One setting for all: metric, topology, uniformity, approach structure

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Abstract

For a complete lattice $\mathbf{V}$ which, as a category, is monoidal closed, and for a suitable **Set**-monad T we consider $(\mathsf{T},\mathbf{V})$-algebras and introduce $(\mathsf{T},\mathbf{V})$-proalgebras, in generalization of Lawvere's presentation of metric spaces and Barr's presentation of topological spaces. In this lax-algebraic setting, uniform spaces appear as proalgebras. Since the corresponding categories behave functorially both in T and in $\mathbf{V}$, one establishes a network of functors at the general level which describe the basic connections between the structures mentioned by the title. Categories of $(\mathsf{T},\mathbf{V})$-algebras and of $(\mathsf{T},\mathbf{V})$-proalgebras turn out to be topological over **Set**.

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Key words: $\mathbf{V}$-matrix, $\mathbf{V}$-promatrix, $(\mathsf{T},\mathbf{V})$-algebra, $(\mathsf{T},\mathbf{V})$-proalgebra, co-Kleisli composition, ordered set, metric space, topological space, uniform space, approach space, prometric space, protopological space, proapproach space, topological category.

0 Introduction

Since the late sixties it has been known that monads over the category **Set**, via their Eilenberg-Moore construction [9], describe precisely the varieties of general algebras (with arbitrarily many infinitary operations and free algebras, see for example [13, 10]), and therefore provide a common categorical description not only of the standard categories of algebra, such as groups, rings, R-modules, R-algebras, etc., but also of some categories outside the realm of algebra, such as the category of compact Hausdorff spaces. The equational description of this latter category by Manes [16] in terms of the "operation" that sends an ultrafilter to a point of convergence satisfying two basic "equations" fully explained many of the algebraic properties of **CompHaus** and enjoyed wide-spread attention. By comparison, there was only moderate interest in Barr's subsequent observation that, when relaxing the operation to a relation and the equalities to inequalities, the Eilenberg-Moore construction actually describes precisely the category **Top** of all topological spaces, in terms of two simple axioms on a convergence relation between ultrafilters and points [2]. It is the aim of this paper to show that, with one additional ingredient to Barr's

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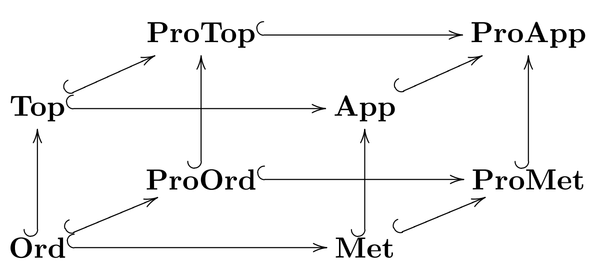
presentation, one is able to describe uniformly all structures that seem to matter in topology, namely metrics, topologies, uniformities and approach structures (as introduced by [14]), and to display the basic functors connecting them in general terms.

This one additional ingredient is an arbitrary complete lattice $\mathbf{V}$ with a monoidal-closed structure that takes the place of the 2-element chain which implicitly governs the axioms defining topological spaces. Lawvere in his fundamental paper [12] considered for $\mathbf{V}$ the extended real half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ (ordered opposite to the natural order) and displayed individual metric spaces as $\mathbf{V}$-categories. In [7], we combined Barr's and Lawvere's ideas and introduced $(\mathsf{T}, \mathbf{V})$-algebras, for a monad T on $\mathbf{Set}$ and a symmetric monoidal-closed category $\mathbf{V}$, obtaining topological spaces for $\mathsf{T} = \mathsf{U}$ the ultrafilter monad and $\mathbf{V} = \mathbf{2}$, premetric (=∞pq-metric) spaces for $\mathsf{T} = \mathsf{1}$ the identity monad and $\mathbf{V} = \overline{\mathbb{R}}_+$, and approach spaces via the natural combination of the previous two structures, with $\mathsf{T} = \mathsf{U}$ and $\mathbf{V} = \overline{\mathbb{R}}_+$. However, we were not able to include uniformities in this setting, although their inclusion in the setting given in [4] indicated that it should be possible to do so.

This paper fills the gap. Instead of considering sets X with a single $\mathbf{V}$-relational Eilenberg-Moore structure $TX \nrightarrow X$, we define $(\mathsf{T}, \mathbf{V})$-*proalgebras* as sets which come with a directed set of $\mathbf{V}$-relational structures $TX \nrightarrow X$. The category of *quasi-uniform spaces* is equivalent to the category of $(\mathsf{1}, \mathbf{2})$-proalgebras, denoted here by $\mathbf{ProOrd}$ since $(\mathsf{1}, \mathbf{2})$-algebras form precisely the category $\mathbf{Ord}$ of preordered sets. Likewise, the category of $(\mathsf{1}, \overline{\mathbb{R}}_+)$-proalgebras is denoted by $\mathbf{ProMet}$ since $(\mathsf{1}, \overline{\mathbb{R}}_+)$-algebras form precisely the category $\mathbf{Met}$ of premetric spaces; it is closely related to the category of *approach-uniform spaces* considered in [15]. By further exploiting the fact that the formation of the categories of $(\mathsf{T}, \mathbf{V})$-algebras and of $(\mathsf{T}, \mathbf{V})$-proalgebras behaves functorially in both T and $\mathbf{V}$, we arrive at a commutative diagram which, together with the various adjoints to the embeddings, not only comprehensively describes the basic connections between the fundamental topological structures already mentioned, but also introduces two new players: *protopological spaces* ($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \mathbf{2}$), not to be confused though closely related with pretopological and pseudotopological spaces as discussed in [11], as well as *proapproach spaces* ($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \overline{\mathbb{R}}_+$). They turn out to be useful when describing some of the connections between the previous categories.



The horizontal embeddings in the diagram above are both reflective and coreflective, a fact that was observed in [14] for $\mathbf{Top} \hookrightarrow \mathbf{App}$, all arising from the both reflective and coreflective embedding $\mathbf{2} \to \overline{\mathbb{R}}_+$. The vertical embeddings are coreflective, with the coreflector always induced by the monad morphism $\mathsf{1} \to \mathsf{U}$. The diagonal embeddings are coreflective as well, with the coreflector always given by "meet". The "induced topology" functor of (quasi-)uniform spaces factors through the coreflector of **Top** $\hookrightarrow$ **ProTop**.

For our $(\mathsf{T}, \mathbf{V})$-algebras and $(\mathsf{T}, \mathbf{V})$-proalgebras, the two basic axioms of an Eilenberg-Moore algebra, namely the unit and the associativity laws, look more like reflexivity and transitivity conditions. We show that, with a formally inverted Kleisli-composition law, the two axioms may also be presented as extensitivity and idempotency conditions. For a topological space, this is exactly the transition from its convergence structure to its Kuratowski closure operation. We extend a result of [4] and give a general proof that the categories occuring in the diagram above are topological over **Set**, by showing the existence of initial structures w.r.t. the underlying **Set**-functors. Furthermore, in the examples considered here we describe explicitly the 2-categorical structure of categories of $(\mathsf{T}, \mathbf{V})$-algebras as given in [7] and extend it naturally to categories of $(\mathsf{T}, \mathbf{V})$-proalgebras.

Finally we point out that, like $(\mathsf{T}, \mathbf{V})$-algebras, also $(\mathsf{T}, \mathbf{V})$-proalgebras may be considered more generally when $\mathbf{V}$ is an arbitrary symmetric monoidal category with coproducts preserved by tensor in each variable, not just a lattice, for the price that one then has to deal with a considerable number of coherence issues which make the treatment considerably more cumbersome (as indicated in [7] and [6] in the case of $(\mathsf{T}, \mathbf{V})$-algebras). But even in this more general context it is possible to prove significant results. For example, the paper [6] shows the local cartesian closedness of categories of $(\mathsf{T}, \mathbf{V})$-algebras which are only reflexive, not necessarily transitive, and thereby provides an important step towards a characterization of exponentiable maps in the category of all $(\mathsf{T}, \mathbf{V})$-algebras. Another type of maps which is notoriously difficult to describe, namely the class of effective descent morphisms, is characterized in [5] for certain cases.

1 Categories of V-matrices

1.1 Hypothesis. Let $\mathbf{V}$ be a complete lattice which, when considered as a category, is symmetric monoidal-closed. Hence, there are a distinguished element $k \in \mathbf{V}$ and an associative and commutative binary operation $\otimes$ on $\mathbf{V}$ for which k is neutral and which preserves suprema in each variable:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \otimes b_i.$$

1.2 Examples. (1) Each frame (=complete lattice in which binary meets distribute over arbitrary joins) is symmetric monoidal-closed, with $\otimes$ given by binary meet and $k = \top$ the top element. In particular, the two-element chain

$$\mathbf{2} = \{\bot, \top\}$$

carries this structure.

