Mathematics: Beauty and the Beast

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EXAMPLE 1: Prime Numbers

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, …

$40 = 2 \cdot 20 = 2 \cdot 2 \cdot 10 = 2 \cdot 2 \cdot 2 \cdot 5$

$2006 = 2 \cdot 1003 = 2 \cdot 17 \cdot 59$

The largest known prime number (as of December 2005) has 9,152,052 digits. (It's the 43rd Mersenne prime number.)

Euclid (~300BC): There are infinitely many prime numbers.

“Whenever you give me a finite list p_1, p_2, ….., p_n of n primes, then I can give you (in principle) another prime p that is not yet in the list.”

$N = p_1 \cdot p_2 \cdot \ldots . \cdot p_n + 1$

N has a prime factor p.

That prime factor p cannot be one of $p_1, p_2, \ldots, p_n$,

for if it were, p would not only be a divisor of N,

but also of $N - 1 = p_1 \cdot p_2 \cdot \ldots . \cdot p_n$: impossible!

Twin primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, …

Are there infinitely many twin primes?

J. G. van der Corput (1939):
There are infinitely many triples of primes in arithmetic progression.

Ben Green and Terence Tao (2004): There exist sequences of primes in arithmetic progression of any given length.

RSA Cryptography

(after Rivest, Shamir and Adleman, 1977; slightly earlier: Ellis, Cocks and Williamson of the British Secret Service)

- Choose two large prime numbers (of 100 digits, say), that's the secret key.
- Form their product (a 200-digit number), that's the public key.
- Use the public key to encrypt messages.
- Decoding is possible only with the secret key.

CONCLUSIONS 1

-Mathematics has beauty.

-Ancient notions and proofs are as fresh today as 23 centuries ago.

-While technology may help to break computational records, the essence of fundamental mathematical thought seldom relies on it.

-“Pure” mathematics, studied often only for the sake of curiosity, elegance and beauty, suddenly finds crucial applications to science or economic development.

-Eugene Wigner: The unreasonable effectiveness of mathematics in the natural sciences.

EXAMPLE 2

Diameter of sphere: 1 meter	$(2r)$
Length of equator: π meters	$(2\pi\, r)$
Surface: π square meters	$(4\pi\, r^2)$
Volume: $1/6\ \pi$ cubic meters	$(4/3\ \pi\, r^3)$

$\pi = 3.14159265358979323846\ldots$

Kate Bush: “π” (“Aerial”)

Sweet and gentle sensitive man

With an obsessive nature and deep fascination

For numbers

And a complete infatuation with the calculation

Of Pi

Oh he love, he love, he love

He does love his numbers

And they run, they run, they run him

In a great big circle

In a circle of infinity

3.1415926535 897932

3846 264 338 3279 …. (oops – first error at the 54th digit!)

$P = (1 - (1/2)^2) \cdot (1 – (1/3)^2) \cdot (1 – (1/5)^2) \cdot (1 – (1/7)^2 \cdot \ldots.$

$= \quad (3/4) \quad \cdot \quad (8/9) \quad \cdot \quad (24/25) \quad \cdot \quad (48/49) \quad \cdot \ldots$

$1/P = 1/6\ \pi^2$

Luc Lemaire:

“Some facetious god of mathematics has encoded the length of a circle in the list of prime numbers, totally unrelated a priori.”

Riemann's Zeta Function:

In the definition for the number P, replace the squares by an arbitrary (complex) variable z and obtain

$$\zeta(z) = 1/P$$

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \ldots = 0$$

Riemann's Hypothesis (1859):

All other zeroes of this function are located on the vertical line of the complex plane that intersects the x-axis at ½.

CONCLUSIONS 2:

-Mathematics has magic.

-There are obscure questions that appear to be “at the centre of spider web of mathematical fields and theories”.

-Mathematics is not so much about specific fields like algebra, geometry, probability theory, etc, but about the relations between different areas, allowing us to use methods of one to solve problems of the other.

EXAMPLE 3

Pythagorean triples: ? $x^2 + y^2 = z^2$?

$$3^2 + 4^2 = 5^2$$

$$5^2 + 12^2 = 13^2$$

$$20^2 + 21^2 = 29^2$$

? $x^3 + y^3 = z^3$?

$$6^3 + 8^3 = 9^3 - 1$$

Fermat's Last "Theorem" (1637): For n = 3, 4, 5,…, there are no integer solutions to the Pythagorean equation.

Confirmed for the first time by Andrew Wiles in 1994 (=1637 + 357) !

CONCLUSIONS 3:

-Mathematics is as much about good problems as it is about good solutions.

-Very good problems may take a long time to be solved, if ever.

