsf{P}^\dagger\mathsf{P}X}$. We explain the commutativity of the following diagram:



①: Equation (6.iv).

②: The definition of λ.

③: Λ satisfies the $\mathbb{P}$-unit law (b) strictly.

④: Since $\mathsf{y}^!_X = \sup_{\mathsf{P}X} \dashv \mathsf{y}_{\mathsf{P}X}$, $(\mathsf{y}^!_X)_{\mathsf{i}}$ is a left adjoint in $\mathcal{Q}$-**Cat**, by Lemma 2.7. Thus

$$\begin{aligned}
(\mathsf{y}^!_X)_{\mathsf{ii}} \cdot \mathsf{y}^{\mathsf{i}}_{\mathsf{P}^\dagger\mathsf{PP}X} &= (\mathsf{y}^!_X)_{\mathsf{ii}} \cdot (\sup_{\mathsf{P}^\dagger\mathsf{PP}X})_{\mathsf{i}} && \text{(Lemma 2.7)}\\
&= (\sup_{\mathsf{P}^\dagger\mathsf{P}X})_{\mathsf{i}} \cdot (\mathsf{y}^!_X)_{\mathsf{i}!\mathsf{i}} && \text{(Lemma 2.9)}\\
&= \mathsf{y}^{\mathsf{i}}_{\mathsf{P}^\dagger\mathsf{P}X} \cdot (\mathsf{y}^!_X)_{\mathsf{i}!\mathsf{i}}. && \text{(Lemma 2.7)}
\end{aligned}$$

⑤: Follows from an application of Lemma 2.11(4) to $\mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X} : \mathsf{P}^\dagger\mathsf{P}X \longrightarrow \mathsf{PP}^\dagger\mathsf{P}X$.

⑥: From $\sup_{\mathsf{P}^\dagger\mathsf{P}X} \dashv \mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X}$ one has

$$\begin{aligned}
(\mathsf{y}^{\mathsf{i}}_{\mathsf{P}^\dagger\mathsf{P}X})_! \cdot \lambda^\dagger_{\mathsf{PP}^\dagger\mathsf{P}X} &= (\sup_{\mathsf{P}^\dagger\mathsf{P}X})_{\mathsf{i}!} \cdot \lambda^\dagger_{\mathsf{PP}^\dagger\mathsf{P}X} && \text{(Lemma 2.7)}\\
&= \lambda^\dagger_{\mathsf{P}^\dagger\mathsf{P}X} \cdot (\sup_{\mathsf{P}^\dagger\mathsf{P}X})_{!\mathsf{i}} && (\lambda^\dagger \text{ satisfies (a)})\\
&= \lambda^\dagger_{\mathsf{P}^\dagger\mathsf{P}X} \cdot ((\mathsf{y}_{\mathsf{P}^\dagger\mathsf{P}X})_!)^{\mathsf{i}}. && \text{(Lemma 2.7)}
\end{aligned}$$

⑦: $\lambda^\dagger$ satisfies the $\mathbb{P}$-unit law (b) strictly.

⑧: Since $\lambda_X^{\mathsf{i}} \dashv (\lambda_X)_{\mathsf{i}}$, Lemma 2.9 implies

$$(\lambda_X)_{\mathsf{i}} \cdot \mathsf{s}^{\dagger}_{\mathsf{PP}X} = (\lambda_X)_{\mathsf{i}} \cdot \inf_{\mathsf{P}^{\dagger}\mathsf{PP}X} = \inf_{\mathsf{P}^{\dagger}\mathsf{PP}X} \cdot (\lambda_X)_{\mathsf{ii}} = \mathsf{s}^{\dagger}_{\mathsf{PP}X} \cdot (\lambda_X)_{\mathsf{ii}}.$$

⑨: $\lambda^{\dagger}$ satisfies the $\mathbb{P}^{\dagger}$-multiplication law (e) strictly.

■

7.2. Theorem. $(\Lambda^{\dagger}, \mathcal{Q})$-**Alg** $\cong \mathcal{Q}$-**Cls**.

Proof. **Step 1.** We show that if (X, p) is a $(\Lambda^{\dagger}, \mathcal{Q})$-algebra, then

$$p = \inf_{\mathsf{P}X} \cdot p_{\mathsf{i}} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})_{\mathsf{i}}. \tag{7.vi}$$

Indeed, with (7.ii) and (7.iii) one may translate the conditions (f) and (g) respectively as

$$\mathsf{y}_X \leqslant p \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \mathsf{y}_X \quad \text{and} \quad \sup_{\mathsf{P}X} \cdot p_! \cdot \Lambda^{\dagger}_X \cdot p_{!_{\mathsf{i}}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} \cdot \mathsf{y}^{\mathsf{i}}_{\mathsf{P}^{\dagger}\mathsf{P}X}.$$