(2) [12] Let $\overline{\mathbb{R}}_+ = [0, \infty]$ be the extended real (half-)line, provided with the order opposite to the natural order (so that $\bigvee_{i \in I} a_i = \inf_{i \in I} a_i$ is the natural infimum of the elements a_i), and with $\otimes = +$

the addition (extended by $a + \infty = \infty + a = \infty$) and $k = 0$. In this way we consider $\overline{\mathbb{R}}_+$ as a symmetric monoidal-closed lattice.

For future reference we remark that the embedding (considered as a functor between "thin" categories, i.e. preordered sets)

$$E : \mathbf{2} \to \overline{\mathbb{R}}_+,\ \bot \mapsto \infty,\ \top \mapsto 0,$$

has both a right adjoint retraction

$$R : \overline{\mathbb{R}}_+ \to \mathbf{2},\ (0 < x < \infty) \mapsto \bot,$$

and a left adjoint retraction

$$L : \overline{\mathbb{R}}_+ \to \mathbf{2},\ (0 < x < \infty) \mapsto \top.$$

1.3 V-matrices. The category $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-*matrices* has as its objects sets, and its morphisms $r : X \nrightarrow Y$ are functions $r : X \times Y \to \mathbf{V}$, often written as families $r = (r(x,y))_{x\in X,\, y\in Y}$; the composite arrow of r followed by $s : Y \nrightarrow Z$ is given by matrix multiplication

$$(sr)(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z),$$

and the identity arrow $1_X : X \nrightarrow X$ is the diagonal matrix with values k in the main diagonal and all other values $\bot$, the bottom element of $\mathbf{V}$.

The hom-sets of $\mathrm{Mat}(\mathbf{V})$ are partially ordered by

$$r \le r' \;\Leftrightarrow\; \forall x \in X \;\; \forall y \in Y \;:\; r(x,y) \le r'(x,y),$$

compatibly with composition. Hence, $\mathrm{Mat}(\mathbf{V})$ is actually a *2-category*. In addition, $\mathrm{Mat}(\mathbf{V})$ has an order-preserving *involution*, given by matrix transposition: the transpose $r^\circ : Y \nrightarrow X$ of $r : X \nrightarrow Y$ is defined by $r^\circ(y,x) = r(x,y)$, $x \in X$, $y \in Y$, and satisfies

$$(sr)^\circ = r^\circ s^\circ,\ \ (1_X)^\circ = 1_X.$$

Finally, there is a functor

$$\mathbf{Set} \to \mathrm{Mat}(\mathbf{V})$$

which maps objects identically and treats $f : X \to Y$ in $\mathbf{Set}$ as a matrix $f : X \nrightarrow Y$, putting

$$f(x,y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{else.} \end{cases}$$

When we write $f : X \to Y$ in $\mathrm{Mat}(\mathbf{V})$, it is understood that f is a set map considered as a matrix in this way. In the 2-category $\mathrm{Mat}(\mathbf{V})$ f plays the role of a map (in the sense of Lawvere), satisfying the inequalities $1_X \le f^\circ f$ and $ff^\circ \le 1_Y$. We also note that the matrix composition becomes a lot simpler when one of the players is a map:

$$(sf)(x,z) = s(f(x),z), \quad (gr)(x,z) = \bigvee_{y\,:\,g(y)=z} r(x,y)$$

for $f : X \to Y$, $s : Y \nrightarrow Z$, $r : X \nrightarrow Y$ and $g : Y \to Z$; furthermore, with $t : X \nrightarrow Z$ one has the adjunction rules

$$\frac{t \leq sf}{tf^\circ \leq s} \qquad \frac{gr \leq t}{r \leq g^\circ t}. \tag{1}$$

1.4 Examples. (1) For $\mathbf{V} = \mathbf{2}$, $\mathrm{Mat}(\mathbf{V})$ is the 2-category $\mathrm{Rel}(\mathbf{Set})$ whose objects are sets and whose morphisms are relations $r : X \nrightarrow Y$ which, when we write xry instead of $r(x, y) = \top$, compose as usual as

$$x(sr)z \;\Leftrightarrow\; \exists y \;:\; xry \;\&\; ysz.$$

(2) For $\mathbf{V} = \overline{\mathbb{R}}_+$, the morphisms $r : X \nrightarrow Y$ of $\mathrm{Mat}(\mathbf{V})$ are functions providing for $x \in X$ and $y \in Y$ a (generalized) "distance" $r(x, y) \in \overline{\mathbb{R}}_+$, with composite distances given by

$$(sr)(x, z) = \inf_{y \in Y} (r(x, y) + s(y, z));$$

1_X puts a discrete structure on X. Alternatively, one may think of r as of a *fuzzy relation* from X to Y.

1.5 Changing V. Let $\mathbf{W}$ be, like $\mathbf{V}$, a symmetric monoidal-closed complete lattice, and let $F : \mathbf{V} \to \mathbf{W}$ be a lax morphism of monoidal categories, i.e. a monotone function satisfying

$$Fx \otimes Fy \leq F(x \otimes y) \text{ and } l \leq Fk$$

for all $x, y \in \mathbf{V}$, with l denoting the $\otimes$-neutral element in $\mathbf{W}$. Then F induces a lax functor

$$\mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{W})$$

which maps objects identically and sends $r : X \times Y \to \mathbf{V}$ to Fr. Hence, for all $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$,

$$(Fs)(Fr) \leq F(sr) \;\text{ and }\; 1_X \leq F1_X$$

in $\mathrm{Mat}(\mathbf{W})$. More generally, $f \leq Ff$ for every **Set**-map $f : X \to Y$, and the triangle

$$\begin{array}{ccc} & \mathbf{Set} & \\ \swarrow & & \searrow \\ \mathrm{Mat}(\mathbf{V}) & \longrightarrow & \mathrm{Mat}(\mathbf{W}) \end{array}$$

commutes if $Fk = l$. But even without the latter condition, one always has $(Fs)(Ff) = F(sf)$, whereas the more general equality $(Fs)(Fr) = F(sr)$ would require $F : \mathbf{V} \to \mathbf{W}$ to preserve the tensor product strictly as well as suprema. Of course, the lax extension of F commutes with the involution:

$$F(r^\circ) = (Fr)^\circ.$$

The functors E, L, R of 1.2(2) preserve the tensor product, and E, L preserve joins, but R not so.

2 Categories of V-promatrices

2.1 Preamble. The completion Pro$A$ of a partially ordered set A under down-directed infima is given by its down-directed subsets $D \subseteq A$ (so that every finite subset of D has a lower bound in D, in particular $D \neq \emptyset$), preordered by

$$D \leq E \;:\Leftrightarrow\; \forall e \in E \;\; \exists d \in D \;:\; d \leq e.$$

(This is a special case of the well-known construction of the *procategory* Pro$\mathbf{A}$ for a category $\mathbf{A}$; see, for example, [17].) There is a natural embedding

$$A \to \mathrm{Pro}A, \;\; x \mapsto \{x\},$$

which has a right adjoint if and only if A has all down-directed infima.

2.2 V-promatrices. For $\mathbf{V}$ as in 1.1 one constructs the category

$$\mathrm{ProMat}(\mathbf{V})$$

having objects sets, with hom-sets given by the formula

$$\mathrm{ProMat}(\mathbf{V})(X,Y) = \mathrm{Pro}(\mathrm{Mat}(\mathbf{V})(X,Y)).$$

Hence, a morphism $R : X \nrightarrow Y$ in ProMat($\mathbf{V}$) is a down-directed set of morphisms $r : X \nrightarrow Y$ in Mat($\mathbf{V}$); its composite with $S : Y \nrightarrow Z$ is the set

$$SR = \{sr \,|\, r \in R,\; s \in S\}$$

of composites sr taken in Mat($\mathbf{V}$). The composition is compatible with the preorder of the hom-sets, whence ProMat($\mathbf{V}$) *is a* 2*-category.* There is a natural 2-functor

$$\mathrm{Mat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{V})$$

which maps objects identically and interprets $r : X \nrightarrow Y$ as $\{r\} : X \nrightarrow Y$. Its right adjoints at the hom-level (see 2.1) define a lax functor

$$\Lambda : \mathrm{ProMat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$$

which sends $R : X \nrightarrow Y$ to its meet $\bigwedge R$, taken pointwise.

Trivially, the involution of Mat($\mathbf{V}$) extends to ProMat($\mathbf{V}$) via

$$R^\circ = \{r^\circ \,|\, r \in R\}.$$

2.3 Changing V. For a morphism $F : \mathbf{V} \to \mathbf{W}$ as in 1.5, its lax extension $\mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{W})$ extends further to a lax functor

$$\mathrm{ProMat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{W}),$$

sending $R : X \nrightarrow Y$ to $FR := \{Fr \,|\, r \in R\} : X \nrightarrow Y$ and commuting with the involution.

