LAX ALGEBRA MEETS TOPOLOGY

DIRK HOFMANN AND WALTER THOLEN

Celebrating Dikran Dikranjan's sixtieth birthday

Abstract. We combine two research directions of the past decade, namely the development of a lax-algebraic framework for categories of interest to topologists and analysts, and the exploration of key topological concepts, like separation and compactness, in an abstract category which comes equipped with an axiomatic notion "closed" or "proper" map. Hence, we present various candidates for such notions in the context of the category of lax $(\mathbb{T},\mathsf{V})$-algebras, with a Set-monad $\mathbb{T}=(T,e,m)$ laxly extended to the category of sets and V-valued relations, for a quantale V. Suitable categories of ordered sets, metric spaces, topological spaces, closure spaces, and approach spaces all fit into this framework and allow for applications of the the general theory.

1. Introduction

Combining the Manes-Barr presentation of topological spaces as the relational algebras with respect to the ultrafilter monad (extended to relations of sets; see [Manes, 1969; Barr, 1970]) with Lawvere's interpretation of metric spaces as small categories enriched over the extended non-negative real half-line ([Lawvere, 1973]), Clementino and the authors of this paper developed a general lax-algebraic framework which turned out to be especially suitable for categories of interest to topologists and analysts ([Clementino and Hofmann, 2003; Clementino and Tholen, 2003; Clementino *et al.*, 2004b]). At various levels of generality, the lax-algebraic setting was shown in particular to allow for an efficient treatment of special types of maps, such as proper maps, open maps, descent maps, effective descent maps, triqotient maps, exponential maps, etc. ([Janelidze and Sobral, 2002; Clementino and Hofmann, 2002; Clementino *et al.*, 2005]).

Parallel to these developments one can trace back proposals for the treatment of the topological concepts of separation and compactness in a category endowed with some notion of closure or closedness, early instances of which were given by Penon [Penon, 1972], Manes [Manes, 1974] (extensively recalled in [Manes, 2010]), and Herrlich, Salicrup and Strecker [Herrlich *et al.*, 1987]. However, once the appropriate categorical notion of *closure operator* had been coined by Dikranjan and Giuli [Dikranjan and Giuli, 1987] it became immediately clear that such an operator provides a convenient structure on a category in order to pursue topological concepts; see, in particular, [Dikranjan and Giuli, 1989], [Clementino and Tholen, 1996], [Clementino *et al.*, 1996]. However,

2010 *Mathematics Subject Classification.* 18D20, 18C15, 18A99, 18B30.

Key words and phrases. monad, lax extension, lax algebra, compact object, separated object, perfect morphism.

The first author acknowledges partial financial assistance by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT and the project MONDRIAN (under the contract PTDC/EIA-CCO/108302/2008).

The second author acknowledges partial financial assistance by the Natural Sciences and Engineering Council of Canada and the hospitality of the Max Plack Institute for Mathematics in Bonn during the Fall of 2010.

the paper [Tholen, 1999] showed that most of the essential results may be obtained in Penon's original setting, and it led to a rather comprehensive categorical presentation of the compactness-separation theme in [Clementino *et al.*, 2004a], for a category equipped with a proper factorisation system and a distinguished class $\mathcal{F}$ of "closed" morphisms, from which one obtains a class $\mathcal{P}$ of "proper" morphisms through stabilisation under pullback.

In this article we merge the two developments just described and present candidates $\mathcal{P}$ which, based on various equivalent lax-algebraic characterisations of morphisms, offer themselves naturally, in the same way as compact Hausdorff spaces distinguish themselves as the strict algebras amongst lax, and this approach leads to the intended classes in the role model Top.

Hence, in Section 2 we carefully recall the lax-algebraic setting as given by Seal [Seal, 2005], paying particular attention to the role of *(bi)modules* of lax algebras. We then give an outline of notions of compactness and separation in a category which comes equipped with a *topology* (a term which we hope the reader can accept despite the presence of Grothendieck's more famous and important notion!), i.e., with a class of morphisms playing the role of proper maps, which is only required to contain the isomorphisms and be closed under composition and stable under pullback. Unlike the approaches taken in [Clementino *et al.*, 1996] and [Clementino *et al.*, 2004a], the setting of Section 3 does not require the presence of a factorisation system. Finally in Sections 4 and 5 we discuss in some detail the candidates for notions of proper map which arise naturally from Section 2, first at the general level, and then in terms of examples, from order, topology, metric spaces and approach spaces. In particular, we introduce a Dikranjan-Giuli closure operator which helps us to conveniently characterise some important types of morphisms.

2. Lax algebra

By "algebra" (in the strict sense) we refer to the study of varieties of general algebras in the sense of Birkhoff, except that we allow operations to be infinitary, but we do require the existence of free algebras. It is known since the late 1960s that such varieties are equivalently described as the Eilenberg–Moore categories with respect to a monad $\mathbb{T} = (T, e, m)$ on the category Set of sets. Recall that the Eilenberg–Moore category $\mathsf{Set}^{\mathbb{T}}$ has as objects $\mathbb{T}$-*algebras*, i.e., sets X which come with a single (generalised) "operation" $a : TX \to X$ satisfying two basic laws:

$$1_X = a \cdot e_X \qquad \text{and} \qquad a \cdot Ta = a \cdot m_X; \tag{ALG}$$

morphisms are $\mathbb{T}$-*homomorphisms* $f : (X, a) \to (Y, b)$, i.e., maps $f : X \to Y$ which satisfy

$$f \cdot a = b \cdot Tf. \tag{HOM}$$

"Algebra" becomes "lax algebra" (as used in this paper) when we replace "=" by "$\leqslant$" in (ALG) and (HOM). But for "$\leqslant$" to make sense, we replace mappings of sets by relations, in fact, by V-relations, for a suitable lattice V.

Hence, we let V be a *unital quantale*, i.e., a complete lattice with a binary associative operation $\otimes$ and a $\otimes$-neutral element k such that $\otimes$ preserves suprema in each variable. The category $\mathsf{V}\text{-}\mathsf{Rel}$ has as objects sets, and a morphism $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ is a V-*relation* given by a map $r : X \times Y \to \mathsf{V}$; its composition with $s : Y \rightarrow\!\!\!\mapsto Z$ is defined by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} s(y, z) \otimes r(x, y).$$

The hom-sets of $\mathsf{V}\text{-}\mathsf{Rel}$ are ordered pointwise ($r \leqslant r' \iff \forall x \in X\; y \in Y \;:\; r(x, y) \leqslant r'(x, y)$) such that $\mathsf{V}\text{-}\mathsf{Rel}$ becomes a quantaloid, i.e., all hom-sets have suprema that are preserved by composition from either side. Considering the monoid $(\mathsf{V}, \otimes, k)$ as a one-object quantaloid and interpreting $v \in \mathsf{V}$ as $v : 1 \rightarrow\!\!\!\mapsto 1$ (with a singleton set 1), one obtains a full and faithful homomorphism $\mathsf{V} \to \mathsf{V}\text{-}\mathsf{Rel}$ of quantaloids, i.e., a functor that preserves suprema. More importantly, there is a functor

$$\mathsf{Set} \to \mathsf{V}\text{-}\mathsf{Rel},\ (f : X \to Y) \mapsto (f_\circ : X \rightarrow\!\!\!\mapsto Y),$$

with $f_\circ(x, y) = k$ if $f(x) = y$ and $f_\circ(x, y) = \bot$ (the bottom element of V) else. This functor is faithful if, and only if, $\bot < k$ or, equivalently, $|\mathsf{V}| > 1$; in this case we may safely *write f instead of* $f_\circ$. The V-relation $f_\circ$ has a right adjoint $f^\circ : Y \rightarrow\!\!\!\mapsto X$ in the 2-categorical sense (so that $1_X \leqslant f^\circ \cdot f_\circ$ and $f_\circ \cdot f^\circ \leqslant 1_Y$), given by $f^\circ(y, x) = f_\circ(x, y)$. Hence, there is also a functor

$$\mathsf{Set}^{\mathrm{op}} \to \mathsf{V}\text{-}\mathsf{Rel},\ f \mapsto f^\circ.$$

Of course, not just $f_\circ$ but every V-relation $r : X \rightarrow\!\!\!\mapsto Y$ has a converse $r^\circ : Y \rightarrow\!\!\!\mapsto X$, but note that one obtains an involution $(-)^\circ : (\mathsf{V}\text{-}\mathsf{Rel})^{\mathrm{op}} \to \mathsf{V}\text{-}\mathsf{Rel}$ only if V is commutative.

Next we consider a *lax extension* $\hat{\mathbb{T}} = (\hat{T}, e, m)$ of the monad $\mathbb{T}$ to $\mathsf{V}\text{-}\mathsf{Rel}$ (see [Seal, 2005]), i.e., a lax functor $\hat{T} : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ which coincides with T on objects, so that

(0) $\hat{T}X = TX$,
(1) $r \leqslant r' \Rightarrow \hat{T}r \leqslant \hat{T}r'$,
(2) $\hat{T}s \cdot \hat{T}r \leqslant \hat{T}(s \cdot r)$,

and which satisfies

(3) $Tf \leqslant \hat{T}f$, $(Tf)^\circ \leqslant \hat{T}(f^\circ)$,
(4) $e_Y \cdot r \leqslant \hat{T}r \cdot e_X$,
(5) $m_Y \cdot \hat{T}\hat{T}r \leqslant \hat{T}r \cdot m_X$,

for all r, r', s, f as above. Of course, (4) and (5) mean that $e : 1_{\mathsf{V}\text{-}\mathsf{Rel}} \to \hat{T}$ and $m : \hat{T}\hat{T} \to \hat{T}$ become op-lax transformations (in the 2-categorical sense), while (3) makes the diagrams

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Rel} & \xrightarrow{\hat{T}} & \mathsf{V}\text{-}\mathsf{Rel} \\ {\scriptstyle (-)_\circ}\uparrow & & \uparrow{\scriptstyle (-)_\circ} \\ \mathsf{Set} & \xrightarrow[T]{} & \mathsf{Set} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Rel} & \xrightarrow{\hat{T}} & \mathsf{V}\text{-}\mathsf{Rel} \\ {\scriptstyle (-)^\circ}\uparrow & & \uparrow{\scriptstyle (-)^\circ} \\ \mathsf{Set}^{\mathrm{op}} & \xrightarrow[T^{\mathrm{op}}]{} & \mathsf{Set}^{\mathrm{op}} \end{array}$$

commute laxly; they commute strictly if, and only if, the lax extension is *flat*, that is, if $\hat{T}1_X = T1_X = 1_{TX}$. This fact is easily seen once one has established the identities

(6) $\hat{T}(s \cdot f) = \hat{T}s \cdot \hat{T}f = \hat{T}s \cdot Tf$ and $\hat{T}(f^\circ \cdot r) = \hat{T}(f^\circ) \cdot \hat{T}r = (Tf)^\circ \cdot \hat{T}r$

that hold for any lax extension, for all $f : X \to Y$, $r : X \rightarrow\!\!\!\mapsto Z$, $s : Y \rightarrow\!\!\!\mapsto Z$ (see [Seal, 2005]). For future reference we record from [Tholen, 2009] also the following identity:

(7) $\hat{T}1_X = \hat{T}(e_X^\circ) \cdot m_X^\circ$.