To simplify the above conditions, first note that Lemma 2.13 implies

$$\mathsf{y}_X \leqslant p \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \mathsf{y}_X \iff 1_{\mathsf{P}X} \leqslant p \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X}.$$

Second, from $\Lambda^{\dagger}_X = \mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X} \cdot (\mathsf{y}^{!}_X)_{\mathsf{i}} = \mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X} \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}}$ (see the commutative diagram (7.v)) one has

$$\begin{aligned}
\sup_{\mathsf{P}X} \cdot p_! \cdot \Lambda^{\dagger}_X \cdot p_{!_{\mathsf{i}}} &= \sup_{\mathsf{P}X} \cdot p_! \cdot \mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X} \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{!_{\mathsf{i}}} \\
&= \sup_{\mathsf{P}X} \cdot \mathsf{y}_{\mathsf{P}X} \cdot p \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{!_{\mathsf{i}}} && (\mathsf{y} \text{ is natural}) \\
&= p \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{!_{\mathsf{i}}}
\end{aligned}$$

and, moreover,

$$\begin{aligned}
& p \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{!_{\mathsf{i}}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} \cdot \mathsf{y}^{\mathsf{i}}_{\mathsf{P}^{\dagger}\mathsf{P}X} \\
\iff\ & p \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{!_{\mathsf{i}}} \cdot (\mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X})_{\mathsf{i}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} && (\mathsf{y}^{\mathsf{i}}_{\mathsf{P}^{\dagger}\mathsf{P}X} \dashv (\mathsf{y}_{\mathsf{P}^{\dagger}\mathsf{P}X})_{\mathsf{i}}) \\
\iff\ & p \cdot (\sup_{\mathsf{P}X})_{\mathsf{i}} \cdot (\mathsf{y}_{\mathsf{P}X})_{\mathsf{i}} \cdot p_{\mathsf{i}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} && (\mathsf{y} \text{ is natural}) \\
\iff\ & p \cdot p_{\mathsf{i}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}}.
\end{aligned}$$

Therefore, (X, p) is a $(\Lambda^{\dagger}, \mathcal{Q})$-algebra if, and only if,

(f) $1_{\mathsf{P}X} \leqslant p \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X}$ and

(g) $p \cdot p_{\mathsf{i}} \leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}}$.

It follows that

$$\begin{aligned}
p &= \inf_{\mathsf{P}X} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot p \\
&= \inf_{\mathsf{P}X} \cdot p_{\mathsf{i}} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}\mathsf{P}X} && (\mathsf{y}^{\dagger} \text{ is natural}) \\
&\leqslant \inf_{\mathsf{P}X} \cdot p_{\mathsf{i}} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})_{\mathsf{i}} && (\mathbb{P}^{\dagger} \text{ is oplax idempotent}) \\
&\leqslant p \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \inf_{\mathsf{P}X} \cdot p_{\mathsf{i}} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})_{\mathsf{i}} && (p \text{ satisfies (f)}) \\
&\leqslant p \cdot p_{\mathsf{i}} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})_{\mathsf{i}} && (\mathsf{y}^{\dagger}_{\mathsf{P}X} \dashv \inf_{\mathsf{P}X}) \\
&\leqslant p \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})_{\mathsf{i}} && (p \text{ satisfies (g)}) \\
&= p. && (\mathsf{y}^{\dagger}_{\mathsf{P}X} \text{ is fully faithful})
\end{aligned}$$

Step 2. As an immediate consequence of (7.vi), p is a right adjoint in $\mathcal{Q}$-**Cat**. For every $\mathcal{Q}$-category X, as one already has

$$\mathcal{Q}\text{-}\mathbf{Dist}(X, \mathsf{P}X) \cong \mathcal{Q}\text{-}\mathbf{Cat}(\mathsf{P}X, \mathsf{P}X) \cong \mathcal{Q}\text{-}\mathbf{Inf}(\mathsf{P}^{\dagger}\mathsf{P}X, \mathsf{P}X)$$

from Lemma 2.14 with the isomorphism given by

$$(\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} \mathsf{P}X) \mapsto (\overleftarrow{\varphi} : \mathsf{P}X \longrightarrow \mathsf{P}X) \mapsto (\varphi^{\downarrow} : \mathsf{P}^{\dagger}\mathsf{P}X \longrightarrow \mathsf{P}X),$$

in order to establish a bijection between $\mathcal{Q}$-closure operations on $\mathsf{P}X$ and $(\Lambda^{\dagger}, \mathcal{Q})$-algebra structures on X, it suffices to prove

- $1_{\mathsf{P}X} \leqslant \overleftarrow{\varphi} \iff \varphi^{\downarrow}$ satisfies (f), and
- $\overleftarrow{\varphi} \cdot \overleftarrow{\varphi} \leqslant \overleftarrow{\varphi} \iff \varphi^{\downarrow}$ satisfies (g)

for all $\mathcal{Q}$-distributors $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} \mathsf{P}X$.