3 $(\mathsf{T}, \mathbf{V})$-algebras

3.1 V-admissible monads. Recall that a monad $\mathsf{T} = (T, e, m)$ of **Set** (or any other category) is given by an endofunctor T together with natural transformations $e : \mathrm{Id} \to T$, $m : TT \to T$ satisfying the unit and associativity laws

$$m(Te) = 1 = m(eT),\ m(Tm) = m(mT).$$

For $\mathbf{V}$ as in 1.1 we call the monad $\mathbf{V}$-*admissible* if $T : \mathbf{Set} \to \mathbf{Set}$ admits a lax extension

$$T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$$

along $\mathbf{Set} \to \mathrm{Mat}(\mathbf{V})$ which makes the transformations e and m op-lax in $\mathrm{Mat}(\mathbf{V})$ and commutes with the involution. Explicitly, the **Set**-monad T allows for an extension

$$(r : X \nrightarrow Y) \mapsto (Tr : TX \nrightarrow TY)$$

which preserves the partial order described in 1.3 and must satisfy

(0) $(Tr)^\circ \le T(r^\circ)$,

(1) $e_Y r \le (Tr)e_X$,

(2) $m_Y(T^2 r) \le (Tr)m_X$,

(3) $(Ts)(Tr) \le T(sr)$

for all $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$. We hasten to remark that in (0) we have in fact an equality (as one easily sees applying the inequality (0) to r° in lieu of r). In pointwise notation, (1)-(3) mean

(1′) $r(x, y) \le Tr(e_X(x), e_Y(y))$,

(2′) $T^2 r(\mathfrak{X}, \mathfrak{Y}) \le Tr(m_X(\mathfrak{X}), m_Y(\mathfrak{Y}))$,

(3′) $Tr(\mathfrak{x}, \mathfrak{y}) \otimes Ts(\mathfrak{y}, \mathfrak{z}) \le T(sr)(\mathfrak{x}, \mathfrak{z})$,

for all $x \in X$, $y \in Y$, $\mathfrak{X} \in T^2X$, $\mathfrak{Y} \in T^2Y$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $\mathfrak{z} \in TZ$. *When $r = f$ is a* **Set**-*map one has*

$$T(sf) \le T(sf)(Tf^\circ)(Tf) \le T(sff^\circ)(Tf) \le (Ts)(Tf),$$

hence (3) *becomes an equality in this case*, reading as $Ts(Tf(\mathfrak{x}), \mathfrak{z}) = T(sf)(\mathfrak{x}, \mathfrak{z})$ in pointwise notation. Likewise, when $s = g$ is a **Set**-map one has the equality

$$(Tg^\circ)(Tr) = T(g^\circ r).$$

3.2 Remark. A lax extension to $\mathrm{Mat}(\mathbf{V})$ of a $\mathbf{V}$-admissible **Set**-monad T need not be unique. For example, the identity monad admits a non-identical lax extension I to $\mathrm{Mat}(\mathbf{3})$, where $\mathbf{3}$ is the 3-element chain, as follows:

$$(Ir)(x, y) = \begin{cases} \bot & \text{if } r(x, y) = \bot, \\ \top & \text{else,} \end{cases}$$

for all $r : X \nrightarrow Y$, $x \in X$, $y \in Y$.

Hence, when talking about a **V**-admissible **Set**-monad T, we always have a fixed lax extension of T to $\mathrm{Mat}(\mathbf{V})$ in mind.

3.3 $(\mathsf{T}, \mathbf{V})$**-algebras.** For a **V**-admissible monad $\mathsf{T} = (T, e, m)$ one forms the category

$$\mathrm{Alg}(\mathsf{T}, \mathbf{V})$$

of (*reflexive and transitive*) $(\mathsf{T}, \mathbf{V})$*-algebras*, as follows: its objects are pairs (X, a) with a set X and a structure $a : TX \nrightarrow X$ in $\mathrm{Mat}(\mathbf{V})$ satisfying the reflexivity and transitivity laws

(4) $1_X \leq a e_X$,

(5) $a(Ta) \leq a m_X$,

which, when expressed pointwise, read as

(4′) $k \leq a(e_X(x), x)$,

(5′) $Ta(\mathfrak{X}, \mathfrak{y}) \otimes a(\mathfrak{y}, z) \leq a(m_X(\mathfrak{X}), z)$,

for all $x, z \in X$, $\mathfrak{y} \in TX$ and $\mathfrak{X} \in T^2X$.

A morphism $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ is a *lax homomorphism*, i.e. a **Set**-map $f : X \to Y$ satisfying

(6) $fa \leq b(Tf)$,

the pointwise version of which reads as

(6′) $a(\mathfrak{x}, y) \leq b(Tf(\mathfrak{x}), f(y))$

for all $\mathfrak{x} \in TX$, $y \in X$. Composition is as in **Set**.

3.4 Co-Kleisli composition. There is another way of thinking of the two fundamental conditions (4), (5). First of all, there is a least $(\mathsf{T}, \mathbf{V})$-algebra structure on each set X, namely e_X°, which in fact defines the left adjoint to the forgetful functor

$$\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}.$$

Now, (4) of 3.3 reads as the *extensivity law*

(4″) $e_X^\circ \leq a$.

With the *co-Kleisli composition*

$$a * b := a(Tb)m_X^\circ$$

for all $a, b : TX \nrightarrow X$, condition (5) presents itself as

(5″) $a * a \leq a$.

Since the co-Kleisli composition is monotone in each variable, so that ($4''$) implies $a = a * e_X^\circ \leq a * a$, (5) in the presence of (4) has become equivalent to the *idempotency* condition $a * a = a$.

Of course, the co-Kleisli composition is in fact the Kleisli composition for the lax comonad (T, e°, m°) of the selfdual 2-category $\mathrm{Mat}(\mathbf{V})$.

3.5 Ordering homomorphisms. We recall from [7] that $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ *actually carries the structure of a 2-category* since its ordinary hom-sets $\mathrm{Alg}(\mathsf{T}, \mathbf{V})((X, a), (Y, b))$ may be compatibly preordered by

$$f \leq g \;:\Leftrightarrow\; gf^\circ \leq be_Y \;\Leftrightarrow\; 1_X \leq g^\circ be_Y f,$$

which, in pointwise notation, read as

$$f \leq g \;\Leftrightarrow\; \forall x \in X \;:\; k \leq b(e_Y f(x), g(x)).$$

Reflexivity follows immediately from (4). For transitivity we observe that when $f \leq g$ and $g \leq h$, with (1) and (5) one obtains $f \leq h$:

$$\begin{aligned} hf^\circ &\leq (hg^\circ)(gf^\circ) \\ &\leq (be_Y)(be_Y) \\ &\leq b(Tb)e_{TY}e_Y \\ &\leq b(m_Y e_{TY})e_Y \\ &= be_Y. \end{aligned}$$

3.6 Change-of-base functors. If the monad T is both $\mathbf{V}$- and $\mathbf{W}$-admissible, so that T extends to an endofunctor of both $\mathrm{Mat}(\mathbf{V})$ and $\mathrm{Mat}(\mathbf{W})$, for a morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal categories as in 1.5 we call T *F-admissible* if the extension $F : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{W})$ satisfies

(7) $TFr \leq FTr$

for all $r : X \nrightarrow Y$. In this case F *induces a 2-functor*

$$\overline{F} : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{T}, \mathbf{W})$$

which maps an object (X, a) to (X, Fa) and leaves morphisms unchanged. This is due to the fact that F preserves the co-Kleisli composition laxly:

(8) $(Fa) * (Fb) \leq F(a * b)$;

indeed,

$$(Fa)(TFb)m_X^\circ \leq (Fa)(FTb)(Fm_X^\circ) \leq F(a(Tb)m_X^\circ).$$

Consequently, $(Fa) * (Fa) \leq F(a * a) \leq Fa$, which shows preservation of ($5''$) by $\overline{F}$. Also, $e_X^\circ \leq a$ gives immediately $e_X^\circ \leq Fe_X^\circ \leq Fa$, hence preservation of ($4''$) follows. Similarly one deals with the homomorphism condition (6) and preservation of the preorder 3.5.