Although a monad $\mathbb{T}$ may have distinct lax extensions, in what follows we always assume $\mathbb{T}$ to come with a fixed lax extension $\hat{\mathbb{T}}$ which we consider as a part of the syntax privided by $\mathbb{T}$ and V.

A *lax* $(\mathbb{T},\mathsf{V})$-*algebra*, also called a $(\mathbb{T},\mathsf{V})$-*category*, is a set X with a V-relation $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ with

$$1_X \leqslant a \cdot e_X \qquad \text{and} \qquad a \cdot \hat{T}a \leqslant a \cdot m_X. \tag{alg}$$

A morphism $f : (X, a) \to (Y, b)$ of lax $(\mathbb{T},\mathsf{V})$-algebras, also called a $(\mathbb{T},\mathsf{V})$-*functor*, is a map $f : X \to Y$ with

$$f \cdot a \leqslant b \cdot Tf. \tag{hom}$$

By (6), since $b \cdot \hat{T}f = b \cdot \hat{T}1_X \cdot Tf \leqslant b \cdot \hat{T}b \cdot Te_X \cdot Tf \leqslant b \cdot m_X \cdot Te_X \cdot Tf = b \cdot Tf \leqslant b \cdot \hat{T}f$ one sees easily that (hom) is equivalent to $f \cdot a \leqslant b \cdot \hat{T}f$. In this paper we are interested in a number of other equivalent formulations of (hom), and for that it is useful to recall the notion of (bi)module (also called distributor or profunctor) as follows.

Given $(\mathbb{T},\mathsf{V})$-categories $X = (X, a)$ and $Y = (Y, b)$, a $(\mathbb{T},\mathsf{V})$-*module* from X to Y is a V-relation $\varphi : TX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ with

$$\varphi \cdot \hat{T}a \leqslant \varphi \cdot m_X \qquad \text{and} \qquad b \cdot \hat{T}\varphi \leqslant \varphi \cdot m_X; \tag{mod}$$

we write $\varphi : X \multimap Y$ in this case. Clearly, $a : X \multimap X$ is a $(\mathbb{T},\mathsf{V})$-module. With the *Kleisli composition*

$$\psi \circ \varphi := \psi \cdot \hat{T}\varphi \cdot m_X^\circ$$

for any V-relations $\varphi : TX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $\psi : TY \relbar\joinrel\mapstochar\joinrel\rightarrow X$, we can rewrite (mod) equivalently as

$$\varphi \circ a \leqslant \varphi \qquad \text{and} \qquad b \circ \varphi \leqslant \varphi. \tag{mod$_\circ$}$$

Since

$$\begin{aligned}
\varphi &= \varphi \cdot 1_{TX} \leqslant \varphi \cdot \hat{T}1_X \leqslant \varphi \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ = \varphi \circ e_X^\circ \leqslant \varphi \circ a,\\
\varphi &= \varphi \cdot 1_{TX} = \varphi \cdot e_{TX}^\circ \cdot m_X^\circ \leqslant e_Y^\circ \cdot \hat{T}\varphi \cdot m_X^\circ = e_Y^\circ \circ \varphi \leqslant b \circ \varphi,
\end{aligned}$$

condition (mod) can in fact be written equivalently as

$$\varphi \circ a = \varphi \qquad \text{and} \qquad b \circ \varphi = \varphi.$$

In particular, we may rewrite condition (alg) as

$$e_X^\circ \leqslant a \qquad \text{and} \qquad a \circ a \leqslant a, \tag{alg$_\circ$}$$

or even as

$$1_X^\# \leqslant a \qquad \text{and} \qquad a \circ a \leqslant a$$

where $1_X^\# := e_X^\circ \cdot \hat{T}1_X$ is the discrete $(\mathbb{T},\mathsf{V})$-structure on the set X. (In fact, $X \mapsto (X, 1_X^\#)$ is left adjoint to the forgetful functor $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$.) To wit, $e_X^\circ \leqslant a$ implies $1_X^\# \leqslant a \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ \leqslant a \cdot \hat{T}a \cdot m_X^\circ = a \circ a \leqslant a$.

Remark 2.1. For future reference we remark that from the adjunctions $f_\circ \dashv f^\circ$ and $(Tf)_\circ \dashv (Tf)^\circ$ one trivially obtains the equivalence of each of the following conditions, with (i)=(hom):

(i) $f \cdot a \leqslant b \cdot Tf$,
(ii) $a \cdot (Tf)^\circ \leqslant f^\circ \cdot b$,

(iii) $a \leqslant f^\circ \cdot b \cdot Tf$,
(iv) $f \cdot a \cdot (Tf)^\circ \leqslant b$.

Since the Kleisli composition may neither be associative nor return a $(\mathbb{T},\mathsf{V})$-module $\psi \circ \varphi$ when ψ and φ are $(\mathbb{T},\mathsf{V})$-modules, it is important to collect the following assertions which show that $(\mathbb{T},\mathsf{V})$-functors act on $(\mathbb{T},\mathsf{V})$-modules from either side. (For an axiomatisation, see [Wood, 1982].)

Proposition 2.2. *Consider $(\mathbb{T},\mathsf{V})$-functors f,g,h,j and a $(\mathbb{T},\mathsf{V})$-module ψ as in*

$$U \xrightarrow{g} X \xrightarrow{f} Y \overset{\psi}{\multimap} Z \xleftarrow{h} W \xleftarrow{j} R$$

Then, with $Y=(Y,b)$,

$$f_* := b \cdot Tf : X \multimap Y, \qquad\qquad f^* := f^\circ \cdot b : Y \multimap X$$

are $(\mathbb{T},\mathsf{V})$-modules with $1_X^* = (1_X)_* = a$ (where $X=(X,a)$) and

$$(f\cdot g)_* = f_* \circ g_*, \qquad\qquad (f\cdot g)^* = g^* \circ f^*.$$

More generally,

$$\psi \circ f_* = \psi \cdot Tf : X \multimap Z, \qquad\qquad h^* \circ \psi = h^\circ \cdot \psi : Y \multimap W$$

are $(\mathbb{T},\mathsf{V})$-modules, such that $\psi \circ 1_Y^* = \psi = 1_Z^* \circ \psi$ and

$$(\psi \circ f_*) \circ g_* = \psi \circ (f_* \circ g_*), \quad j^* \circ (h^* \circ \psi) = (j^* \circ h^*) \circ \psi, \quad h^* \circ (\psi \circ f_*) = (h^* \circ \psi) \circ f_*.$$

Proof. With $Z=(Z,c)$ and $W=(W,d)$ we show the assertions about $\psi \circ f_*$ and $h^* \circ \psi$; the remaining statements follow easily. First,

$$\begin{aligned}\psi \circ f_* = \psi \cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ = \psi \cdot \hat{T}b \cdot TTf \cdot m_X^\circ \leqslant \psi \cdot \hat{T}b \cdot m_Y^\circ \cdot Tf = (\psi \circ b)\cdot Tf = \psi \cdot Tf \\ \leqslant \psi \cdot Tf \cdot \hat{T}1_X = \psi \cdot Tf \cdot \hat{T}(e_X^\circ)\cdot m_X^\circ \leqslant \psi \cdot \hat{T}(f\cdot e_X^\circ)\cdot m_X^\circ \\ \leqslant \psi \cdot \hat{T}(f\cdot a)\cdot m_X^\circ \leqslant \psi \cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ = \psi \circ f_*.\end{aligned}$$

Since also

$$\begin{aligned}&(\psi \cdot Tf)\cdot \hat{T}a \leqslant \psi \cdot \hat{T}(f\cdot a) \leqslant \psi \cdot \hat{T}(b\cdot Tf) \leqslant \psi \cdot \hat{T}b \cdot TTf \leqslant \psi \cdot m_Y \cdot TTf = (\psi\cdot Tf)\cdot m_X,\\ &c \cdot \hat{T}(\psi \cdot Tf) = c \cdot \hat{T}\psi \cdot TTf \leqslant \psi \cdot m_Y \cdot TTf = (\psi \cdot Tf)\cdot m_X,\end{aligned}$$

$\psi \circ f_* = \psi \cdot Tf$ is a $(\mathbb{T},\mathsf{V})$-module. Second, quite easily one has

$$\begin{aligned}&h^* \circ \psi = h^\circ \cdot c \cdot \hat{T}\psi \cdot m_Y^\circ = h^\circ \cdot (c \circ \psi) = h^\circ \cdot \psi,\\ &(h^\circ \cdot \psi)\cdot \hat{T}b \leqslant (h^\circ \cdot \psi)\cdot m_Y^\circ,\\ &d \cdot \hat{T}(h^\circ \cdot \psi) = d \cdot (Th)^\circ \cdot \hat{T}\psi \leqslant h^\circ \cdot c \cdot \hat{T}\psi \leqslant (h^\circ \cdot \psi)\cdot m_Y,\end{aligned}$$

which confirms that $h^* \circ \psi = h^\circ \cdot \psi$ is a $(\mathbb{T},\mathsf{V})$-module. □

Remark 2.3. We note that, as mere V-relations, f_* and f^* are defined already when $f : X \to Y = (Y,b)$ is just a mapping of sets. For future reference we record the inequalities

$$\begin{aligned}&f\cdot\varphi \leqslant b \cdot e_Y \cdot f \cdot \varphi = b \cdot Tf \cdot e_X \cdot \varphi \leqslant b \cdot Tf \cdot \hat{T}\varphi \cdot e_{TX} \leqslant b \cdot Tf \cdot \hat{T}\varphi \cdot m_X^\circ = f_* \circ \varphi,\\ &\varphi \cdot (Tg)^\circ = \varphi \cdot (Tg)^\circ \cdot (Te_Z)^\circ \cdot m_Z^\circ \leqslant \varphi \cdot (Tg)^\circ \cdot \hat{T}c \cdot m_Z^\circ = \varphi \cdot \hat{T}(g^\circ \cdot c)\cdot m_Z^\circ = \varphi \circ g^*,\end{aligned}$$

where $g : W \to Z = (Z,c)$ is a mapping and $\varphi : TW \relbar\!\!\!\mapsto X$ a V-relation. We also note the useful identity

$$f^* \circ f_* = f^\circ \cdot b \cdot Tf,$$

which follows form Proposition 2.2: $f^* \circ f_* = f^* \cdot Tf = f^\circ \cdot b \cdot Tf$. Finally, if we are also given $h : Z \to W = (W,d)$ and $j : Y \to (X,a)$, then

$$j^* \circ (\varphi \circ g^*) = (j^* \circ \varphi) \circ g^*, \qquad f_* \circ (\varphi \circ h_*) \leqslant (f_* \circ \varphi) \circ h_*,$$

the proof of which is quite straightforward.