First, the equivalence ($1_{\mathsf{P}X} \leqslant \overleftarrow{\varphi} \iff \varphi^{\downarrow}$ satisfies (f)) is trivial since $\overleftarrow{\varphi} = \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X}$.

Second, $\overleftarrow{\varphi} \cdot \overleftarrow{\varphi} \leqslant \overleftarrow{\varphi} \iff \varphi^{\downarrow}$ satisfies (g). Indeed,

$$\begin{aligned}
\overleftarrow{\varphi} \cdot \overleftarrow{\varphi} \leqslant \overleftarrow{\varphi} &\iff \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \leqslant \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} && (\overleftarrow{\varphi} = \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X}) \\
&\iff \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \cdot \varphi^{\downarrow} \leqslant \varphi^{\downarrow} && (\text{Lemma } 2.13) \\
&\iff \varphi^{\downarrow} \cdot (\varphi^{\downarrow})_{\mathsf{i}} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}\mathsf{P}X} \leqslant \varphi^{\downarrow} = \varphi^{\downarrow} \cdot \inf_{\mathsf{P}^{\dagger}\mathsf{P}X} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}^{\dagger}\mathsf{P}X} && (\mathsf{y}^{\dagger} \text{ is natural}) \\
&\iff \varphi^{\downarrow} \cdot (\varphi^{\downarrow})_{\mathsf{i}} \leqslant \varphi^{\downarrow} \cdot \inf_{\mathsf{P}^{\dagger}\mathsf{P}X} = \varphi^{\downarrow} \cdot (\mathsf{y}^{\dagger}_{\mathsf{P}X})^{\mathsf{i}} && (\text{Lemma } 2.13) \\
&\iff \varphi^{\downarrow} \text{ satisfies (g).}
\end{aligned}$$

Step 3. $f : (X, \overleftarrow{\varphi}) \longrightarrow (Y, \overleftarrow{\psi})$ is a continuous $\mathcal{Q}$-functor if, and only if, $f : (X, \varphi^{\downarrow}) \longrightarrow (Y, \psi^{\downarrow})$ satisfies (h). Indeed,

$$\begin{aligned}
f_! \cdot \overleftarrow{\varphi} \leqslant \overleftarrow{\psi} \cdot f_! &\iff f_! \cdot \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \leqslant \psi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}Y} \cdot f_! \\
&\iff f_! \cdot \varphi^{\downarrow} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} \leqslant \psi^{\downarrow} \cdot f_{!_{\mathsf{i}}} \cdot \mathsf{y}^{\dagger}_{\mathsf{P}X} && (\mathsf{y}^{\dagger} \text{ is natural}) \\
&\iff f_! \cdot \varphi^{\downarrow} \leqslant \psi^{\downarrow} \cdot f_{!_{\mathsf{i}}}, && (\text{Lemma } 2.13)
\end{aligned}$$

which completes the proof. ∎

8. Distributive laws of $\mathbb{T}$ over $\mathbb{P}$ versus lax extensions of $\mathbb{T}$ to $\mathcal{Q}$-**Dist**

In this section, for an arbitrary 2-monad $\mathbb{T}$ on $\mathcal{Q}$-**Cat**, we outline the bijective correspondence between distributive laws of $\mathbb{T}$ over $\mathbb{P}$ and so-called lax extensions of $\mathbb{T}$ to $\mathcal{Q}$-**Dist**. The techniques adopted here generalize their discrete counterparts as given in [30].

Given a 2-functor $T : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$, a *lax extension* of T to $\mathcal{Q}$-**Dist** is a lax functor

$$\hat{T} : \mathcal{Q}\text{-}\mathbf{Dist} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dist}$$

that coincides with T on objects and satisfies the extension condition (3) below. Explicitly, $\hat{T}$ is given by a family

$$(\hat{T}\varphi : TX \relbar\!\!\circ\!\!\rightarrow TY)_{\varphi\in\mathcal{Q}\text{-}\mathbf{Dist}(X,Y),\ X,Y\in\mathrm{ob}(\mathcal{Q}\text{-}\mathbf{Cat})} \tag{8.i}$$

of $\mathcal{Q}$-distributors such that

(1) $\varphi \preceq \varphi' \Longrightarrow \hat{T}\varphi \preceq \hat{T}\varphi'$,

(2) $\hat{T}\psi \circ \hat{T}\varphi \preceq \hat{T}(\psi \circ \varphi)$,

(3) $(Tf)_* \preceq \hat{T}(f_*)$, $\quad (Tf)^* \preceq \hat{T}(f^*)$,

for all $\mathcal{Q}$-distributors $\varphi, \varphi' : X \relbar\!\!\circ\!\!\rightarrow Y$, $\psi : Y \relbar\!\!\circ\!\!\rightarrow Z$ and $\mathcal{Q}$-functors $f : X \longrightarrow Y$.