3.7 Algebraic functors. Let us now consider $\mathbf{V}$-admissible monads $\mathsf{T} = (T, e, m)$ and $\mathsf{S} = (S, d, n)$ of $\mathbf{Set}$ with a morphism $j : \mathsf{S} \to \mathsf{T}$ of monads, i.e. a natural transformation $j : S \to T$

satisfying

$$jd = e \text{ and } jn = mj^2 \text{ (with } j^2 = Tj \cdot jS = jT \cdot Sj).$$

If the extensions of T and S to $\mathrm{Mat}(\mathbf{V})$ make j op-lax, so that

(9) $j_Y(Sr) \leq (Tr)j_X$

for all $r : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{V})$, which in pointwise notation reads as

(9′) $Sr(\mathfrak{x}, \mathfrak{y}) \leq Tr(j_X(\mathfrak{x}), j_Y(\mathfrak{y}))$

for all $\mathfrak{x} \in SX$, $\mathfrak{y} \in SY$, then j *induces a 2-functor*

$$J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V}),$$

sending (X, a) to (X, aj_X) and mapping morphisms identically. Since $1_X \leq ae_X = (aj_X)d_X$, (X, aj_X) remains reflexive, while its transitivity follows from (3), (9), (5); indeed,

$$\begin{array}{rcl} (aj_X)S(aj_X) & = & aj_X(Sa)(Sj_X) \\ & \leq & a(Ta)j_{TX}(Sj_X) \\ & \leq & am_X j_{TX}(Sj_X) \\ & = & (aj_X)n_X. \end{array}$$

A morphism $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ becomes a morphism $f : (X, aj_X) \to (Y, bj_Y)$ since

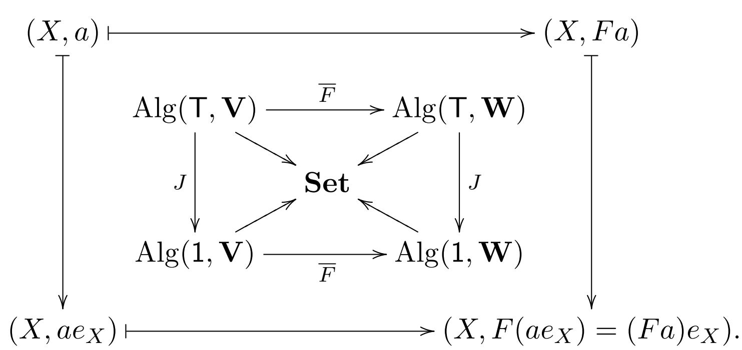
$$f(aj_X) \leq b(Tf)j_X = (bj_Y)(Sf).$$

One easily sees that the preorder on the hom-sets is preserved as well.

Often we consider the case $\mathsf{S} = \mathsf{1} = (\mathrm{Id}, 1, 1)$; then necessarily $j = e$, and we obtain a 2-functor

$$J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{1}, \mathbf{V}), \quad (X, a) \mapsto (X, ae_X).$$

3.8 Proposition. *For every morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal lattices as in* 1.5 *and every F-admissible* **Set***-monad* T *there is a commutative diagram of 2-functors*



An example of this situation is considered in Section 5.

4 $(\mathsf{T},\mathbf{V})$-proalgebras

4.1 Monad extension. For $\mathbf{V}$ as in 1.1 and a $\mathbf{V}$-admissible **Set**-monad $\mathsf{T}=(T,e,m)$, the lax extension $T:\mathrm{Mat}(\mathbf{V})\to\mathrm{Mat}(\mathbf{V})$ admits a further extension

$$\begin{array}{rcl} T:\mathrm{ProMat}(\mathbf{V}) & \to & \mathrm{ProMat}(\mathbf{V}) \\ (R:X\nrightarrow Y) & \mapsto & (TR:=\{Tr\,|\,r\in R\}:X\nrightarrow Y), \end{array}$$

which automatically satisfies the conditions

(0) $(TR)^\circ\leq T(R^\circ)$,

(1) $e_Y R\leq (TR)e_X$,

(2) $m_Y(T^2R)\leq (TR)m_X$,

(3) $(TS)(TR)\leq T(SR)$

for all $R:X\nrightarrow Y$, $S:Y\nrightarrow Z$, *with equality holding when R is a map* (more precisely: a singleton set $\{f\}$ containing a map f; here notationally we don't distinguish between $\{f\}$ and f).

4.2 $(\mathsf{T},\mathbf{V})$-proalgebras. For $\mathbf{V}$ as in 1.1 and a $\mathbf{V}$-admissible monad $\mathsf{T}=(T,e,m)$ of **Set**, a $(\mathsf{T},\mathbf{V})$*-proalgebra* (X,A) is a set X with a morphism $A:TX\nrightarrow X$ in $\mathrm{ProMat}(\mathbf{V})$ satisfying the reflexivity and transitivity conditions

(4) $1_X\leq Ae_X$,

(5) $A(TA)\leq Am_X$.

This means that A is a down-directed set of morphisms $TX\nrightarrow X$ in $\mathrm{Mat}(\mathbf{V})$ satisfying the conditions

(4′) $\forall a\in A\,:\,1_X\leq ae_X$,

(5′) $\forall a\in A\;\;\exists b\in A\,:\,b(Tb)\leq am_X$,

which are expressed pointwise as in 3.3, and in terms of the co-Kleisli composition as

(4″) $\forall a\in A\,:\,e_X^\circ\leq a$,

(5″) $\forall a\in A\,\exists b\in A\,:\,b*b\leq a$.

A *lax homomorphism* $f:(X,A)\to(Y,B)$ of $(\mathsf{T},\mathbf{V})$-proalgebras is a **Set**-map $f:X\to Y$ satisfying

(6) $fA\leq B(Tf)$,

meaning that

(6′) $\forall b\in B\;\;\exists a\in A\,:\,fa\leq b(Tf)$, to be expressed pointwise as in 3.3. With composition of **Set**-maps, this defines the ordinary category

$$\mathrm{ProAlg}(\mathsf{T},\mathbf{V}).$$

4.3 Ordering and coreflection. When we preorder the hom-sets $\mathrm{ProAlg}(\mathsf{T},\mathbf{V})((X,A),(Y,B))$ by

$$f \le g \ :\Leftrightarrow \ \forall b \in B \ \ gf^\circ \le be_Y$$

it is easy to see that $\mathrm{ProAlg}(\mathsf{T},\mathbf{V})$ *becomes a 2-category*: the only slightly critical part is to check that $f \le g$ implies $hf \le hg$ for every morphism $h : (Y,B) \to (Z,C)$; but for all $c \in C$ there is $b \in B$ such that

$$1_X \le g^\circ be_Y f \le g^\circ h^\circ hbe_Y f \le (hg)^\circ c(Th)e_Y f = (hg)^\circ ce_Z(hf),$$

as desired. Furthermore, the full embedding

$$\mathrm{Alg}(\mathsf{T},\mathbf{V}) \hookrightarrow \mathrm{ProAlg}(\mathsf{T},\mathbf{V})$$

is obviously a 2-functor. More importantly, *there is a 2-functor*

$$\Lambda : \mathrm{ProAlg}(\mathsf{T},\mathbf{V}) \to \mathrm{Alg}(\mathsf{T},\mathbf{V}), \ \ (X,A) \mapsto (X,\Lambda A),$$

which is right adjoint to the embedding, with $\Lambda A = \bigwedge A$ (see 2.2). Indeed, $1_X \le Ae_X$ implies $1_X \le (\Lambda A)e_X$, and from $A(TA) \le Am_X$ one obtains for all $a \in A$ some $b \in A$ with

$$(\Lambda A)(T\Lambda A) \le (\Lambda A)(\Lambda TA) \le b(Tb) \le am_X,$$

hence $(\Lambda A)(T\Lambda A) \le \Lambda(Am_X) = (\Lambda A)m_X$. Similarly one shows that a morphism $f : (X,A) \to (Y,B)$ becomes a morphism $f : (X,\Lambda A) \to (Y,\Lambda B)$, with the preorder being preserved, as well as right adjointness.

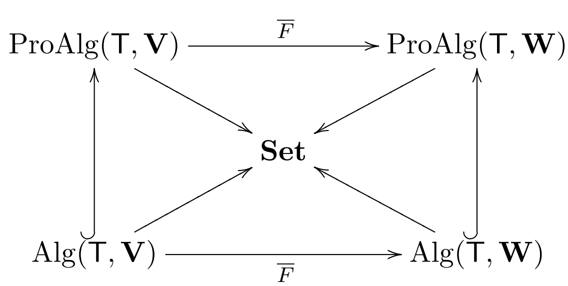
4.4 Change-of-base functors. Let now $F : \mathbf{V} \to \mathbf{W}$ be as in 1.5, and consider an F-admissible monad T as in 3.6. Then the extension 2.3 $F : \mathrm{ProMat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{V})$ satisfies

(7) $TFR \le FTR$

for all $R : X \nrightarrow Y$. This condition enables us to extend $\overline{F}$ of 3.6 along the embedding $\mathrm{Alg}(\mathsf{T},\mathbf{V}) \hookrightarrow \mathrm{ProAlg}(\mathsf{T},\mathbf{W})$ to *obtain a 2-functor*

$$\overline{F} : \mathrm{ProAlg}(\mathsf{T},\mathbf{V}) \to \mathrm{ProAlg}(\mathsf{T},\mathbf{W}),$$

which maps an object (X,A) to (X,FA) and leaves morphisms unchanged. The verifications are as in 3.4. We obtain a commutative diagram of 2-functors:



The diagram remains commutative if the vertical embeddings are replaced by their right adjoints Λ, provided that $F : \mathbf{V} \to \mathbf{W}$ preserves infima.