Corollary 2.4. *A $(\mathbb{T},\mathsf{V})$-functor $f : X \to Y$ satisfies*

$$1_X^* \leqslant f^* \circ f_* \qquad \textit{and} \qquad f_* \circ f^* \leqslant 1_Y^*.$$

Proof. The first inequality means $1_X^* = a \leqslant f^\circ \cdot b \cdot Tf$, which is equivalent to (hom) (see Remark 2.1), while the second inequality holds for any map f:

$$f_* \circ f^* = b \cdot Tf \cdot \hat{T}(f^\circ \cdot b) \cdot m_Y^\circ = b \cdot Tf \cdot (Tf)^\circ \cdot \hat{T}b \cdot m_Y^\circ \leqslant b \cdot \hat{T}b \cdot m_Y^\circ = b \circ b = b = 1_Y^*. \qquad \square$$

Not only in Corollary 2.4, but also in Proposition 2.2, whenever we used (hom), it was necessary to do so:

Corollary 2.5. *For lax $(\mathbb{T},\mathsf{V})$-algebras $X = (X,a)$, $Y = (Y,b)$ and any mapping $f : X \to Y$, in addition to* (i)-(iv) *of Remark 2.1, each of the following inequalities is equivalent to* (hom)*:*

(v) $f_* \cdot \hat{T}a \leqslant f_* \cdot m_X$,
(vi) $a \cdot \hat{T}(f^*) \leqslant f^* \cdot m_Y$,
(vii) $f_* \circ a \leqslant f_*$,
(viii) $a \circ f^* \leqslant f^*$.

Proof. That (hom) implies (vii) and (viii) follows from Proposition 2.2, and the equivalences (v) $\iff$ (vii) and (vi) $\iff$ (viii) follow trivially from the adjunctions $(m_X)_\circ \dashv m_X^\circ$ and $(m_Y)_\circ \dashv m_Y^\circ$. Finally, from Remark 2.3 one has

$$f \cdot a \leqslant f_* \circ a \qquad \text{and} \qquad a \cdot (Tf)^\circ \leqslant a \circ f^*,$$

which shows (vii)$\Rightarrow$(i) and (viii)$\Rightarrow$(ii) of Remark 2.1, respectively. $\square$

Corollary 2.6. *In the setting of Corollary 2.5, the following statements are equivalent: f is a $(\mathbb{T},\mathsf{V})$-functor; f_* is a $(\mathbb{T},\mathsf{V})$-module; f^* is a $(\mathbb{T},\mathsf{V})$-module; $1_X^* \leqslant f^* \circ f_*$ & $f_* \circ f^* \leqslant 1_Y^*$.*

Finally we mention that the hom-sets of $(\mathbb{T},\mathsf{V})$-Cat inherit the order of $(\mathbb{T},\mathsf{V})$-modules via

$$f \leqslant f' :\iff f^* \leqslant (f')^*,$$

for $f, f' : (X,a) \to (Y,b)$, which makes $(\mathbb{T},\mathsf{V})$-Cat a 2-category. (Compatibility with composition is guaranteed by Proposition 2.2.)

Remark 2.7. For the proof of the following equivalences one may apply the (in)equalities of Remark 2.3, the details of which must be left to the reader:

$$\begin{aligned} f \leqslant f' &\iff f'_* \leqslant f_* \iff 1_X^* \leqslant (f')^* \circ f_* \iff f^* \circ f'_* \leqslant 1_Y^* \\ &\iff \forall x \in X,\, \mathfrak{y} \in TY \,:\, b(\mathfrak{y}, f(x)) \leqslant b(\mathfrak{y}, f'(x)) \\ &\iff \forall \mathfrak{x} \in TX,\, y \in Y \,:\, b(Tf(\mathfrak{x}), y) \leqslant b(Tf'(\mathfrak{x}), y) \\ &\iff \forall x \in X \,:\, k \leqslant b(e_Y(f(x)), f'(x)). \end{aligned}$$

In the 2-category $(\mathbb{T}, \mathsf{V})$-Cat there is now a notion of adjunction: a $(\mathbb{T}, \mathsf{V})$-functor $g : Y \to X$ is *right adjoint* if there is a $(\mathbb{T}, \mathsf{V})$-functor $f : X \to Y$ with $1_X \leqslant g \cdot f$ and $f \cdot g \leqslant 1_Y$; one writes $f \dashv g$ in this case. From Proposition 2.2 and Remark 2.3 one obtains:

$$\begin{aligned} f \dashv g &\iff 1_X^* \leqslant f^* \circ g^* \;\&\; g^* \circ f^* \leqslant 1_Y^* \\ &\iff g^* = f_* \qquad\qquad (\text{since } 1_X^* \leqslant f^* \circ f_* \;\&\; f_* \circ f^* \leqslant 1_Y^*) \\ &\iff \forall \mathfrak{x} \in TX,\, y \in Y \,:\, a(\mathfrak{x}, g(y)) = b(Tf(\mathfrak{x}), y). \end{aligned}$$

In what follows we are interested in those $(\mathbb{T}, \mathsf{V})$-functors which satisfy any of (i)-(vi) of Remark 2.1 and Corollary 2.5, *with "$\leqslant$" replaced by "$=$"*. We put

$$\begin{aligned} &\mathcal{P}_{\mathrm{i}} := \{f : (X, a) \to (Y, b) \mid f \cdot a = b \cdot Tf\}, \\ &\mathcal{P}_{\mathrm{ii}} := \{f : (X, a) \to (Y, b) \mid a \cdot (Tf)^\circ = f^\circ \cdot b\}, \\ &\text{etc.} \end{aligned}$$

Essential stability properties of these classes will be shown in Section 4. (We note that there is no point in studying the analogously-defined classes $\mathcal{P}_{\mathrm{vii}}$ and $\mathcal{P}_{\mathrm{viii}}$: since every $(\mathbb{T}, \mathsf{V})$-functor f satisfies $f_* \circ a = f_*$ and $a \circ f^* = f^*$, these are the classes of all $(\mathbb{T}, \mathsf{V})$-functors.)

3. Separation and compactness in a category

Throughout this section we consider a finitely-complete category X and call a class $\mathcal{P}$ of morphisms in X a *topology* on X if

- $\mathcal{P}$ contains all isomorphisms,
- $\mathcal{P}$ is closed under composition,
- $\mathcal{P}$ is stable under pullback.

For another topology $\mathcal{S}$ on X which satisfies the right cancellation condition

$$p \cdot s \in \mathcal{S},\, s \in \mathcal{S} \;\Rightarrow\; p \in \mathcal{S},$$

we call the topology $\mathcal{P}$ an $\mathcal{S}$-*topology* on X if the cancellation condition

- $p \cdot s \in \mathcal{P},\, s \in \mathcal{S} \;\Rightarrow\; p \in \mathcal{P}$

holds. Every topology $\mathcal{P}$ is an $\mathrm{Iso}\mathsf{X}$-topology (with $\mathrm{Iso}\mathsf{X}$ the class of all isomorphisms), and the hypothesis on $\mathcal{S}$ means precisely that $\mathcal{S}$ is an $\mathcal{S}$-topology.

The role model of this setting is the class $\mathcal{P}$ of *proper* (= stably closed = closed with compact fibres) maps, which is an $\mathcal{S}$-topology on the category Top of topological spaces for $\mathcal{S}$ the class of surjective maps in Top. Other important examples of topologies on Top are given by the classes of open or of exponentiable maps.

In the setting of [Clementino *et al.*, 2004a] where one is axiomatically given a class of $\mathcal{F}$ of "closed" morphisms in a finitely-complete category X endowed with a proper factorisation system $(\mathcal{E}, \mathcal{M})$, one may choose $\mathcal{P}$ and $\mathcal{S}$ to contain those morphisms that belong *stably* to $\mathcal{F}$ and $\mathcal{E}$, respectively, and obtain an $\mathcal{S}$-topology $\mathcal{P}$ on X. In particular, if as in [Clementino *et al.*, 1996] X comes with a closure operator c, one lets $\mathcal{F}$ be the class of morphisms for which taking images preserves the closure operator c, provided that c is weakly hereditary.

For a topology $\mathcal{P}$ on X we refer to the morphisms of $\mathcal{P}$ also as the $\mathcal{P}$-*proper* morphisms of X. An object X in X is $\mathcal{P}$-*compact* if $X \to 1$ (= the terminal object in X) is in $\mathcal{P}$. A morphism $f : X \to Y$ is $\mathcal{P}$-*separated* if the morphism $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is in $\mathcal{P}$, and an object X is $\mathcal{P}$-*separated* if $X \to 1$ is $\mathcal{P}$-separated; equivalently, if $\delta_X : X \to X \times X$ lies in $\mathcal{P}$. Morphisms that are $\mathcal{P}$-proper and $\mathcal{P}$-seperated are called $\mathcal{P}$-*perfect*.