It is useful to present the following equivalent conditions of (3), which can be proved analogously to their discrete versions in [30], by straightforward calculation:

8.1. Lemma. *Given a family (8.i) of $\mathcal{Q}$-distributors satisfying (1) and (2), the following conditions are equivalent when quantified over the variables occurring in them ($f : X \longrightarrow Y$, $\varphi : Z \relbar\!\!\circ\!\!\rightarrow Y$, $\psi : Y \relbar\!\!\circ\!\!\rightarrow Z$):*

(i) $1^*_{TX} \preceq \hat{T}(1^*_X)$, $\quad \hat{T}(f^* \circ \varphi) = (Tf)^* \circ \hat{T}\varphi$.

(ii) $1^*_{TX} \preceq \hat{T}(1^*_X)$, $\quad \hat{T}(\psi \circ f_*) = \hat{T}\psi \circ (Tf)_*$.

(iii) $(Tf)_* \preceq \hat{T}(f_*)$, $\quad (Tf)^* \preceq \hat{T}(f^*)$ $\quad$ (i.e., *$\hat{T}$ satisfies (3)).*

8.2. Proposition. *Lax extensions of a 2-functor $T : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ to $\mathcal{Q}$-**Dist** correspond bijectively to lax natural transformations $T\mathsf{P} \longrightarrow \mathsf{P}T$ satisfying the lax $\mathbb{P}$-unit law the lax $\mathbb{P}$-multiplication law.*

Proof. **Step 1.** For each $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$ satisfying (a), (b) and (c), $\Phi(\lambda) := \hat{T} = (\hat{T}\varphi)_\varphi$ with $\overleftarrow{\hat{T}\varphi} := \lambda_X \cdot T\overleftarrow{\varphi}$ is a lax extension of T to $\mathcal{Q}$-**Dist**.

$$\begin{array}{cccc}
\Phi(\lambda) = \hat{T} : & \mathcal{Q}\text{-}\mathbf{Dist}(X,Y) & \longrightarrow & \mathcal{Q}\text{-}\mathbf{Dist}(TX,TY) \\[1em]
& (\overleftarrow{\varphi} : Y \longrightarrow \mathsf{P}X) & \mapsto & \begin{array}{ccc} TY & \xrightarrow{\ \overleftarrow{\hat{T}\varphi}\ } & \mathsf{P}TX \\ & {\scriptstyle T\overleftarrow{\varphi}}\searrow \quad \nearrow{\scriptstyle \lambda_X} & \\ & T\mathsf{P}X & \end{array}
\end{array}$$

Indeed, (1) follows immediately from the 2-functoriality of T. For (2), just note that

$$\begin{aligned}
\overleftarrow{\hat{T}\psi \circ \hat{T}\varphi} &= \mathsf{y}^{!}_{TX} \cdot (\overleftarrow{\hat{T}\varphi})_{!} \cdot \overleftarrow{\hat{T}\psi} && \text{(Lemma 2.12(1))}\\
&= \mathsf{y}^{!}_{TX} \cdot (\lambda_X)_{!} \cdot (T\overleftarrow{\varphi})_{!} \cdot \lambda_Y \cdot T\overleftarrow{\psi} &&\\
&\leqslant \mathsf{y}^{!}_{TX} \cdot (\lambda_X)_{!} \cdot \lambda_{\mathsf{P}X} \cdot T(\overleftarrow{\varphi}_{!}) \cdot T\overleftarrow{\psi} && (\lambda \text{ satisfies (a)})\\
&\leqslant \lambda_X \cdot T\mathsf{y}^{!}_{X} \cdot T(\overleftarrow{\varphi}_{!}) \cdot T\overleftarrow{\psi} && (\lambda \text{ satisfies (c)})\\
&= \lambda_X \cdot T(\overleftarrow{\psi \circ \varphi}) && \text{(Lemma 2.12(1))}\\
&= \overleftarrow{\hat{T}(\psi \circ \varphi)}.
\end{aligned}$$

For (3), it suffices to check Lemma 8.1(i). Since λ satisfies (b), it follows easily that

$$\overleftarrow{1^*_{TX}} = \mathsf{y}_{TX} \leqslant \lambda_X \cdot T\mathsf{y}_X = \lambda_X \cdot T\overleftarrow{1^*_X} = \overleftarrow{\hat{T}(1^*_X)}.$$

For the second identity, Lemma 2.12(1) implies

$$\overleftarrow{\hat{T}(f^* \circ \varphi)} = \lambda_X \cdot T(\overleftarrow{f^* \circ \varphi}) = \lambda_X \cdot T\overleftarrow{\varphi} \cdot Tf = \overleftarrow{\hat{T}\varphi} \cdot Tf = \overleftarrow{(Tf)^* \circ \hat{T}\varphi}.$$

Step 2. For every lax extension $\hat{T}$ of T, $\Psi(\hat{T}) := \lambda = (\lambda_X)_X$ with

$$\lambda_X := \overleftarrow{\hat{T}\varepsilon_X} = \overleftarrow{\hat{T}(\mathsf{y}_X)_*} : T\mathsf{P}X \longrightarrow \mathsf{P}TX$$

is a lax natural transformation satisfying the $\mathbb{P}$-unit law and the $\mathbb{P}$-multiplication law.