4.5 Algebraic functors. For a morphism $j : \mathsf{S} \to \mathsf{T}$ of $\mathbf{V}$-admissible monads satisfying 3.5(8) we automatically have

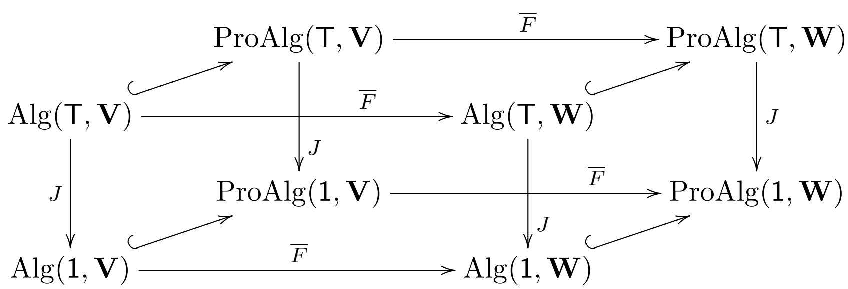
(8) $j_Y(SR) \leq (TR)j_X$

for all $R : X \nrightarrow Y$ in ProMat($\mathbf{V}$). The 2-functor $J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V})$ may therefore be extended to a 2-functor

$$J : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathrm{ProAlg}(\mathsf{S}, \mathbf{V})$$

sending (X, A) to (X, Aj_X) and leaving morphisms unchanged. As in 3.8, the case $\mathsf{S} = \mathsf{1} = (\mathrm{Id}, 1, 1)$ with $j = e$ is of particular importance, and with 4.3 and 4.4, we can extend the commutative diagram obtained in 3.8, as follows:

4.6 Theorem. *For every morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal lattices as in* 1.5 *and every F-admissible* **Set***-monad T there is a commutative diagram of 2-functors which also commutes with the underlying* **Set***-functors:*



If the (diagonal) embeddings are replaced by their right adjoints Λ, the vertical faces remain commutative while the top- and bottom faces commute if F preserves infima.

An example of this situation is considered in the next section.

5 Examples

5.1 Ordered sets. For $\mathbf{V} = \mathbf{2}$ and $\mathsf{T} = \mathsf{1}$, conditions 3.2(4′), (5′) translate into the reflexivity and transitivity conditions for a relation a on X, and (6′) expresses preservation of the relation. Hence, Alg($\mathsf{1}, \mathbf{2}$) is the category **Ord** of preordered sets. Denoting the preorders by $\leq$, we see that 3.5 puts the pointwise preorder on the hom-sets:

$$f \leq g \;\Leftrightarrow\; \forall x \in X : \;\; f(x) \leq g(x).$$

5.2 Metric spaces. For $\mathbf{V} = \overline{\mathbb{R}}_+$ and $\mathsf{T} = \mathsf{1}$, the $(\mathsf{T}, \mathbf{V})$-algebra structure $a : X \times X \to [0, \infty]$ must satisfy the conditions

$$a(x, x) = 0 \;\text{ and }\; a(x, z) \leq a(x, y) + a(y, z)$$

for all $x, y, z \in X$. A lax homomorphism $f : (X, a) \to (Y, b)$ is a non-expansive map:

$$b(f(x), f(y)) \leq a(x, y)$$

for all $x, y \in X$. Hence, $\mathrm{Alg}(\mathbf{1}, \overline{\mathbb{R}}_+)$ is the category **Met** of *premetric spaces* (called metric spaces in [12] and ∞pq-metric spaces in [14]). The hom-sets are preordered via 3.5 by

$$f \leq g \;\Leftrightarrow\; \forall x \in X \;:\; b(f(x), g(x)) = 0.$$

The embedding $E : \mathbf{2} \to \overline{\mathbb{R}}_+$ of 1.2(2) gives with 3.6 the 2-functor

$$\overline{E} : \mathbf{Ord} \to \mathbf{Met}$$

which maps $(X, \leq)$ to the premetric space (X, d) with $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = \infty$ otherwise. The two adjoints $L \dashv E \dashv R$ give adjoints $\overline{L} \dashv \overline{E} \dashv \overline{R}$, providing a premetric space (X, d) with the preorders given by

$$\overline{L} \;:\; x \leq y \;\Leftrightarrow\; d(x, y) < \infty, \quad \overline{R} \;:\; x \leq y \;\Leftrightarrow\; d(x, y) = 0.$$

5.3 Uniform spaces. An object in $\mathrm{ProAlg}(\mathbf{1}, \mathbf{2})$ is a set X which comes with a down-directed (w.r.t. $\subseteq$) set A of relations on X which are reflexive and satisfy the transitivity condition 4.2(5″)

$$\forall a \in A \;\; \exists b \in A \;:\; bb \subseteq a.$$

(with the usual relational product, see 1.4(1)); a morphism $f : (X, A) \to (Y, B)$ satisfies the condition

$$\forall b \in B \;\; \exists a \in A \;:\; (f \times f)(a) \subseteq b,$$

as 4.2(6′) and 3.3(6′) show. Hence, $\mathrm{ProAlg}(\mathbf{1}, \mathbf{2})$ is the category **ProOrd** of *pro-ordered sets* which is obviously equivalent to the category **QUnif** of *quasi-uniform spaces*. (A quasi-uniformity A on X is usually required to be not just a filter base but a filter on $X \times X$; in this paper we do not distinguish between **ProOrd** and **QUnif**.) The preorder on the hom-sets in **ProOrd** is given by

$$f \leq g \;\Leftrightarrow\; \forall b \in B \;:\; (f \times g)\Delta_X \subseteq b,$$

with Δ_X the diagonal in $X \times X$. According to 4.3, the embedding

$$\mathbf{Ord} \to \mathbf{ProOrd}, \;\; (X, a) \mapsto (X, \{a\}),$$

has a right adjoint Λ which preorders a quasi-uniform space (X, A) by

$$x \leq y \;\Leftrightarrow\; \forall a \in A \;:\; (x, y) \in a.$$

5.4 Prometric spaces. An object in $\mathrm{ProAlg}(\mathbf{1}, \overline{\mathbb{R}}_+)$ equips a set X with an up-directed (w.r.t. the pointwise natural order of $[0, \infty]$-valued functions) set A of ("distance") functions $a : X \times X \to [0, \infty]$ satisfying the conditions

$$\forall a \in A \;\; \forall x \in X \;:\; a(x, x) = 0,$$

$$\forall a \in A \ \exists b \in A \ \forall x, y, z \in X \ : \ a(x,z) \leq b(x,y) + b(y,z);$$

a morphism $f : (X, A) \to (Y, B)$ satisfies

$$\forall b \in B \ \exists a \in A \ \forall x, y \in X \ : \ b(f(x), f(y)) \leq a(x,y).$$

The resulting category $\mathbf{ProMet} = \mathrm{ProAlg}(1, \overline{\mathbb{R}}_+)$ of *prometric spaces* contains the category **AQUnif** of *approach-quasi-uniform spaces* as considered by Lowen and Windels [15] (which satisfy an additional saturation condition for the structure A) as a full subcategory. Its hom-sets are preordered by

$$f \leq g \ \Leftrightarrow \ \forall b \in B \ \forall x \in X \ : \ b(f(x), g(x)) = 0.$$

The right adjoint Λ to the embedding

$$\mathbf{Met} \to \mathbf{ProMet}, \ \ (X, d) \mapsto (X, \{d\}),$$

provides a prometric space (X, A) with the premetric

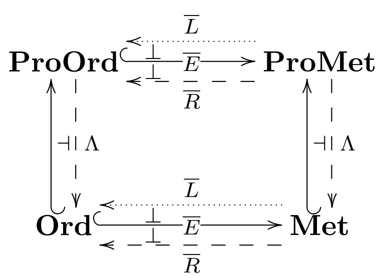
$$d(x,y) := \sup\{a(x,y) \,|\, a \in A\}.$$

E of 1.2(2) induces the 2-functor

$$\overline{E} : \mathbf{ProOrd} \to \mathbf{ProMet}$$

which equips a quasi-uniform space (X, A) with the set $\overline{A} = \{\overline{a} \,|\, a \in A\}$ of distance functions $\overline{a}$ with $\overline{a}(x,y) = 0$ if $(x,y) \in a$ and $\overline{a}(x,y) = \infty$ otherwise. $\overline{E}$ is both a full reflective and coreflective embedding, with adjoints $\overline{L} \dashv \overline{E} \dashv \overline{R}$, induced by $L \dashv E \dashv R$, where $\overline{L}$ assigns to a prometric space (X, A) the quasi-uniformity $\{\{(x,y) \,|\, a(x,y) < \infty\} \,|\, a \in A\}$, and $\overline{R}$ the quasi-uniformity $\{\{(x,y) \,|\, a(x,y) = 0\} \,|\, a \in A\}$.

We have thus described the diagram



which commutes with respect to both the solid and the dashed arrows.