We note that when $\mathcal{P}$ is an $\mathcal{S}$-topology on X, $\Sigma_B^{-1}\mathcal{P}$ is a $\Sigma_B^{-1}\mathcal{S}$-topology on $\mathsf{X}/B = (\mathsf{X} \downarrow B)$, the comma category of morphisms with codomain B, with $Sigma_B$ the forgetful functor to X. *A morphism $f : X \to Y$ in X is $\mathcal{P}$-proper ($\mathcal{P}$-separated) if and only if f (as an object in X/Y) is $\Sigma_B^{-1}\mathcal{P}$-compact ($\Sigma_B^{-1}\mathcal{P}$-separated).*

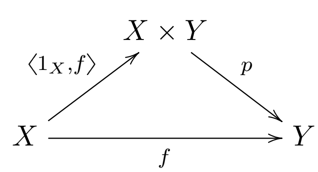
Proposition 3.1. *For a topology $\mathcal{P}$ and an object X, the following conditions are equivalent:*

(i) *X is $\mathcal{P}$-compact;*
(ii) *every morphism $f : X \to Y$ with Y $\mathcal{P}$-separated is $\mathcal{P}$-proper;*
(iii) *there is a $\mathcal{P}$-proper morphism $f : X \to Y$ with Y $\mathcal{P}$-compact;*
(iv) *the projection $X \times Y \to Y$ is $\mathcal{P}$-proper for all objects Y;*
(v) *$X \times Y$ is $\mathcal{P}$-compact for every $\mathcal{P}$-compact object Y.*

Furthermore, if $\mathcal{P}$ is an $\mathcal{S}$-topology, the following condition is also equivalent to (i)*:*

(vi) *for every morphism $f : X \to Y$ in $\mathcal{S}$, Y is $\mathcal{P}$-compact.*

Proof. (i)⇒(ii): In the graph factorisation



$\langle 1_X, f \rangle$ is in $\mathcal{P}$ as a pullback of δ_Y, and p is in $\mathcal{P}$ as a pullback of $X \to 1$.
(ii)⇒(iii): Consider $Y = 1$.
(iii)⇒(i): $(X \to 1) = (X \xrightarrow{f} Y \to 1)$.
(i)⇒(iv): p is a pullback of $X \to 1$.
(iv)⇒(v): $(X \times Y \to 1) = (X \times Y \to Y \to 1)$.
(v)⇒(i): Consider $Y = 1$.
(i)⇒(vi): See (iii)⇒(i).
(vi)⇒(i): Consider $f = 1_X$. □

Corollary 3.2. *For a topology $\mathcal{P}$, let the composite morphism $q \cdot p$ be $\mathcal{P}$-proper. Then, if q is $\mathcal{P}$-separated, also p is $\mathcal{P}$-proper.*

Proof. Apply Proposition 3.1 (i)⇒(ii) to the morphism $p : (q \cdot p) \to q$ in X/Z (if $q : Y \to Z$ in X). □

Lemma 3.3. *For a topology $\mathcal{P}$ on X,*

$$\mathcal{P}' := \{f \mid f \text{ is } \mathcal{P}\text{-separated}\}$$

is a topology which contains all monomorphisms of X and satisfies $(g \cdot f \in \mathcal{P}' \Rightarrow f \in \mathcal{P}')$. Moreover, if $\mathcal{P}$ is an $\mathcal{S}$-topology, then $\mathcal{P}'$ is a $(\mathcal{P} \cap \mathcal{S})$-topology on X.

Proof. See Proposition 4.2 of [Clementino *et al.*, 2004a]. □

Corollary 3.4. *For a topology $\mathcal{P}$ and an object X, the following conditions are equivalent:*

(i) *X is $\mathcal{P}$-separated;*
(ii) *every morphism $f : X \to Y$ is $\mathcal{P}$-separated;*
(iii) *there is a $\mathcal{P}$-separated morphism $f : X \to Y$ with Y $\mathcal{P}$-separated;*
(iv) *the projection $X \times Y \to Y$ is $\mathcal{P}$-separated for all objects Y;*
(v) *$X \times Y$ is $\mathcal{P}$-separated for every $\mathcal{P}$-separated object Y.*

Furthermore, if $\mathcal{P}$ is an $\mathcal{S}$-topology, the following condition is also equivalent to (i)*:*

(vi) *for every $\mathcal{P}$-proper morphism $f : X \to Y$ in $\mathcal{S}$, Y is $\mathcal{P}$-separated.*

Proof. Apply Proposition 3.1 to $\mathcal{P}'$ in lieu of $\mathcal{P}$, observing that "$\mathcal{P}'$-proper" means "$\mathcal{P}$-separated" and "$\mathcal{P}'$-compact" means "$\mathcal{P}$-separated", and that all objects and morphisms are $\mathcal{P}'$-separated. □

Remark 3.5. We may augment the list of equivalent conditions in Corollary 3.4 by

(vii) *for every equaliser diagram $E \xrightarrow{u} Z \rightrightarrows X$, u is $\mathcal{P}$-proper.*

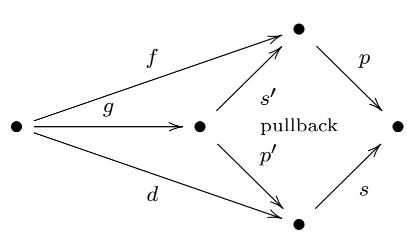
Indeed, such equalisers u are precisely the pullbacks of $\delta_X : X \to X \times X$.

Corollary 3.6. *For a topology $\mathcal{P}$ on X, the full subcategory of $\mathcal{P}$-Sep of $\mathcal{P}$-separated objects and $\mathcal{P}$-$\mathsf{CompSep}$ of $\mathcal{P}$-compact $\mathcal{P}$-separated objects are closed under finite limits in X. When $f : X \to Y$ is a monomorphism in X, or just $\mathcal{P}$-separated, then $Y \in \mathcal{P}$-Sep implies $X \in \mathcal{P}$-Sep, and when f is also $\mathcal{P}$-proper, then $Y \in \mathcal{P}$-$\mathsf{CompSep}$ implies $X \in \mathcal{P}$-$\mathsf{CompSep}$.* □

In what follows, let $\mathcal{P}$ and $\mathcal{S}$ be topologies on X, with $\mathcal{S}$ satisfying the right cancellation condition. We call a morphism d in X *$(\mathcal{P}, \mathcal{S})$-dense* if in every factorisation $d = p \cdot f$ with $p \in \mathcal{P}$ one has $p \in \mathcal{S}$, and we denote by $\mathcal{D} = \mathcal{D}_{\mathcal{P},\mathcal{S}}$ the class of $(\mathcal{P}, \mathcal{S})$-dense morphisms in X. Trivially, one has $(d \cdot g \in \mathcal{D} \Rightarrow d \in \mathcal{D})$. Furthermore, for composable morphisms

$$s \in \mathcal{S},\ d \in \mathcal{D} \Rightarrow s \cdot d \in \mathcal{D}. \qquad (*)$$

Indeed, if $s \cdot d = p \cdot f$ with $p \in \mathcal{P}$, consider the diagram



Since $p' \in \mathcal{P}$, $d \in \mathcal{D}$, one obtains $p' \in \mathcal{S}$ and $s \cdot p' = p \cdot s' \in \mathcal{S}$, hence $p \in \mathcal{S}$. As a consequence of $(*)$ one has the equivalences

$$\mathcal{S} \subseteq \mathcal{D} \iff \mathrm{Iso}\mathsf{X} \subseteq \mathcal{D} \iff \text{ every retraction that lies in } \mathcal{P} \text{ is also in } \mathcal{S}.$$

For a morphism $f : X \to Y$ in X let $f^* : \mathsf{X}/Y \to \mathsf{X}/X$ denote the pullback functor.

Definition 3.7. A morphism $f : X \to Y$ in X is $(\mathcal{P}, \mathcal{S})$-*open* if, for every pullback $g : U \to V$ of f, g^* preserves $(\mathcal{P}, \mathcal{S})$-density; that is, if for all pullback diagrams

$$\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
{\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle d} \\
U & \xrightarrow{g} & V \\
\downarrow & & \downarrow \\
X & \xrightarrow[f]{} & Y
\end{array}$$

$d \in \mathcal{D}$ implies $d' = g^*(d) \in \mathcal{D}$. Let $\mathcal{O} = \mathcal{O}_{\mathcal{P},\mathcal{S}}$ denote the class of all $(\mathcal{P}, \mathcal{S})$-open morphisms in X.

Lemma 3.8. *If $\mathcal{P}$ is an $\mathcal{S}$-topology, so is $\mathcal{O}$.*

Proof. Clearly, $\mathcal{O}$ is a topology. Since any pullback of $f \cdot s$ with $s \in \mathcal{S}$ is of the form $f' \cdot s'$with $s' \in \mathcal{S}$, it suffices to show that f^* preserves $(\mathcal{P}, \mathcal{S})$-density when $(f \cdot s)^*$ does. But for the pullback diagrams

$$\begin{array}{ccccc}
\bullet & \xrightarrow{s'} & \bullet & \xrightarrow{f'} & \bullet \\
{\scriptstyle d''}\downarrow & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} \\
\bullet & \xrightarrow[s]{} & \bullet & \xrightarrow[f]{} & \bullet
\end{array}$$

$d \in \mathcal{D}$ implies $d'' \in \mathcal{D}$, which gives $s \cdot d'' = d' \cdot s' \in \mathcal{D}$ by $(*)$ and, hence, $d' \in \mathcal{D}$. □

In the role model $\mathsf{X} = \mathsf{Top}$, $\mathcal{P} = \{\text{proper}\}$, $\mathcal{S} = \{\text{surjective}\}$, $\mathcal{D}$ is the class of dense maps (those continuous $f : X \to Y$ with $\overline{f(X)} = Y$), and $\mathcal{O}$ is the class of open maps (preservation of openess for subsets). $\mathcal{O}$-separated maps are the locally injective maps $f : X \to Y$ (so that every $x \in X$ has a neighbourhood U with $f|_U$ injective), which are local homeomorphism when they are also open. Every space is $\mathcal{O}$-compact, and $\mathcal{O}$-separated spaces are precisely the discrete ones.