(a) $(Tf)_{!} \cdot \lambda_X \leqslant \lambda_Y \cdot T(f_{!})$ for all $\mathcal{Q}$-functors $f : X \longrightarrow Y$. Indeed,

$$\begin{aligned}
(Tf)_{!} \cdot \lambda_X &= \overleftarrow{\hat{T}(\mathsf{y}_X)_* \circ (Tf)^*} && \text{(Lemma 2.12(1))}\\
&\leqslant \overleftarrow{\hat{T}(\mathsf{y}_X)_* \circ (Tf)^* \circ \hat{T}(1^*_Y)} && \text{(Lemma 8.1(i))}\\
&= \overleftarrow{\hat{T}(\mathsf{y}_X)_* \circ \hat{T}(f^*)} && \text{(Lemma 8.1(i))}\\
&\leqslant \overleftarrow{\hat{T}((\mathsf{y}_X)_* \circ f^*)} && (\hat{T} \text{ satisfies (2)})\\
&= \overleftarrow{\hat{T}((f_{!})^* \circ (\mathsf{y}_Y)_*)} && \text{(Lemma 2.11(4))}\\
&= \overleftarrow{(Tf_{!})^* \circ \hat{T}(\mathsf{y}_Y)_*} && \text{(Lemma 8.1(i))}\\
&= \lambda_Y \cdot T(f_{!}). && \text{(Lemma 2.12(1))}
\end{aligned}$$

(b) $\mathsf{y}_{TX} \leqslant \lambda_X \cdot T\mathsf{y}_X$. Indeed,

$$\begin{aligned}
\mathsf{y}_{TX} = \overleftarrow{1^*_{TX}} &\leqslant \overleftarrow{\hat{T}(1^*_X)} && \text{(Lemma 8.1(i))}\\
&= \overleftarrow{\hat{T}(\mathsf{y}^*_X \circ (\mathsf{y}_X)_*)} && (\mathsf{y}_X \text{ is fully faithful})\\
&= \overleftarrow{(T\mathsf{y}_X)^* \circ \hat{T}(\mathsf{y}_X)_*} && \text{(Lemma 8.1(i))}\\
&= \lambda_X \cdot T\mathsf{y}_X. && \text{(Lemma 2.12(1))}
\end{aligned}$$

(c) $\mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X} \leqslant \lambda_X \cdot T\mathsf{s}_X$. Indeed,

$$\begin{aligned}
\mathsf{s}_{TX} \cdot (\lambda_X)_! \cdot \lambda_{\mathsf{P}X} &= \overleftarrow{\hat{T}(\mathsf{y}_{\mathsf{P}X})_* \circ \hat{T}(\mathsf{y}_X)_*} && \text{(Lemma 2.12(1))}\\
&\leqslant \overleftarrow{\hat{T}((\mathsf{y}_{\mathsf{P}X})_* \circ (\mathsf{y}_X)_*)} && \text{($\hat{T}$ satisfies (2))}\\
&= \overleftarrow{\hat{T}(((\mathsf{y}_X)_!)_* \circ (\mathsf{y}_X)_*)} && \text{(y is natural)}\\
&= \overleftarrow{\hat{T}((\mathsf{y}^!_X)^* \circ (\mathsf{y}_X)_*)} && ((\mathsf{y}_X)_! \dashv \mathsf{y}^!_X)\\
&= \overleftarrow{(T\mathsf{y}^!_X)^* \circ \hat{T}(\mathsf{y}_X)_*} && \text{(Lemma 8.1(i))}\\
&= \lambda_X \cdot T\mathsf{s}_X. && \text{(Lemma 2.12(1))}
\end{aligned}$$

Step 3. Φ and Ψ are inverses to each other. For each $\lambda : T\mathsf{P} \longrightarrow \mathsf{P}T$, $\Psi\Phi(\lambda) = \lambda$ since

$$(\Psi\Phi(\lambda))_X = \overleftarrow{\Phi(\lambda)(\mathsf{y}_X)_*} = \lambda_X \cdot T\overleftarrow{(\mathsf{y}_X)_*} = \lambda_X \cdot T1_{\mathsf{P}X} = \lambda_X.$$

Conversely, for every lax extension $\hat{T}$, one has

$$\overleftarrow{(\Phi\Psi(\hat{T}))\varphi} = \overleftarrow{\hat{T}(\mathsf{y}_X)_*} \cdot T\overleftarrow{\varphi} = \overleftarrow{(T\overleftarrow{\varphi})^* \circ \hat{T}(\mathsf{y}_X)_*} = \overleftarrow{\hat{T}(\overleftarrow{\varphi}^* \circ (\mathsf{y}_X)_*)} = \overleftarrow{\hat{T}\varphi},$$

where the last three equalities follow respectively from Lemmas 2.12(1), 8.1(i) and 2.10(4). ■