5.5 Topological spaces. The ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ assigns to a set X the set UX of ultrafilters on X; for $f : X \to Y$, the map $Uf : UX \to UY$ takes an ultrafilter $\mathfrak{x}$ on X to its image $f(\mathfrak{x})$ defined by $(B \in f(\mathfrak{x}) \Leftrightarrow f^{-1}(B) \in \mathfrak{x})$. Since U preserves finite coproducts, there is a uniquely determined monad structure e, m on U (see [3]), given by

$$(A \in e_X(x) \ \Leftrightarrow \ x \in A), \ \ (A \in m_X(\mathfrak{X}) \ \Leftrightarrow \ A^\sharp \in \mathfrak{X}),$$

with $A^\sharp := \{\mathfrak{x} \in UX \,|\, A \in \mathfrak{x}\}$, for all $x \in X$, $A \subseteq X$, $\mathfrak{X} \in UUX$. As G. Janelidze observed, the monad $\mathsf{U} = (U, e, m)$ is naturally induced by the adjunction

$$(\mathbf{BooA})^{\mathrm{op}} \underset{\longleftarrow}{\overset{\top}{\longrightarrow}} \mathbf{Set}$$

with both adjoints represented by 2, the two-element set or Boolean algebra. The lax extension of U to $U : \mathrm{Rel}(\mathbf{Set}) \to \mathrm{Rel}(\mathbf{Set})$ transforms $r : X \nrightarrow Y$ into $Ur : UX \nrightarrow UY$ defined by

$$\mathfrak{x}(Ur)\mathfrak{y} \Leftrightarrow \forall A \in \mathfrak{x}\ \ \forall B \in \mathfrak{y}\ \ \exists x \in A\ \ \exists y \in B\ \ :\ \ x\,r\,y$$

for all $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$. Briefly, the **Set**-monad U is **2**-admissible.

A $(\mathsf{U}, \mathbf{2})$-algebra is a set X with a relation $a : UX \nrightarrow X$ which, when we write $(\mathfrak{x} \to x\ :\Leftrightarrow \mathfrak{x}\,a\,x)$ and $(\mathfrak{X} \to \mathfrak{x} \Leftrightarrow \mathfrak{X}\,(Ua)\,\mathfrak{x})$, must satisfy the reflexivity and transitivity conditions

$$\dot{x} := e_X(x) \to x,\ \ (\mathfrak{X} \to \mathfrak{y}\ \&\ \mathfrak{y} \to z\ \Rightarrow\ m_X(\mathfrak{X}) \to z)$$

for all $x, z \in X$, $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$. These are exactly the convergence structures defining a topology on X. Morphisms in $\mathrm{Alg}(\mathsf{U}, \mathbf{2})$ preserve the convergence structures, i.e. are continuous maps. Hence, $\mathrm{Alg}(\mathsf{U}, \mathbf{2})$ is (isomorphic to) the category **Top** of topological spaces (see [2]). It may be considered as a 2-category when we preorder its hom-sets by

$$\begin{aligned} f \leq g \quad &\Leftrightarrow \quad \forall x \in X\ :\ f(\dot{x}) \to g(x) \\ &\Leftrightarrow \quad \forall x \in X\ :\ g(x) \in \overline{f(x)}. \end{aligned}$$

The unique monad morphism $j = e : 1 \to \mathsf{U}$ induces the 2-functor

$$J : \mathbf{Top} \to \mathbf{Ord}$$

which provides a topological space X with the "specialization order" given by $(x \leq y \Leftrightarrow \dot{x} \to y \Leftrightarrow y \in \overline{x})$ for all $x, y \in X$. J has a left adjoint which embeds **Ord** as a full coreflective subcategory into **Top**: it provides a preordered set $(X, \leq)$ with the topology whose open sets are generated by the down sets $\downarrow x = \{z \in X \,|\, z \leq x\}$, $x \in X$.

5.6 Protopological spaces. We wish to give an easy description of the category $\mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$. Recall that a *pretopology* (or Čech closure operation) on a set X is an extensive and finitely-additive function $c : PX \to PX$; hence, $M \subseteq c(M)$, $c(\emptyset) = \emptyset$, $c(M \cup N) = c(M) \cup c(N)$ for all $M, N \subseteq X$. Pretopologies on X are ordered pointwise: $c \leq d$ if $c(M) \subseteq d(M)$ for all $M \subseteq X$. A *protopology* on X is a down-directed set C of pretopologies on X with the transitivity property

$$\forall c \in C\ \ \exists d \in C\ \ :\ \ dd \leq c.$$

Continuity of a map $f : (X, C) \to (Y, D)$ means

$$\forall d \in D\ \ \exists c \in C\ \ \forall M \subseteq X\ \ :\ \ f(c(M)) \subseteq d(f(M)).$$

This defines the category **ProTop** of protopological spaces which can be made into a 2-category by

$$f \leq g \;:\Leftrightarrow\; \forall d \in D \;\; \forall x \in X \;:\; g(x) \in d(\{f(x)\}).$$

In order to see that **ProTop** is equivalent to $\mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$ one follows the same procedure that shows $\mathbf{Top} \cong \mathrm{Alg}(\mathsf{U}, \mathbf{2})$: every pretopology c on X defines a "convergence relation" $a : UX \nrightarrow X$ via

$$\mathfrak{x}\, a\, x \;:\Leftrightarrow\; \forall M \in \mathfrak{x} \;:\; x \in c(M),$$

which satisfies the reflexivity but not necessarily the transitivity condition; hence a is a *pseudotopology* on X. Conversely, every pseudotopology a defines a pretopology c via

$$x \in c(M) \;:\Leftrightarrow\; \exists \mathfrak{x} \in UX \; (M \in \mathfrak{x} \;\&\; \mathfrak{x}\, a\, x).$$

The resulting maps

$$\{\,\text{pretopologies on } X\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\,\text{pseudotopologies on } X\}$$

which satisfy $\psi\varphi = \mathrm{id}$ and $\mathrm{id} \leq \varphi\psi$ (and describe the category of pretopological spaces as a full reflective subcategory of the category of pseudotopological spaces), have an important algebraic property: *they are homomorphisms with respect to the ordinary composition of pretopologies* (*as closure operations*) *and to the co-Kleisli composition of pseudotopologies as introduced in* 3.4:

$$\varphi(\mathrm{id}) = e_X^\circ, \quad \varphi(cd) = \varphi(c) * \varphi(d),$$
$$\psi(e_X^\circ) = \mathrm{id}, \quad \psi(a * b) = \psi(a)\psi(b)$$

(see [8] for details). This homomorphic behaviour helps to prove that ϕ and ψ induce a category equivalence

$$\mathbf{ProTop} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{ProAlg}(\mathsf{U}, \mathbf{2}).$$

The only non-trivial point is to see that, for $(X, A) \in \mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$, the identity map is actually a morphism $\Phi\Psi(X, A) \to (X, A)$. Indeed, for every $a \in A$, we have $b \in A$ with $b * b \leq a$; now it is not difficult to show that every pseudotopology b satisfies $\phi(\psi(b)) \leq b * b$.

The full embedding $\mathbf{Top} \hookrightarrow \mathbf{ProTop}$ has a left adjoint Λ which provides a protopological space (X, C) with a topology whose Kuratowski closure operation is given by

$$\overline{M} = \bigcap \{c(M) \,|\, c \in C\}$$

for all $M \subseteq X$. The 2-functor $J : \mathbf{Top} \to \mathbf{Ord}$ of 5.5 can be extended to a 2-functor

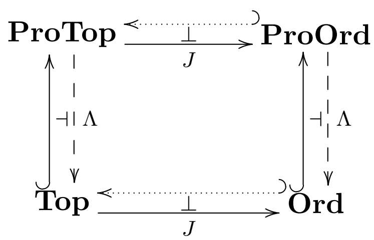
$$J : \mathbf{ProTop} \to \mathbf{ProOrd}$$

which provides a protopological space (X, C) with the quasi-uniformity given by the sets $\{(x, y) \,|\, y \in c(\{x\})\}$, $c \in C$. J has a left adjoint which embeds **ProOrd** as a full reflective subcategory into **ProTop**, as follows: for a quasi-uniformity A on X consider the protopology $\hat{A} = \{\hat{a} \,|\, a \in A\}$ on X with

$$\hat{a}(M) = \{y \in X \,|\, \exists x \in M \;:\; (x, y) \in a\}$$

for all $M \subseteq X$.