In general, we call morphisms in $\mathcal{O}' = (\mathcal{O}_{\mathcal{P},\mathcal{S}})'$ *locally* $(\mathcal{P}, \mathcal{S})$-*injective*, and those in $\mathcal{O} \cap \mathcal{O}'$ *local* $(\mathcal{P}, \mathcal{S})$-*homeomorphism*. Objects in $\mathcal{O}$-Sep are called $(\mathcal{P}, \mathcal{S})$-*discrete*, and $\mathcal{O}$-compact objects $\mathcal{S}$-*inhabited*. This last terminology is motivated by the following:

Remarks 3.9. (1) Assume that for $d_i \in \mathcal{D} = \mathcal{D}_{\mathcal{P},\mathcal{S}}$ also any small-indexed coproduct $\sum_i d_i$ exists and is in $\mathcal{D}$, and let X be an object such that, for all objects U, the morphism

$$s_U : \sum_{x:1 \to X} U \to X \times U$$

whose x-th restriction to U is $\langle x, 1_U\rangle : U \to X \times U$, lies in $\mathcal{S}$. Then X is $\mathcal{O}$-compact.
Indeed, considering the diagram

$$\begin{array}{ccc}
\sum_x U & \xrightarrow{\sum_x d} & \sum_i V \\
{\scriptstyle s_U}\downarrow & & \downarrow{\scriptstyle s_V} \\
X \times U & \xrightarrow[1_X \times d]{} & X \times V
\end{array}$$

one obtains for $d \in \mathcal{D}$ first $s_V \cdot (\sum_x d) = (1_X \times d) \cdot s_U \in \mathcal{D}$, and then $1_X \times d = (X \to 1)^*(d) \in \mathcal{D}$.
(2) $\mathcal{D}$ is closed under the formation of coproducts if $\mathcal{S}$ is, and if X is extensive (see [Carboni *et al.*, 1993]), in the infinitary sense). Indeed, considering $d = \sum_i d_i$ with all $d_i : U_i \to V_i$ in $\mathcal{D}$, and assuming $d = p \cdot f$ with $p \in \mathcal{P}$ we can build the diagram

$$\begin{array}{ccccc}
U_i & \xrightarrow{f_i} & W_i & \xrightarrow{p_i} & V_i \\
\downarrow & & \downarrow & \text{pullback} & \downarrow \\
\sum_i U_i & \xrightarrow[f]{} & W & \xrightarrow[p]{} & \sum_i V_i
\end{array}$$

with $p_i \cdot f_i = d_i$ and $p_i \in \mathcal{P}$ for all i. Hence, $p_i \in \mathcal{S}$ since $d_i \in \mathcal{D}$, and $p = \sum_i p_i \in \mathcal{S}$ by extensivity and hypothesis on $\mathcal{S}$.

Corollary 3.10. *For an $\mathcal{S}$-topology $\mathcal{P}$ on X, the following conditions are equivalent:*

(i) *X is $(\mathcal{P}, \mathcal{S})$-discrete;*
(ii) *every morphism $f : X \to Y$ is locally $(\mathcal{P}, \mathcal{S})$-injective;*
(iii) *there is a locally $(\mathcal{P}, \mathcal{S})$-injective morphism $f : X \to Y$ with Y $(\mathcal{P}, \mathcal{S})$-discrete;*
(iv) *the projection $X \times Y \to Y$ is locally $(\mathcal{P}, \mathcal{S})$-injective for all objects Y;*
(v) *$X \times Y$ is $(\mathcal{P}, \mathcal{S})$-discrete for every $(\mathcal{P}, \mathcal{S})$-discrete object Y;*
(vi) *for every $(\mathcal{P}, \mathcal{S})$-open morphism $f : X \to Y$ in $\mathcal{S}$, Y is $(\mathcal{P}, \mathcal{S})$-discrete.*

Proof. Apply Corollary 3.4 with $\mathcal{O}$ in lieu of $\mathcal{P}$. □

Similarly, one may obtain characteristic properties of $\mathcal{S}$-inhabited objects from Proposition 3.1.

In order for us to define notions of “Tychonoff” and “local compactness” in our setting, we should have a suitable notion of “subobject”. Calling a morphism m *$\mathcal{S}$-extremal* if m does not factor as $m = f \cdot s$ with $s \in \mathcal{S}$ unless s is an isomorphism, we let $\mathcal{M} = \mathcal{M}_{\mathcal{S}}$ be the class of all morphisms that are *stably* $\mathcal{S}$-extremal. Trivially, $(g \cdot m \in \mathcal{M} \Rightarrow m \in \mathcal{M})$ and $\mathcal{S} \cap \mathcal{M} \subseteq \mathrm{Iso}\mathsf{X}$; for “$\supseteq$” one needs split monomorphisms in $\mathcal{S}$ to be isomorphisms. Furthermore, $\mathcal{M}$ is closed under composition if $\mathcal{S}$ satisfies the strong cancellation condition $(s \cdot t \in \mathcal{S} \Rightarrow s \in \mathcal{S})$. Finally, if every regular epimorphism lies in $\mathcal{S}$ and X has coequalisers of kernel pairs, then every morphism in $\mathcal{M}$ is a monomorphism and, in particular, $\mathcal{P}$-separated.

We call morphisms in the class $(\mathcal{P} \cap \mathcal{P}') \cdot \mathcal{M}$ *$(\mathcal{P}, \mathcal{S})$-Tychonoff*, and morphisms in the class $(\mathcal{P} \cap \mathcal{P}') \cdot (\mathcal{O} \cap \mathcal{M})$ *locally $(\mathcal{P}, \mathcal{S})$-perfect*. Hence, $f : X \to Y$ is $(\mathcal{P}, \mathcal{S})$-Tychonoff (locally $(\mathcal{P}, \mathcal{S})$-perfect) if it is the restriction of a $\mathcal{P}$-perfect morphism $p : Z \to Y$ along a ($(\mathcal{P}, \mathcal{S})$-open) morphism $m : X \to Z$ in $\mathcal{M}$: $f = p \cdot m$. We denote by $(\mathcal{P}, \mathcal{S})$-$\mathsf{Tych}$ the full subcategory of X of objects X for which $X \to 1$ is $(\mathcal{P}, \mathcal{S})$-Tychonoff, and by $(\mathcal{P}, \mathcal{S})$-$\mathsf{LocCompSep}$ the full subcategory of X of those X with $X \to 1$ locally $(\mathcal{P}, \mathcal{S})$-perfect. Hence, $X \in (\mathcal{P}, \mathcal{S})$-$\mathsf{Tych}$ if X is presentable as $m : X \to K$ with

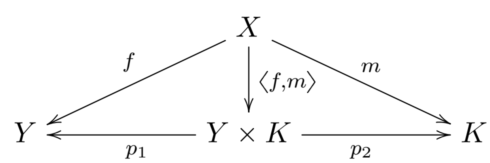
$K \in \mathcal{P}$-$\mathsf{CompSep}$ and $m \in \mathcal{M}$, while $X \in (\mathcal{P}, \mathcal{S})$-$\mathsf{LocCompSep}$ means that m can be chosen to be in $\mathcal{O} \cap \mathcal{M}$.

We note that $(\mathcal{P} \cap \mathcal{P}') \cdot \mathcal{M}$ and $(\mathcal{P} \cap \mathcal{P}') \cdot (\mathcal{O} \cap \mathcal{M})$ are both stable under pullback. Although they may not enjoy the other required properties of an $\mathcal{S}$-topology, we are able to prove propositions about them in the style of Proposition 3.1.

Proposition 3.11. *Let $\mathcal{P}$ and $\mathcal{S}$ be topologies on X, with $\mathcal{S}$ satisfying the strong cancellation condition ($s \cdot t \in \mathcal{S} \Rightarrow s \in \mathcal{S}$). Then the following conditions are equivalent for an object X:*

(i) $X \in (\mathcal{P}, \mathcal{S})$-$\mathsf{Tych}$;
(ii) *every morphism $f : X \to Y$ is $(\mathcal{P}, \mathcal{S})$-Tychonoff;*
(iii) *there is a $(\mathcal{P}, \mathcal{S})$-Tychonoff morphism $f : X \to Y$ with $Y \in \mathcal{P}$-$\mathsf{CompSep}$;*
(iv) *the projection $X \times Y \to Y$ is $(\mathcal{P}, \mathcal{S})$-Tychonoff for all objects Y;*
(v) $X \times Y (\mathcal{P}, \mathcal{S})$-$\mathsf{Tych}$ *for all* $Y \in (\mathcal{P}, \mathcal{S})$-$\mathsf{Tych}$.

Proof. (i)⇒(ii): With $m : X \to K$ in $\mathcal{M}$ and $K \in \mathcal{P}$-$\mathsf{CompSep}$, we consider the diagram



Since $p_2 \cdot \langle f, m\rangle \in \mathcal{M}$ also $\langle f, m\rangle \in \mathcal{M}$, and since $K \in \mathcal{P}$-$\mathsf{CompSep}$, $p_1 \in \mathcal{P} \cap \mathcal{P}'$ by Proposition 3.1 and Corollary 3.4.
(ii)⇒(iii): Choose $Y = 1$.
(iii)⇒(i): By hypothesis, $f = p \cdot m$ with $m \in \mathcal{M}$ and $p : Z \to Y$ in $\mathcal{P} \cap \mathcal{P}'$. But since $Y \in \mathcal{P}$-$\mathsf{CompSep}$, also $Z \in \mathcal{P}$-$\mathsf{CompSep}$.
(i)⇒(iv): $(X \times Y \to Y)$ is a pullback of $X \to 1$.
(i)⇒(v): If $m : X \to K$, $n : Y \to L$ in $\mathcal{M}$ with $K, L \in \mathcal{P}$-$\mathsf{CompSep}$, then also $m \times n = (m \times 1_L) \cdot (1_X \times n)$ in $\mathcal{M}$ and $K \times L \in \mathcal{P}$-$\mathsf{CompSep}$.
(iv)⇒(i),(v)⇒(i): $Y = 1$. □

In Top, a locally closed set A in a space X is the intersection of an open set O and a closed set F in X. Hence, the inclusion map $A \to O \to X$ can be rewritten as $A \to F \to X$, which switches the order of open and closed maps. The "rewriting hypothesis" of the following proposition formulates this observation in general and is crucial for the validity of the proposition.