For a 2-monad $\mathbb{T} = (T, m, e)$ on $\mathcal{Q}$-**Cat**, a lax extension $\hat{T}$ of T to $\mathcal{Q}$-**Dist** becomes a lax extension of the 2-monad $\mathbb{T}$ if it further satisfies

(4) $\varphi \circ e^*_X \preceq e^*_Y \circ \hat{T}\varphi$,

(5) $\hat{T}\hat{T}\varphi \circ m^*_X \preceq m^*_Y \circ \hat{T}\varphi$

for all $\mathcal{Q}$-distributors $\varphi : X \to\!\!\!\!\!\circ\;\, Y$. By adjunction, (4) and (5) may be equivalently expressed as

(4') $(e_Y)_* \circ \varphi \preceq \hat{T}\varphi \circ (e_X)_*$,

(5') $(m_Y)_* \circ \hat{T}\hat{T}\varphi \preceq \hat{T}\varphi \circ (m_X)_*$.

8.3. Theorem. *Lax extensions of a 2-monad* $\mathbb{T} = (T, m, e)$ *on* $\mathcal{Q}$-**Cat** *to* $\mathcal{Q}$-**Dist** *correspond bijectively to distributive laws of* $\mathbb{T}$ *over* $\mathbb{P}$.

Proof. With Proposition 8.2 at hand, it suffices to prove

- $\hat{T}$ satisfies (4) $\iff$ λ satisfies (d), and
- $\hat{T}$ satisfies (5) $\iff$ λ satisfies (e)

for every lax extension $\hat{T}$ of the 2-functor T and $\lambda = \Psi(\hat{T})$ with $\lambda_X = \overleftarrow{\hat{T}(\mathsf{y}_X)_*}$: $T\mathsf{P}X \longrightarrow \mathsf{P}TX$.

First, ($\hat{T}$ satisfies (4) $\iff$ λ satisfies (d)). Since Lemma 2.12(1) and the naturality of e imply

$$(e_X)_! \cdot \overleftarrow{\varphi} = \overleftarrow{\varphi \circ e_X^*} \quad \text{and} \quad \lambda_X \cdot e_{\mathsf{P}X} \cdot \overleftarrow{\varphi} = \lambda_X \cdot T\overleftarrow{\varphi} \cdot e_Y = \overleftarrow{\hat{T}\varphi} \cdot e_Y = \overleftarrow{e_Y^* \circ \hat{T}\varphi}$$

for all $\varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y$, it follows that

$$\begin{aligned}(e_X)_! \leqslant \lambda_X \cdot e_{\mathsf{P}X} &\iff \forall \varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y : \ (e_X)_! \cdot \overleftarrow{\varphi} \leqslant \lambda_X \cdot e_{\mathsf{P}X} \cdot \overleftarrow{\varphi} \\ &\iff \forall \varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y : \ \varphi \circ e_X^* \preceq e_Y^* \circ \hat{T}\varphi.\end{aligned}$$

Second, ($\hat{T}$ satisfies (5) $\iff$ λ satisfies (e)). Similarly as above, one has

$$(m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \cdot TT\overleftarrow{\varphi} = (m_X)_! \cdot \lambda_{TX} \cdot T\overleftarrow{\hat{T}\varphi} = (m_X)_! \cdot \overleftarrow{\hat{T}\hat{T}\varphi} = \overleftarrow{\hat{T}\hat{T}\varphi \circ m_X^*}$$

and

$$\lambda_X \cdot m_{\mathsf{P}X} \cdot TT\overleftarrow{\varphi} = \lambda_X \cdot T\overleftarrow{\varphi} \cdot m_Y = \overleftarrow{\hat{T}\varphi} \cdot m_Y = \overleftarrow{m_Y^* \circ \hat{T}\varphi}$$

by Lemma 2.12(1) and the naturality of m. Consequently,

$$\begin{aligned}&(m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \leqslant \lambda_X \cdot m_{\mathsf{P}X} \\ \iff &\forall \varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y : \ (m_X)_! \cdot \lambda_{TX} \cdot T\lambda_X \cdot TT\overleftarrow{\varphi} \leqslant \lambda_X \cdot m_{\mathsf{P}X} \cdot TT\overleftarrow{\varphi} \\ \iff &\forall \varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y : \ \hat{T}\hat{T}\varphi \circ m_X^* \preceq m_Y^* \circ \hat{T}\varphi.\end{aligned}$$