In summary, in 5.5 and 5.6 we have described the diagram



which commutes with respect to the solid arrows; also, the two full embeddings **Ord** $\hookrightarrow$ **ProTop** described by the diagram coincide. Let us also remark that the composite

$$\mathbf{ProOrd} \hookrightarrow \mathbf{ProTop} \xrightarrow{\Lambda} \mathbf{Top}$$

is nothing but the induced-topology functor of quasi-uniform spaces which provides a quasi-uniform space (X, A) with the Kuratowski closure operation given by

$$y \in \overline{M} \iff \forall a \in A \, \exists x \in M \; : \; (x, y) \in a.$$

5.7 Approach spaces. The objects of $\mathrm{Alg}(\mathsf{U}, \overline{\mathbb{R}}_+)$ are sets X which come with a function $a : UX \times X \to [0, \infty]$ satisfying the reflexivity and transitivity conditions

$$a(\overset{\bullet}{x}, x) = 0, \quad a(m_X(\mathfrak{X}), z) \leq Ua(\mathfrak{X}, \mathfrak{y}) + a(\mathfrak{y}, z)$$

for all $x, z \in X$, $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$, with

$$Ua(\mathfrak{X}, \mathfrak{y}) \quad = \quad \sup_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \; \inf_{\substack{\mathfrak{x} \in \mathcal{A} \\ y \in B}} \; a(\mathfrak{x}, y).$$

A morphism $f : (X, a) \to (Y, b)$ must satisfy

$$b(f(\mathfrak{x}), f(x)) \leq a(\mathfrak{x}, x)$$

for all $x \in X$, $\mathfrak{x} \in UX$. As observed in [4, 7], this is precisely Lowen's category of approach spaces which becomes a 2-category via

$$f \leq g \iff \forall x \in X \; : \; b(\overset{\bullet}{f(x)}, g(x)) = 0.$$

The monad morphism $j = e : 1 \to \mathsf{U}$ induces the 2-functor

$$J : \mathbf{App} \to \mathbf{Met},$$

providing an approach space (X, a) with the premetric d given by $d(x, y) = a(\overset{\bullet}{x}, y)$. It is the right adjoint to the full embedding **Met** $\hookrightarrow$ **App** described by [14] which puts on a premetric space (X, d) the approach structure

$$a(\mathfrak{x}, y) = \inf_{F \in \mathfrak{x}} \sup_{x \in F} d(x, y).$$

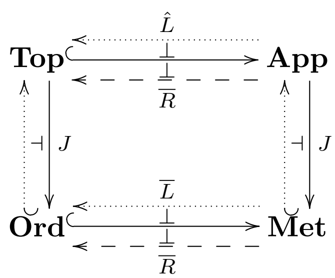
The full reflective and coreflective embedding

$$\overline{E} : \mathbf{Top} \hookrightarrow \mathbf{App}$$

is also described by [14]; as observed in [7], it is induced by $L \dashv E \dashv R$ although the situation is more complicated than in 5.4. $\overline{E}$ provides a topological space X with the approach structure a defined by $a(\mathfrak{x}, x) = 0$ if $\mathfrak{x}$ converges to x, and $a(\mathfrak{x}, x) = \infty$ otherwise. Its right adjoint $\overline{R}$ puts on an approach space (X, a) the topology which lets $\mathfrak{x}$ converge to x precisely when $a(\mathfrak{x}, x) = 0$. But while U is R-admissible, it fails to be L-admissible. Nevertheless, it is useful to consider for an approach space (X, a) the adjoint (X, La) as in 3.4, which tries to let $\mathfrak{x}$ converge to x precisely when $a(\mathfrak{x}, x) < \infty$. This structure satisfies the reflexivity but not the transitivity condition for topologies defined via convergence. In other words, (X, La) is just a pseudotopological space to which, however, one may apply the reflector of $\mathbf{Top} \hookrightarrow \mathbf{PsTop}$ to obtain the topological space $\hat{L}(X, a)$. The resulting functor $\hat{L}$ is left adjoint to $\overline{E}$, as observed in [7].

Incidently, the reflector $\mathbf{PsTop} \to \mathbf{Top}$ is obtained by iterating the endofunctor $(X, b) \mapsto (X, b * b)$ transfinitely (see [4]), another useful application of the co-Kleisli composition 3.4.

We have thus described the diagram below which commutes with respect to both the solid and the dotted arrows, but not the dashed arrows; also, the two full embeddings $\mathbf{Ord} \hookrightarrow \mathbf{App}$ described by it coincide.



5.8 Proapproach spaces.

We call the objects (X, A) of $\mathrm{ProAlg}(\mathsf{U}, \overline{\mathbb{R}}_+) = \mathbf{ProApp}$ *proapproach spaces.* These are sets with an up-directed (w.r.t. the pointwise natural order of $[0, \infty]$-valued functions) set A of functions $a : UX \times X \to [0, \infty]$ satisfying the conditions

$$\forall a \in A \;\; \forall x \in X \;:\; a(\dot{x}, x) = 0,$$

$$\forall a \in A \;\; \exists b \in B \;\; \forall \mathfrak{X} \in UUX,\; \mathfrak{y} \in UX,\; z \in X \;:\; a(m_X(\mathfrak{X}), z) \leq Ub(\mathfrak{X}, \mathfrak{y}) + b(\mathfrak{y}, z),$$

with

$$Ub(\mathfrak{X}, \mathfrak{y}) \;=\; \sup_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \; \inf_{\substack{\mathfrak{x} \in \mathcal{A} \\ y \in B}} \; b(\mathfrak{x}, y);$$

a morphism $f : (X, A) \to (Y, B)$ satisfies

$$\forall b \in B \;\; \exists a \in A \;\; \forall \mathfrak{x} \in UX,\; y \in X \;:\; b(f(\mathfrak{x}), f(y)) \leq a(\mathfrak{x}, y).$$

Its hom-sets are preordered by

$$f \leq g \;\Leftrightarrow\; \forall b \in B \;\; \forall x \in X \;:\; b(\dot{f(x)}, g(x)) = 0.$$

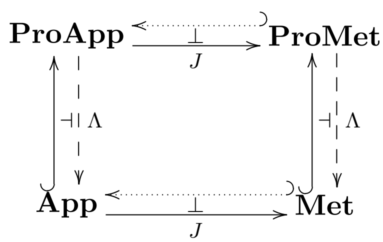
The right adjoint Λ to the full embedding **App** $\hookrightarrow$ **ProApp** provides a proapproach space (X, A) with the approach structure given by

$$d(\mathfrak{x}, y) = \sup\{a(\mathfrak{x}, y) \mid a \in A\}.$$

The 2-functor $J : (X, A) \mapsto (X, \{a(e_X \times 1_X) \mid a \in A\})$ is right adjoint to the full embedding

$$\mathbf{ProMet} \hookrightarrow \mathbf{ProApp}$$

which extends the embedding **Met** $\hookrightarrow$ **App** "structure by structure". This describes the commutative diagram which "lifts" the diagram given in 5.6:



Finally we consider the full embedding

$$\overline{E} : \mathbf{ProTop} \hookrightarrow \mathbf{ProApp}$$

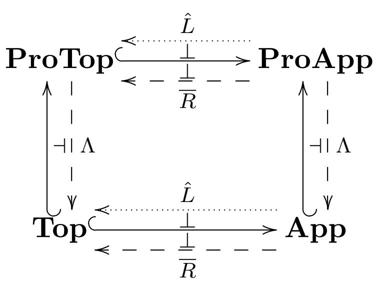
induced by $E : \mathbf{2} \to [0, \infty]$; it extends $\overline{E} : \mathbf{Top} \to \mathbf{App}$, again "structure by structure", providing a protopological space (X, C) with the proapproach structure $\tilde{C} = \{\tilde{c} \mid c \in C\}$, where $\tilde{c}(\mathfrak{x}, x) = 0$ if $\mathfrak{x}$ converges to x in the pretopology c, and $\tilde{c}(\mathfrak{x}, x) = \infty$ otherwise. Its right adjoint $\overline{R}$ defines for a proapproach space (X, A) a protopology $A^* = \{a^* \mid a \in A\}$ with

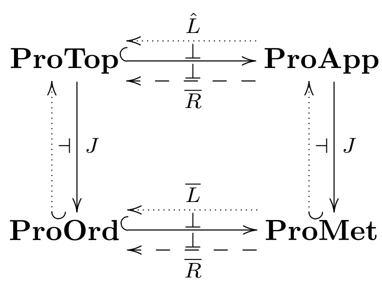
$$a^*(M) = \{x \in X \mid \exists \mathfrak{x} \in UX \ (M \in \mathfrak{x} \ \& \ a(\mathfrak{x}, x) = 0)\}$$

for all $M \subseteq X$.

$\overline{E}$ also has a left adjoint $\hat{L}$ whose construction we can only sketch, as follows: *pseudo-protopological spaces* are sets with a down-directed set of pseudotopologies; with morphisms as in $\mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$, they form the category **PsProTop** in which **ProTop** $\simeq \mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$ is reflective. Using the definition of $\overline{L}$ as in 4.3 one obtains a functor $\overline{L}$: **ProApp** $\to$ **PsProTop** which, when composed with the reflector of **ProTop** $\hookrightarrow$ **PsProTop**, gives us $\hat{L}$.