Proposition 3.12. *Let $\mathcal{P}$ and $\mathcal{S}$ be as in Proposition 3.11, and assume that every composite morphism $m \cdot p$ with $p \in \mathcal{P} \cap \mathcal{P}' \cap \mathcal{M}$ and $m \in \mathcal{O} \cap \mathcal{M}$ is locally $(\mathcal{P}, \mathcal{S})$-perfect. Then the following conditions are equivalent for X:*

(i) $X \in (\mathcal{P}, \mathcal{S})$-$\mathsf{LocCompSep}$;
(ii) *every morphism $f : X \to Y$ with $Y \in \mathcal{P}$-Sep is locally $(\mathcal{P}, \mathcal{S})$-perfect;*
(iii) *there is a locally $(\mathcal{P}, \mathcal{S})$-perfect morphism $f : X \to Y$ with $Y \in \mathcal{P}$-$\mathsf{CompSep}$;*
(iv) *the projection $X \times Y \to Y$ is locally $(\mathcal{P}, \mathcal{S})$-perfect for all objects Y;*
(v) $X \times Y \in (\mathcal{P}, \mathcal{S})$-$\mathsf{LocCompSep}$ *for all* $Y \in (\mathcal{P}, \mathcal{S})$-$\mathsf{LocCompSep}$.

Proof. (i)⇒(ii): Proceeding as in Proposition 3.11 (i)⇒(ii), one decomposes $\langle f, m\rangle \in \mathcal{M}$ as

$$X \xrightarrow{\langle f,1_X\rangle} Y \times X \xrightarrow{1_Y \times m} Y \times K.$$

Then $\langle f, 1_X\rangle \in \mathcal{M}$, and $\langle f, 1_X\rangle \in \mathcal{P} \cap \mathcal{P}'$ as a pullback of δ_Y; furthermore, $1 \times m \in \mathcal{O} \cap \mathcal{M}$ as a pullback of $m \in \mathcal{O} \cap \mathcal{M}$. By hypothesis then, $\langle f, m\rangle$ is locally $(\mathcal{P}, \mathcal{S})$-perfect, and so is $f = p_1 \cdot \langle f, m\rangle$ since $p_1 : Y \times K \to Y$ lies in $\mathcal{P} \cap \mathcal{P}'$.
All other steps can be taken as in Proposition 3.11. □

Corollary 3.13. *Under the hypothesis of Proposition 3.11, if the composite morphism $q \cdot p$ is locally $(\mathcal{P}, \mathcal{S})$-perfect with q $\mathcal{P}$-separated, then p is locally $(\mathcal{P}, \mathcal{S})$-perfect.*

Proof. Apply Proposition 3.12 (i)⇒(ii) to the comma categories of X. □

4. Topological structures on categories of lax algebras

With $\mathbb{T}$ and V as in Section 2 we explore candidates for topologies $\mathcal{P}$ on the category $(\mathbb{T}, \mathsf{V})$-Cat. Throughout this section we let $\mathcal{S}$ denote the class of surjective $(\mathbb{T}, \mathsf{V})$-functors (which is an $\mathcal{S}$-topology). We start by collecting some easy-to-prove and well-known facts that are being used in the sequel. Then $\mathcal{M}$ (as defined after Corollary 3.10) is the class of embeddings, i.e. injective $(\mathbb{T}, \mathsf{V})$-functors $f : (X, a) \to (Y, b)$ with $a = f^\circ \cdot b \cdot Tf$.

Remarks 4.1. (1) A map $f : X \to Y$ of sets is injective if and only if $f^\circ \cdot f = 1_X$, and f is surjective if and only if $f \cdot f^\circ = 1_Y$ (in V-Rel); the latter statement requires $|\mathsf{V}| > 1$, which we assume henceforth. We also make use of the Axiom of Choice which makes surjections split epimorphisms in Set and therefore being preserved by T.
(2) The following conditions are equivalent:

(i) V is a frame, i.e. $v \wedge \bigvee_i w_i = \bigvee_i v \wedge w_i$, for $v, w_i \in \mathsf{V}$;
(ii) the left Frobenius law $f \cdot (\varphi \wedge f^\circ \cdot \psi = f \cdot \varphi \wedge \psi$ holds in V-Rel, for all $f : X \to Y$, $\varphi : Z \relbar\joinrel\mapsto X$, $\psi : Z \relbar\joinrel\mapsto Y$;
(iii) the right left Frobenius law $(\varphi \wedge \psi \cdot f) \cdot f^\circ = \varphi \cdot f^\circ \wedge \psi$ holds in V-Rel, for all $f : X \to Y$, $\varphi : X \relbar\joinrel\mapsto Z$, $\psi : Y \relbar\joinrel\mapsto Z$.

(3) A pullback diagram in $(\mathbb{T}, \mathsf{V})$-Cat

$$\begin{array}{ccc} (P,d) & \xrightarrow{q} & (Y,b) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Z,c) \end{array} \qquad (*)$$

is constructed at the level of Set, with $d = (p^\circ \cdot a \cdot Tp) \wedge (q^\circ \cdot b \cdot Tq)$.
(4) For any commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \qquad (\dagger)$$

in Set one has $q\cdot p^\circ \leqslant g^\circ\cdot f$. Diagram (†) is a *Beck-Chevalley square* (or *BC-square*) if $g^\circ\cdot f\leqslant q\cdot p^\circ$; equivalently, if $(*)$ is a weak pullback diagram, that is, if $P\xrightarrow{\langle p,q\rangle} X\times_Z Y$ is surjective. *T satisfies the Beck-Chevalley condition (BC)* if T preserves (BC)-squares; equivalently, if T maps (weak) pullback diagrams to weak pullback diagrams.

Proposition 4.2. *Let V be a frame. Then*

$$\mathcal{P}_i=\{f:(X,a)\to(Y,b)\mid f\cdot a=b\cdot Tf\}$$

is an $\mathcal{S}$-topology on $(\mathbb{T},\mathsf{V})$-Cat, and so is

$$\mathcal{P}_{ii}=\{f:(X,a)\to(Y,b)\mid a\cdot(Tf)^\circ=f^\circ\cdot b\},$$

provided that T satisfies (BC).

Proof. The following calculations show that if $f\in\mathcal{P}_{\mathrm{i}}$ ($\mathcal{P}_{\mathrm{ii}}$) in diagram $(*)$, then also $q\in\mathcal{P}_{\mathrm{i}}$ ($\mathcal{P}_{\mathrm{ii}}$, respectively):

$$\begin{aligned}
b\cdot Tq&=(b\wedge b)\cdot Tq\\
&\leqslant((g^\circ\cdot c\cdot Tg)\wedge b)\cdot Tq\\
&\leqslant(g^\circ\cdot c\cdot Tg\cdot Tq)\wedge b\cdot Tq\\
&=(g^\circ\cdot c\cdot Tf\cdot Tp)\wedge b\cdot Tq\\
&=(g^\circ\cdot f\cdot a\cdot Tp)\wedge b\cdot Tq\\
&=(q\cdot p^\circ\cdot a\cdot Tp)\wedge b\cdot Tq\\
&=q\cdot((p^\circ\cdot a\cdot Tp)\wedge(q^\circ\cdot b\cdot Tq))\\
&=q\cdot d,
\end{aligned}
\qquad
\begin{aligned}
q^\circ\cdot b&=q^\circ\cdot(b\wedge b)\\
&\leqslant q^\circ\cdot((g^\circ\cdot c\cdot Tg)\wedge b)\\
&\leqslant(q^\circ\cdot g^\circ\cdot c\cdot Tg)\wedge q^\circ\cdot b\\
&=(p^\circ\cdot f^\circ\cdot c\cdot Tg)\wedge q^\circ\cdot b\\
&=(p^\circ\cdot a\cdot(Tf)^\circ\cdot Tg)\wedge q^\circ\cdot b\\
&=(p^\circ\cdot a\cdot Tp\cdot(Tq)^\circ)\wedge q^\circ\cdot b\\
&=((p^\circ\cdot a\cdot Tp)\wedge(q^\circ\cdot b\cdot Tq))\cdot(Tq)^\circ\\
&=d\cdot(Tq)^\circ.
\end{aligned}$$

All other verifications are straightforward (and don't require the additional hypotheses). □

In what follows we try to describe the morphisms in $\mathcal{P}_{\mathrm{i}}$ and $\mathcal{P}_{\mathrm{ii}}$ using a Dikranjan-Giuli closure operator. For simplicity, throughout we assume that V is a frame and T satisfies (BC). Since a $(\mathbb{T},\mathsf{V})$-functor $f:(X,a)\to(Y,b)$ belongs to $\mathcal{P}_{\mathrm{i}}$ whenever $b\cdot Tf\leqslant f\cdot a$, we can state its characteristic property as

$$\forall\mathfrak{x}\in TX,\ y\in Y\ :\ b(Tf(\mathfrak{x}),y)\leqslant\bigvee_{x\in f^{-1}(y)}a(\mathfrak{x},x).$$

In particular, for a $(\mathbb{T},\mathsf{V})$-category $X=(X,a)$ and a subset M (structured by the restriction of a), the inclusion map $i:M\hookrightarrow X$ is in $\mathcal{P}_{\mathrm{i}}$ if and only if

$$\forall\mathfrak{x}\in TM,\ x\in X\ :\ a(Ti(\mathfrak{x}),x)>\bot\ \Rightarrow\ x\in M.$$

This description motivates the introduction of a closure operator on X, where we put

$$x\in\overline{M}:\iff\ \exists\mathfrak{x}\in TM\ :\ a(Ti(\mathfrak{x}),x)>\bot$$

for every $M\subseteq X$ and every $x\in X$. Certainly one has $M\subseteq\overline{M}$, $(M\subseteq N\ \Rightarrow\ \overline{M}\subseteq\overline{N})$, and every $(\mathbb{T},\mathsf{V})$-functor preserves this closure in the sense that $x\in\overline{M}$ implies $f(x)\in\overline{f(M)}$. However, we note that $\overline{(-)}$ is in general not idempotent. In terms of this closure operator, $i:M\hookrightarrow X$ is in $\mathcal{P}_{\mathrm{i}}$ if and only if $\overline{M}=M$, and it is now not hard to see that $i:M\hookrightarrow X$ is $\mathcal{P}_{\mathrm{i}}$-dense if and only if

$\operatorname{cl} M = X$ (where cl is the idempotent hull of $\overline{(-)}$). Let us call a $(\mathbb{T},\mathsf{V})$-functor $f:(X,a)\to(Y,b)$ *pseudo-open* if

$$\forall x\in X,\ \mathfrak{y}\in TY\ :\ b(\mathfrak{y},f(x))>\bot\ \Rightarrow\ \exists\mathfrak{x}\in TX\ :\ (Tf(\mathfrak{x})=\mathfrak{y}\ \&\ a(\mathfrak{x},x)>\bot).$$