■

A *strict* extension of $T : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ is a 2-functor

$$\hat{T} : \mathcal{Q}\text{-}\mathbf{Dist} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dist}$$

that coincides with T on objects and satisfies

(3∗) $\hat{T}(f^* \circ \varphi) = (Tf)^* \circ \hat{T}\varphi$

for all $f : X \longrightarrow Y$, $\varphi : Z \longrightarrow\!\!\!\!\!\circ\;\, Y$. It is moreover a *strict* extension of the 2-monad $\mathbb{T} = (T, m, e)$ on $\mathcal{Q}$-**Cat** if

(4∗) $\varphi \circ e_X^* = e_Y^* \circ \hat{T}\varphi$,

(5∗) $\hat{T}\hat{T}\varphi \circ m_X^* = m_Y^* \circ \hat{T}\varphi$

for all $\varphi : X \longrightarrow\!\!\!\!\!\circ\;\, Y$. In other words, a lax extension $\hat{T}$ of $\mathbb{T}$ is strict if all the inequalities in (2), Lemma 8.1(i), (4) and (5) are equalities. From the above proofs one immediately sees that strict extensions of $\mathbb{T}$ to $\mathcal{Q}$-**Dist** correspond bijectively to strict distributive laws of $\mathbb{T}$ over $\mathbb{P}$.

For a lax extension $\hat{T}$ of $\mathbb{T}$ we can now define:

8.4. Definition. A $(\mathbb{T},\mathcal{Q})$-*category* (X,α) consists of a $\mathcal{Q}$-category X and a $\mathcal{Q}$-distributor $\alpha : X \to\!\!\!\!\circ\; TX$ satisfying the lax unit and lax multiplication laws

$$1_X^* \preceq e_X^* \circ \alpha \quad \text{and} \quad \hat{T}\alpha \circ \alpha \preceq m_X^* \circ \alpha.$$

A $(\mathbb{T},\mathcal{Q})$-*functor* $f : (X,\alpha) \longrightarrow (Y,\beta)$ is a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ with

$$\alpha \circ f^* \preceq (Tf)^* \circ \beta.$$

$(\mathbb{T},\mathcal{Q})$-categories and $(\mathbb{T},\mathcal{Q})$-functors constitute a 2-category $(\mathbb{T},\mathcal{Q})$-**Cat**, and we write $(\mathbb{T},\hat{T},\mathcal{Q})$-**Cat** to stress the dependency on the chosen extension $\hat{T}$ if there is any danger of ambiguity.

8.5. Theorem. *If λ and $\hat{T}$ are related by the correspondence of Theorem 8.3, then*

$$(\lambda,\mathcal{Q})\text{-}\mathbf{Alg} \cong (\mathbb{T},\hat{T},\mathcal{Q})\text{-}\mathbf{Cat}.$$

Proof. For any $\mathcal{Q}$-category X, as one already has

$$\mathcal{Q}\text{-}\mathbf{Dist}(X,TX) \cong \mathcal{Q}\text{-}\mathbf{Cat}(TX,\mathsf{P}X)$$

with the isomorphism given by

$$(\alpha : X \to\!\!\!\!\circ\; TX) \mapsto (\overleftarrow{\alpha} : TX \longrightarrow \mathsf{P}X),$$

in order to establish a bijection between $(\mathbb{T},\mathcal{Q})$-category structures on X and $(\lambda,\mathcal{Q})$-algebra structures on X, it suffices to prove

- $1_X^* \preceq e_X^* \circ \alpha \iff \mathsf{y}_X \leqslant \overleftarrow{\alpha} \cdot e_X$, and
- $\hat{T}\alpha \circ \alpha \preceq m_X^* \circ \alpha \iff \mathsf{y}_X^! \cdot \overleftarrow{\alpha}_! \cdot \lambda_X \cdot T\overleftarrow{\alpha} \leqslant \overleftarrow{\alpha} \cdot m_X$,

for all $\mathcal{Q}$-distributors $\alpha : X \to\!\!\!\!\circ\; TX$. Indeed, the first equivalence is easy since $\overleftarrow{1_X^*} = \mathsf{y}_X$ and $\overleftarrow{e_X^* \circ \alpha} = \overleftarrow{\alpha} \cdot e_X$ by Lemma 2.12(1). For the second equivalence, just note that $\overleftarrow{m_X^* \circ \alpha} = \overleftarrow{\alpha} \cdot m_X$ and

$$\begin{aligned}
\overleftarrow{\hat{T}\alpha \circ \alpha} &= \overleftarrow{(\hat{T}(\overleftarrow{\alpha}^* \circ (\mathsf{y}_X)_*) \circ \alpha} && \text{(Lemma 2.10(4))}\\
&= \overleftarrow{(T\overleftarrow{\alpha})^* \circ \hat{T}(\mathsf{y}_X)_* \circ \alpha} && \text{(Lemma 8.1(i))}\\
&= \overleftarrow{\hat{T}(\mathsf{y}_X)_* \circ \alpha} \cdot T\overleftarrow{\alpha} && \text{(Lemma 2.12(1))}\\
&= \mathsf{y}_X^! \cdot \overleftarrow{\alpha}_! \cdot \lambda_X \cdot T\overleftarrow{\alpha}, && \text{(Lemma 2.12(1) and } \lambda_X = \overleftarrow{\hat{T}(\mathsf{y}_X)_*})
\end{aligned}$$