In extension of the diagrams given in 5.4 and 5.7 we obtain the following diagrams which commute to the same extent as their predecessors:





We also note that, in analogy to the induced-topology functor of quasi-uniform spaces, one has the induced-approach functor of approach-uniform spaces and, more generally, of prometric spaces, given by

$$\mathbf{ProMet} \hookrightarrow \mathbf{ProApp} \xrightarrow{\Lambda} \mathbf{App}.$$

It assigns to a prometric space (X, A) the approach structure d given by

$$d(\mathfrak{x}, y) = \sup_{a \in A} \inf_{F \in \mathfrak{x}} \sup_{x \in F} a(x, y).$$

6 $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ as a topological category

6.1 Initial structures. Recall that, in order to show topologicity of the underlying **Set**-functor of $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ (with T and $\mathbf{V}$ as in 3.1), by definition we must, for every family $(Y_i, b_i)_{i \in I}$ of $(\mathsf{T}, \mathbf{V})$-algebras (*with no size restriction on* I) and every family $(f_i : X \to Y_i)_{i \in I}$ of **Set**-maps, provide a $(\mathsf{T}, \mathbf{V})$-algebra structure a on the fixed set X (the so-called *initial* structure) such that, for any $(\mathsf{T}, \mathbf{V})$-algebra (Z, c), a **Set**-map $h : Z \to X$ is actually a morphism in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ when all composites $f_i h$ are. First we note the following simple but useful lemma.

6.2 Lemma. (1) *For any morphisms* $r_i : X \nrightarrow Y$ *in* $\mathrm{Mat}(\mathbf{V})$,

$$T(\bigwedge_i r_i) \leq \bigwedge_i Tr_i.$$

(2) *For any morphisms* $f : X \to Y$, $a : TX \nrightarrow X$, $b : TY \nrightarrow Y$ *in* $\mathrm{Mat}(\mathbf{V})$, *if* $fa \leq b(Tf)$ *one has*

$$(Tf)(Ta) \leq (Tb)(T^2 f).$$

Proof. (1) This inequality follows trivially from the fact that T preserves the preorder on the hom-sets of $\mathrm{Mat}(\mathbf{V})$.

(2) Monotonicity of T leads from $fa \leq b(Tf)$ to

$$(Tf)(Ta) \leq T(fa) \leq T(b(Tf)) = (Tb)(T^2 f)$$

when we use the fact that T preserves composition in $\mathrm{Mat}(\mathbf{V})$ strictly whenever the first factor is a map. □

6.3 Proposition. *In the notation of* 6.1, *the initial structure* a *on* X *can be constructed as* $a = \bigwedge_i b_i^*$, *with* $b_i^* := f_i^\circ b_i (Tf_i)$, *which, in pointwise notation, reads as*

$$b_i^*(\mathfrak{x}, y) = b_i(Tf_i(\mathfrak{x}), f_i(y))$$

for all $\mathfrak{x} \in TX$, $y \in X$ *and* $i \in I$.

Proof. 3.3(4) follows from

$$1_X \leq \bigwedge_i 1_X \leq \bigwedge_i f_i^\circ f_i \leq \bigwedge_i f_i^\circ b_i e_{Y_i} f_i = \bigwedge_i f_i^\circ b_i (Tf_i) e_X = (\bigwedge_i b_i^*) e_X.$$

In order to show (5) of 3.3, we apply Lemma 6.2 and obtain:

$$\begin{array}{rcl} a(Ta) & \leq & (\bigwedge_i b_i^*)(\bigwedge_j Tb_j^*) \\ & \leq & \bigwedge_i b_i^*(Tb_i^*) \\ & \leq & \bigwedge_i f_i^\circ b_i(Tf_i)(Tf_i^\circ)(Tb_i)(T^2 f_i) \\ & \leq & \bigwedge_i f_i^\circ b_i(Tb_i)(T^2 f_i) \\ & \leq & \bigwedge_i f_i^\circ b_i m_{Y_i}(T^2 f_i) \\ & \leq & (\bigwedge_i f_i^\circ b_i(Tf_i))m_X. \end{array}$$

Trivially, for every $i \in I$, $f_i : (X, a) \to (Y_i, b_i)$ is a morphism in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$. Given (Z, c) and $h : Z \to X$ as in 6.1, such that $f_i hc \leq b_i(Tf_i)(Th)$ for all $i \in I$, we obtain

$$hc \leq \bigwedge_i f_i^\circ f_i hc \leq \bigwedge_i f_i^\circ b_i(Tf_i)(Th) = a(Th),$$

as desired. □

6.4 Theorem. *For* T *and* $\mathbf{V}$ *as in* 3.1*, the forgetful functor* $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ *is topological, in the sense that it admits all initial structures. It therefore admits also all final structures, has both a left and a right adjoint, and makes* $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ *a complete, cocomplete, wellpowered and cowellpowered category with a generator and a cogenerator. Furthermore, for every monad morphism* $j : \mathsf{S} \to \mathsf{T}$ *as in* 3.5*, the functor* $J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V})$ *has a left adjoint, and for every lax monoidal morphism* $F : \mathbf{V} \to \mathbf{W}$ *as in* 3.4*, the functor* $\overline{F} : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{T}, \mathbf{W})$ *has a left adjoint if* $F : \mathbf{V} \to \mathbf{W}$ *preserves infima.*

Proof. Existence of initial structures was shown in 6.3 and implies existence of final structures (defined dually, see [1]). The initial and final structures for empty families define the left- and right adjoint of $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$, respectively (see also 3.4). Limits and colimits in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ are therefore constructed by putting the initial and final structures on the limits and colimits formed in **Set**, respectively. Wellpoweredness and cowellpoweredness get lifted from **Set** to $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ since every set X admits only a (small) set of structures $a : TX \nrightarrow X$ (smallness of $\mathbf{V}$ is crucial here). To obtain a generator in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$, one applies the left adjoint of the underlying **Set**-functor to a generator of **Set**, and proceeds dually to have a cogenerator. With the explicit formula for the initial structures given in 6.3, it is easy to see that these are always preserved by J, while $\overline{F}$ preserves them when F preserves infima. Consequently, in this case J and $\overline{F}$ preserve all limits, hence have left adjoints by Freyd's Special Adjoint Functor Theorem. □

7 $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$ as a topological category

7.1 Preamble. In this section we show topologicity of the underlying **Set**-functor $P : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ and first confirm the existence of initial structures for *down-directed* structured cones. Hence, we are given a down-directed class I and a functor $D : I \to \mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$ which provides us with objects $D_i = (Y_i, B_i)$ and morphisms $g_{i,j} : (Y_i, B_i) \to (Y_j, B_j)$ whenever

$i \leq j$ in I; furthermore, we have a cone $f : \Delta X \to PD$, i.e. a family of compatible maps $f_i : X \to Y_i$. Putting

$$A := \{b_i^* \mid b \in B_i,\ i \in I\},$$

with b_i^* defined as in 6.3 by $b_i^* = f_i^\circ b(Tf_i)$ we obtain:

7.2 Proposition. *A is the initial structure on X w.r.t. $(f_i)_{i \in I}$.*

Proof. First of all, A is in fact a (small) set since when $\mathbf{V}$ is small also the hom-set $\mathrm{Mat}(\mathbf{V})(TX, X)$ is small. Next we show that A is down-directed. Since $I \neq \emptyset$, also $A \neq \emptyset$. Having $b \in B_i$, $c \in B_j$ with $i, j \in I$, we can find $t \leq i, j$ in I and then $d \in B_t$ such that

$$g_{t,i} d \leq b(Tg_{t,i}),\ \ g_{t,j} d \leq c(Tg_{t,j}).$$

Hence,

$$d_t^* = f_t^\circ d(Tf_t) \leq f_t^\circ g_{t,i}^\circ b(Tg_{t,i})(Tf_t) = f_i^\circ b(Tf_i) = b_i^*,$$

and likewise $d_t^* \leq c_j^*$.

The proof of conditions (4), (5) of 4.2 now proceeds with the same arguments as in the proof of 6.3 and is therefore omitted; likewise the trivial verification that A does indeed have the initiality property. □

We arrive at the following Theorem whose additional assertions are obvious.

7.3 Theorem. *For* T *and* $\mathbf{V}$ *as in* 3.1*, like* $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to$ **Set** *also the forgetful functor* $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to$ **Set** *is topological, and all assertions of Theorem* 6.4 *remain true mutatis mutandis.*

Proof. After Proposition 7.2 it suffices to show the existence of initial structures for *finite* families $(f_i : X \to Y_i)_{i \in I}$ of **Set**-maps with $(\mathsf{T}, \mathbf{V})$-proalgebras (Y_i, B_i). Indeed, given any such family of arbitrary size, we obtain the initial structure on X by applying 7.2 to the cone

$$(f_F : X \to Y_F)_{F \subseteq I \text{ finite}},$$

with $f_F = < f_i >_{i \in F} : X \to Y_F := \prod_{i \in F} Y_i$.

The empty case is taken care of by the left adjoint to P which is constructed just like the left adjoint to $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to$ **Set** (see 3.4). In case $I = \{1, 2\}$ one lets

$$A := \{b_1^* \wedge b_2^* \mid b_1 \in B_1,\ b_2 \in B_2\},$$

with $b_i^* = f_i^\circ b_i(Tf_i)$. Since B_1 and B_2 are down-directed, also A is down-directed, and all other verifications proceed as in 6.3. □

7.4 Corollary. *Each of the categories appearing in the cubic diagram of the Introduction is topological over* **Set***.*

□

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