The following proposition collects some easily-established facts:

Proposition 4.3. *Let* V *be a frame and* T *satisfy (BC), and let* $f:(X,a)\to(Y,b)$ *be a* $(\mathbb{T},\mathsf{V})$-*functor.*

(1) *If* f *is in* $\mathcal{P}_i$*, then* $f(\overline{M})=\overline{f(M)}$ *for all* $M\subseteq X$*.*
(2) *If* f *is in* $\mathcal{P}_{ii}$*, then* f *is pseudo-open, with the converse holding if* $\mathsf{V}=\mathsf{2}=\{\bot<\top\}$*.*
(3) *If* f *is pseudo-open, then* $f^{-1}(\overline{N})=\overline{f^{-1}(N)}$ *for all* $N\subseteq Y$*.*

Proof. We restrict our attention to (3) and show $f^{-1}(\overline{N})\subseteq\overline{f^{-1}(N)}$ and consider the pullback diagram

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{f'} & N \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Let $x\in X$ with $x\in f^{-1}(\overline{N})$. Hence, there is some $\mathfrak{y}\in TN$ with $b(Tj(\mathfrak{y}),f(x))>\bot$, and since f is pseudo-open there is some $\mathfrak{x}\in TX$ with $Tf(\mathfrak{x})=Tj(\mathfrak{y})$ and $a(\mathfrak{x},x)>\bot$. Since T satisfies (BC), there is some $\mathfrak{x}'\in T(f^{-1}(N))$ with $Ti(\mathfrak{x}')=\mathfrak{x}$ and $Tf'(\mathfrak{x}')=\mathfrak{y}$, which implies $x\in\overline{f^{-1}(N)}$. □

Remark 4.4. By transfinite induction, Proposition 4.3 remains true when $\overline{(-)}$ is substituted by its idempotent hull.

Lemma 4.5. *The class of pseudo-open* $(\mathbb{T},\mathsf{V})$*-functors is pullback stable provided that* T *satisfies (BC) and the frame* V *satisfies*

$$\forall u,v\in\mathsf{V}\ :\ u\wedge v=\bot\ \Rightarrow\ (u=\bot\ \text{or}\ v=\bot). \tag{‡}$$

Proof. Let

$$\begin{array}{ccc} (P,d) & \xrightarrow{f'} & (Z,c) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

be a pullback diagram in $(\mathbb{T},\mathsf{V})$-Cat where f is pseudo-open. Let $(x,z)\in P$ and $\mathfrak{z}\in TZ$ with $c(\mathfrak{z},z)>\bot$. Then $b(Tg(\mathfrak{z}),g(z)=f(x))>\bot$ as well and, hence, there exists some $\mathfrak{x}\in TX$ with $Tf(\mathfrak{x})=Tg(\mathfrak{z})$ and $a(\mathfrak{x},x)>\bot$. Since T satisfies (BC), there is some $\mathfrak{p}\in TP$ with $Tf'(\mathfrak{p})=\mathfrak{z}$ and $Tg'(\mathfrak{p})=\mathfrak{x}$, and therefore $d(\mathfrak{w},(x,z)=a(\mathfrak{x},x)\wedge c(\mathfrak{z},z)>\bot$. □

Proposition 4.6. *If* T *satisfies (BC) and the frame* V *satisfies* (‡)*, then every pseudo-open map is* $\mathcal{P}_i$*-open. If, in addition,* $\mathbb{T}=\mathbb{1}$ *is the identity monad (identically extended to* V-Rel*) and* V *satisfies*

$$\forall u,v\in\mathsf{V}\ :\ u\otimes v=\bot\ \Rightarrow\ (u=\bot\ \text{or}\ v=\bot), \tag{§}$$

then also the converse is true.

Note that when $k = \top$ then $u \otimes v \leqslant u \wedge v$, so that (§) implies (‡) in this case.

Proof. The first statement follows directly from Proposition 4.3 and Lemma 4.5. Regarding the second statement, note first that the condition (§) ensures that the closure $\overline{(-)}$ is idempotent. Let $f : (X, a) \to (Y, b)$ be a $\mathcal{P}_{\mathrm{i}}$-open $(\mathbb{1}, \mathsf{V})$-functor, and let $x \in X$ and $y' \in Y$ with $b(y', f(x)) > \bot$. Then $\{y'\}$ is dense in $\{y', f(x)\}$, hence, since $f : f^{-1}\{y', f(x)\} \to \{y', f(x)\}$ reflects denseness, $f^{-1}(y')$ is dense in $f^{-1}\{y', f(x)\}$. Therefore there exists some $x' \in X$ with $f(x') = y'$ and $a(x', x) > \bot$. □

Remarks 4.7. (1) Recall that the monad morphism $e : \mathbb{1} \to \mathbb{T}$ (where $\mathbb{1}$ is the identity monad on Set, identically extended to $\mathsf{V}\text{-}\mathsf{Rel}$) induces a functor

$$(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat} := (\mathbb{1}, \mathsf{V})\text{-}\mathsf{Cat},\ (X, a) \mapsto (X, a \cdot e_X).$$

Hence, a $(\mathbb{T}, \mathsf{V})$-functor $f : (X, a) \to (Y, b)$ becomes a V-functor ($= (\mathbb{1}, \mathsf{V})$-functor) $f : (X, a \cdot e_X) \to (Y, b \cdot e_X)$, which in turn induces V-modules ($= (\mathbb{1}, \mathsf{V})$-modules)

$$(X, a \cdot e_X) \underset{f^{\circledast}}{\overset{f_{\circledast}}{\rightleftarrows}} (Y, b \cdot e_X) \qquad (f_{\circledast} \dashv f^{\circledast})$$

and the conditions (i), (ii) of Remark 2.1 may now be equivalently rewritten as

(i$^{\circledast}$) $f_{\circledast} \cdot a \leqslant b \cdot (Tf)_{\circledast}$,

(ii$^{\circledast}$) $a \cdot (Tf)^{\circledast} \leqslant f^{\circledast} \cdot b$.

Since $a : (TX, \hat{T}1_X) \mathrel{-\!\!\circ\!\!\to} (X, a \cdot e_X)$ is a V-module, and so are $(Tf)_{\circledast} = \hat{T}(f)$ and $(Tf)^{\circledast} = \hat{T}(f^{\circ})$ where $Tf : (TX, \hat{T}1_X) \to (TY, \hat{T}1_Y)$, these inequalities live in the category $\mathsf{V}\text{-}\mathsf{Mod}$ of V-categories and V-modules. (Note that the Kleisli composition for V-modules is simply the composition of V-relations.) We consider the classes (see Proposition 2.2)

$$\mathcal{P}_{\mathrm{i}}^{\circledast} = \{f : (X, a) \to (Y, b) \mid f_{\circledast} \cdot a = b \cdot (Tf)_{\circledast}\} = \{f : (X, a) \to (Y, b) \mid b \cdot e_Y \cdot f \cdot a = b \cdot Tf\},$$

$$\mathcal{P}_{\mathrm{ii}}^{\circledast} = \{f : (X, a) \to (Y, b) \mid a \cdot (Tf)^{\circledast} = f^{\circledast} \cdot b\} = \{f : (X, a) \to (Y, b) \mid a \cdot \hat{T}(f^{\circ}) = f^{\circ} \cdot b\}.$$

Since $f \cdot a \leqslant f_{\circledast} \cdot a$ and $a \cdot Tf^{\circ} \leqslant a \cdot \hat{T}(f^{\circ})$, one has $\mathcal{P}_{\mathrm{i}} \subseteq \mathcal{P}_{\mathrm{i}}^{\circledast}$, $\mathcal{P}_{\mathrm{ii}} \subseteq \mathcal{P}_{\mathrm{ii}}^{\circledast}$; and $\mathcal{P}_{\mathrm{i}}^{\circledast}$ ($\mathcal{P}_{\mathrm{ii}}^{\circledast}$) contains all isomorphisms, is closed under composition, and satisfies ($f \cdot s \in \mathcal{P}_{\mathrm{i}}^{\circledast}, s \in \mathcal{S} \Rightarrow f \in \mathcal{P}_{\mathrm{i}}^{\circledast}$), and analogously for $\mathcal{P}_{\mathrm{ii}}^{\circledast}$. Hence, *the class of morphisms that are stably in $\mathcal{P}_{i}^{\circledast}$ (respectively $\mathcal{P}_{ii}^{\circledast}$) is an $\mathcal{S}$-topology on* $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$.

(2) The structure a of a lax $(\mathbb{T}, \mathsf{V})$-algebra X may also be interpreted as a $(\mathbb{T}, \mathsf{V})$-module $a : (X, 1_X^{\#}) \mathrel{-\!\!\circ\!\!\to} (X, a)$ (for $1_X^{\#}$, see Section 2). With $f_{\#}$, $f^{\#}$ denoting the $(\mathbb{T}, \mathsf{V})$-modules induced by $f : (X, 1_X^{\#}) \to (Y, 1_Y^{\#})$, while f_*, f^* continue to be as in Proposition 2.2, one can now rewrite (i), (ii) equivalently as:

(i*) $f_* \circ a \leqslant b \circ f_{\#}$,

(ii*) $a \circ f^{\#} \leqslant f^* \circ b$.

But since $b \circ f_{\#} = b \cdot Tf = f_*$ (by Proposition 2.2), (i*)=(vii) of Corollary 2.5; and since $f^* \circ b = f^{\circ} \cdot b = f^{\circledast} \cdot b$ and $a \circ f^{\#} = a \cdot (Tf)^{\circ} \cdot \hat{T}(1_Y^{\#}) \cdot m_Y^{\circ} = a \cdot (Tf)^{\circ} \cdot \hat{T}(1_Y) = a \cdot \hat{T}(f^{\circ})$, (ii*)=(ii$^{\circledast}$).