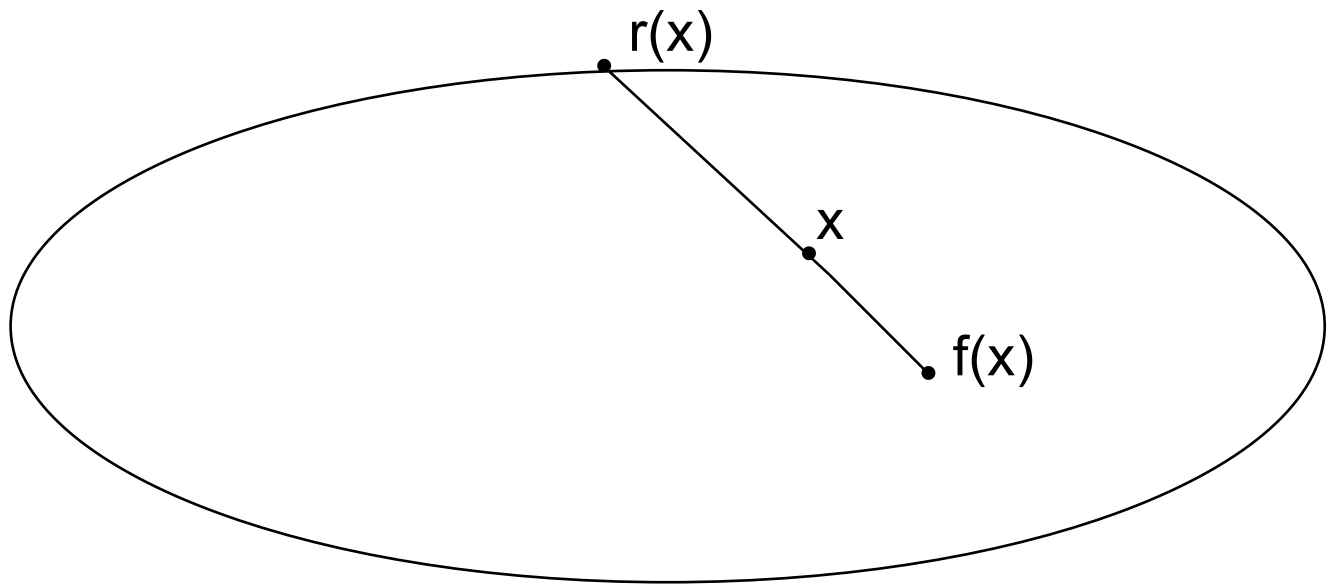
-Easy looking mathematical problems may need a huge "abstract machinery" to be settled.

EXAMPLE 4   Fixed Points (after Lawvere and Schanuel)

Banach's Fixed Point Theorem:

Every "shrinking" self-map has a unique fixed point, and there is a simple algorithm on how to get to that point.

Brouwer's Fixed Point Theorem: Every continuous self-map of the disk/sphere (etc) has at least one fixed point.



Problem reduction: If there is a continuous self-map WITHOUT any fixed points, then there exists a continuous retraction map onto the boundary.

Hence: If we can prove that there exists no continuous retraction map onto the boundary, then there can NOT be any continuous self-map WITHOUT fixed points!

So Brouwer's Fixpoint Theorem is proved, as soon as we can prove that there is no continuous retraction map onto the boundary! (Is it?)

But the statement that there is no continuous retraction map onto the boundary seems to be a lot more plausible than the assertion of Brouwer's Theorem:

Consider a rubber string firmly fastened at two ends and try to pull and push it all to one end without tearing it: impossible!

(A mere plausibility argument, not a mathematical proof!)

CONCLUSIONS 4:

-Most mathematical research manifests itself in a steady output of small improvements to existing knowledge which, taken in isolation, may seem minor.
However, taken in combination, they may well represent a very significant and surprising body of work.

- Abstraction is needed to “get to the bottom of things”.

- Abstraction enables mathematics to become universally applicable.

Luc Lemaire:

How should we remember Carl Wilhelm Ferdinand, Duke of Brunswick? In my dictionary, he is described as a duke soldier who was beaten by the French in Valmy, then again in Jena. But I must say I looked only in a French dictionary. Still, an uninspiring notice. But one day, he got a report from a school teacher that a young boy seemed remarkably gifted in mathematics. The boy was the son of a poor gardener and bricklayer, so his future should have been rather bleak. But the Duke liked mathematics, saw the boy and was convinced by his obvious talent (if not by his good manners). Thus he supported his studies and career throughout his life. The boy's name was Carl Friederich Gauss, and we owe to him (and the Duke) the Gauss law of prime numbers, the Gauss distribution in probability, the Gauss laws of electromagnetism, most of non-Euclidean geometry, and the Gauss approximation in optics.

Obviously, we need more Dukes of Brunswick in our governments!

Saunders Mac Lane (1909 – 2005)

The progress of mathematics is like the difficult exploration of possible trails up a massive infinitely high mountain, shrouded in a heavy mist which will occasionally lift a little to afford new and charming perspectives. This or that route is explored a bit more, and we hope that some will lead higher up, while indeed many routes may join and reinforce each other.