Finally, a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ is a $(\mathbb{T},\mathcal{Q})$-functor $f : (X,\alpha) \longrightarrow (Y,\beta)$ if, and only if, $f : (X,\overleftarrow{\alpha}) \longrightarrow (Y,\overleftarrow{\beta})$ is a lax λ-homomorphism since

$$\alpha \circ f^* \preceq (Tf)^* \circ \beta \iff f_! \cdot \overleftarrow{\alpha} = \overleftarrow{\alpha \circ f^*} \leqslant \overleftarrow{(Tf)^* \circ \beta} = \overleftarrow{\beta} \cdot Tf$$

by Lemma 2.12(1). ∎

8.6. Example.

(1) For the identity 2-monad $\mathbb{I}$ on $\mathcal{Q}$-**Cat**, the identity 2-functor on $\mathcal{Q}$-**Dist** is a strict extension of $\mathbb{I}$ and it is easy to see $(\mathbb{I},\mathcal{Q})$-**Cat** $\cong$ **Mon**($\mathcal{Q}$-**Dist**).

(2) The distributive law λ of $\mathbb{P}$ over itself described in Theorem 4.1 corresponds to the lax extension $\hat{\mathsf{P}}$ of $\mathbb{P}$ (which is strict as an extension of P) with

$$\hat{\mathsf{P}}\varphi := \varphi^{\odot *} : \mathsf{P}X \xrightarrow{\circ} \mathsf{P}Y$$

for $\varphi : X \xrightarrow{\circ} Y$. From Theorem 4.2 one soon knows $(\mathbb{P},\mathcal{Q})$-**Cat** $\cong$ $\mathcal{Q}$-**Cls**.

(3) The strict distributive law $\lambda^{\dagger}$ of $\mathbb{P}^{\dagger}$ over $\mathbb{P}$ given in Theorem 5.1 determines the strict extension $\check{\mathsf{P}}^{\dagger}$ of $\mathbb{P}^{\dagger}$ with

$$\check{\mathsf{P}}^{\dagger}\varphi := (\varphi^{\oplus})_* : \mathsf{P}^{\dagger}X \xrightarrow{\circ} \mathsf{P}^{\dagger}Y.$$

Theorem 5.3 shows that $(\mathbb{P}^{\dagger},\mathcal{Q})$-**Cat** $\cong$ **Mon**($\mathcal{Q}$-**Dist**).

(4) Theorem 6.1 gives the distributive law Λ of $\mathbb{P}\mathbb{P}^{\dagger}$ over $\mathbb{P}$ that corresponds to the lax extension $\widehat{\mathsf{P}\mathsf{P}^{\dagger}}$ of $\mathbb{P}\mathbb{P}^{\dagger}$ (which is strict as an extension of $\mathsf{P}\mathsf{P}^{\dagger}$) with

$$\widehat{\mathsf{P}\mathsf{P}^{\dagger}}\varphi := \hat{\mathsf{P}}\check{\mathsf{P}}^{\dagger}\varphi = ((\varphi^{\oplus})_*)^{\odot *} = \varphi^{\oplus ! *} : \mathsf{P}\mathsf{P}^{\dagger}X \xrightarrow{\circ} \mathsf{P}\mathsf{P}^{\dagger}Y.$$

From Theorem 6.3 one has $(\mathbb{P}\mathbb{P}^{\dagger},\mathcal{Q})$-**Cat** $\cong$ $\mathcal{Q}$-**Int**.

(5) The distributive law $\Lambda^{\dagger}$ of $\mathbb{P}^{\dagger}\mathbb{P}$ over $\mathbb{P}$ (see Theorem 7.1) is related to the lax extension $\widehat{\mathsf{P}^{\dagger}\mathsf{P}}$ of $\mathbb{P}^{\dagger}\mathbb{P}$ (which is strict as an extension of $\mathsf{P}^{\dagger}\mathsf{P}$) with

$$\widehat{\mathsf{P}^{\dagger}\mathsf{P}}\varphi := \check{\mathsf{P}}^{\dagger}\hat{\mathsf{P}}\varphi = (\varphi^{\odot * \oplus})_* = (\varphi^{\odot \mathrm{i}})_* : \mathsf{P}^{\dagger}\mathsf{P}X \xrightarrow{\circ} \mathsf{P}^{\dagger}\mathsf{P}Y.$$

Theorem 7.2 shows that $(\mathbb{P}^{\dagger}\mathbb{P},\mathcal{Q})$-**Cat** $\cong$ $\mathcal{Q}$-**Cls**.

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