Remarks 4.8. (1) $\mathcal{P}_{\mathrm{iii}} := \{f : (X, a) \to (Y, b) \mid a = f^{\circ} \cdot b \cdot Tf\}$ is an $\mathcal{S}$-topology on $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$. In fact, this is the class of U-initial morphisms with respect to the forgetful functor $U : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to$

Set which is topological. Since the structure of the terminal object 1 in $(\mathbb{T},\mathsf{V})$-Cat has constant value $\top$ one sees immediately that the $\mathcal{P}_{\rm iii}$-compact objects are precisely the indiscrete $(\mathbb{T},\mathsf{V})$-algebras, i.e. those (X,a) with $a(\mathfrak{x},x)=\top$ constantly. Furthermore, every morphism (and object) is $\mathcal{P}_{\rm iii}$-separated, and $\mathcal{P}_{\rm iii}$-dense morphisms are already surjective. Consequently, every morphism is also $\mathcal{P}_{\rm iii}$-open, and $\mathcal{P}_{\rm iii}$-$\mathsf{LocCompSep}$ = $\mathcal{P}_{\rm iii}$-$\mathsf{CompSep}$.

(2) $\mathcal{P}_{\rm iv}:=\{f:(X,a)\to(Y,b)\mid f\cdot a\cdot(Tf)^\circ=b\}$ satisfies all requirements for an $\mathcal{S}$-topology except pullback stability. However, if V is a frame and T satisfies (BC), the pullback of a morphism in $\mathcal{P}_{\rm iv}$ along a morphism in $\mathcal{P}_{\rm iii}$ is again in $\mathcal{P}_{\rm iv}$. Still, failure of pullback stability leads us to not pursue this class further in the general context.

(3) In terms of their general properties, the situation is even worse for the classes

$$\mathcal{P}_{\rm v}=\{f:(X,a)\to(Y,b)\mid f_*\cdot\hat{T}a=f_*\cdot m_X\}$$
$$\mathcal{P}_{\rm vi}=\{f:(X,a)\to(Y,b)\mid a\cdot\hat{T}(f^*)=f^*\cdot m_Y\}$$

which may even fail to contain all isomorphisms. However, it is interesting to note that the equivalent statements

$$1_X\in\mathcal{P}_{\rm v}\,,\qquad 1_X\in\mathcal{P}_{\rm vi}\,,\qquad a\cdot\hat{T}(a)=a\cdot m_X,$$

describe precisely those $(\mathbb{T},\mathsf{V})$-algebras $X=(X,a)$ which satisfy the lax "associative law" strictly, and they form an important full subcategory of $(\mathbb{T},\mathsf{V})$-Cat.

5. Applications to order, metric, topology, and approach structure

5.1. Ordered sets as $(\mathbb{1},2)$-categories. Probably the simplest structure appearing as a lax $(\mathbb{T},\mathsf{V})$-algebra is an *ordered set*, which in this paper is a set X equipped with a relation $\leqslant: X\relbar\!\!\!\mapsto X$ subject to

$$x\leqslant x,\qquad (x\leqslant y\ \&\ y\leqslant z)\ \Rightarrow\ (x\leqslant z),$$

for all $x,y,z\in X$. Note that we do not insist on anti-symmetry here, so that our orders ar in fact only preorders. Quite directly one has that ordered sets are precisely the $(\mathbb{1},2)$-categories, where 2 denotes the two-element ordered set $\{\mathsf{false}\leqslant\mathsf{true}\}$ which is a quantale with $\otimes=\&$ (and neutral element $k=\mathsf{true}$), and monotone maps are precisely the $(\mathbb{1},2)$-functors. That is, $\mathsf{Ord}\simeq(\mathbb{1},2)$-$\mathsf{Cat}$. For a monotone map $f:X\to Y$ one has

$$\begin{aligned}
f\in\mathcal{P}_{\rm i}&\iff\forall x\in X,\,y\in Y\ :\ (f(x)\leqslant y)\ \Rightarrow\ (\exists x'\in X\ :\ (x\leqslant x')\ \&\ (f(x')=y))\\
&\iff\forall x\in X\ :\ \uparrow_Y f(x)\subseteq f(\uparrow_X x)\\
&\iff\forall A\subseteq X\ :\ \uparrow_Y f(A)\subseteq f(\uparrow_X A)\\
&\iff\forall y\in A\ :\ f^{-1}(\downarrow_Y y)\subseteq\downarrow_X(f^{-1}(y))\\
&\iff\forall B\subseteq Y\ :\ f^{-1}(\downarrow_Y B)\subseteq\downarrow_X(f^{-1}(B)),
\end{aligned}$$

with $\uparrow_X A=\{x'\in X\mid\exists x\in A\ :\ x\leqslant x'\}$ the up-closure of A in X, $\downarrow_X A=\uparrow_{X^{\rm op}}A$. Consequently, f is $(\mathcal{P}_{\rm i},\mathcal{S})$-dense precisely when $\uparrow_X f(X)=Y$, hence $(\mathcal{P}_{\rm i},\mathcal{S})$-open precisely when $f^{\rm op}:(X,\geqslant)\to(Y,\geqslant)$ is in $\mathcal{P}_{\rm i}$, that is, $f^{-1}(\downarrow_Y B)=\downarrow_X(f^{-1}(B))$ for all $B\subseteq Y$. The $\mathcal{P}_{\rm i}$-separated maps f are characterised by

$$f(x_1)=f(x_2)\ \&\ \exists z\in X\ :\ (z\leqslant x_1\ \&\ z\leqslant x_2)\ \Rightarrow\ x_1=x_2$$

for all $x_1, x_2 \in X$. Consequently, only discrete objects are $\mathcal{P}_{\mathrm{i}}$-separated while every object is $\mathcal{P}_{\mathrm{i}}$-compact.

Since

$$f \in \mathcal{P}_{\mathrm{ii}} \iff f^{\mathrm{op}} \in \mathcal{P}_{\mathrm{i}} \iff f\ (\mathcal{P}_{\mathrm{i}}, \mathcal{S})\text{-open}$$

the corresponding statements for $\mathcal{P}_{\mathrm{ii}}$ are obtained from $\mathcal{P}_{\mathrm{i}}$ by dualisation. In particular,

$$X\ (\mathcal{P}_{\mathrm{i}}, \mathcal{S})\text{-discrete} \iff X\ \mathcal{P}_{\mathrm{ii}}\text{-separated} \iff X = X^{\mathrm{op}}\text{ discrete.}$$

For $\mathcal{P}_{\mathrm{iii}}$ we can refer to Remarks 4.8 (1), and we mention here also $\mathcal{P}_{\mathrm{iv}}$ since it is pullback stable: $f \in \mathcal{P}_{\mathrm{iv}}$ means that $f \times f : X \times X \to Y \times Y$ maps the orders surjectively: $f \times f(\leqslant_X) = \leqslant_Y$. Every monotone map is $(\mathcal{P}_{\mathrm{iv}}, \mathcal{S})$-open, and all $\mathcal{P}_{\mathrm{iv}}$-separated maps are injective. Every non-empty object is $\mathcal{P}_{\mathrm{iv}}$-compact, while $\mathcal{P}_{\mathrm{iv}}$-separated objects have only at most one point.

5.2. **Ordered sets as $(\mathbb{P}, 2)$-categories.** Another presentation of Ord as a catgory of lax $(\mathbb{T}, \mathsf{V})$-algebras uses the powerset monad $\mathbb{P} = (P, e, m)$ on Set, extended to Rel by putting

$$A\, \hat{P}r\, B :\iff \forall x \in A\, \exists y \in B\ :\ x\, r\, y,$$

for $r : X \relbar\joinrel\mapsto Y$ in Rel, $A \subseteq X$ and $B \subseteq Y$. Then, as shown in [Seal, 2005], the category $(\mathbb{P}, 2)$-Cat is isomorphic to Ord: $(X, \leqslant)$ is to be considered as a lax $(\mathbb{P}, 2)$-algebra $(X, \lll)$ via

$$A \lll y \iff \forall x \in A\ :\ x \leqslant y$$

for $A \subseteq X$, $y \in X$. (Note that $\lll_X = e_X^\circ \cdot \hat{P}(\leqslant_X)$, with $e_X : X \to PX$, $x \mapsto \{x\}$.) Now, with this presentation of Ord and

$$\Uparrow_X A = \{x \in A \mid A \lll x\},$$

a monotone map $f : X \to Y$ lies in $\mathcal{P}_{\mathrm{i}}$ precisely when $\Uparrow_Y f(A) \subseteq f(\Uparrow_X A)$ for all $A \subseteq X$. Such maps are necessarily surjective (consider $A = \varnothing$) and preserve the up-closure $\uparrow_X$ (i.e., lie in the class $\mathcal{P}_{\mathrm{i}}$ of Subsection 5.1; consider $A = \{x\}$), but not vice versa. Every monotone map is $\mathcal{P}_{\mathrm{i}}$-open but only injective ones are $\mathcal{P}_{\mathrm{i}}$-separated. X is $\mathcal{P}_{\mathrm{i}}$-compact precisely when X has a top-element, but $\mathcal{P}_{\mathrm{i}}$-separatedness requires $|X| \leqslant 1$.

While the current description of Ord changes the class $\mathcal{P}_{\mathrm{i}}$ (compared to Subsection 5.1), the class $\mathcal{P}_{\mathrm{ii}}$ stays as in 5.1, moreover, one also has $\mathcal{P}_{\mathrm{ii}} = \mathcal{P}_{\mathrm{ii}}^{\circledast}$.

We wish to characterise the morphisms in $\mathcal{P}_{\mathrm{i}}^{\circledast}$ of Remarks 4.7 (1) and claim for $f : X \to Y$ (using the Axiom of Choice)

$$\begin{aligned} f \in \mathcal{P}_{\mathrm{i}}^{\circledast} &\iff f \text{ is left adjoint} \\ &\iff \exists g : Y \to X\ :\ \forall x \in X, y \in Y\ :\ (f(x) \leqslant y \iff x \leqslant g(y)). \end{aligned}$$

Indeed, the defining condition for $f \in \mathcal{P}_{\mathrm{i}}^{\circledast}$ reads in elementwise notation as

$$\forall A \subseteq X,\, y \in Y\ :\ (f(A) \lll y \Rightarrow \exists \tilde{x} \in X\ :\ (A \lll \tilde{x}\ \&\ f(\tilde{x}) \leqslant y))$$

which, given $y \in Y$, we may exploit for $A := \{x \in X \mid f(x) \leqslant y\}$ to obtain $g(y) = \tilde{x}$ with $A \lll \tilde{x}$ and $f(\tilde{x}) \leqslant y$, and that means precisely that g is right adjoint to f.

5.3. **Closure spaces as $(\mathbb{P},2)$-categories.** A different way to extend the powerset monad to Rel uses

$$A\,(\hat{P}r)\,B \iff \forall y \in B\,\exists x \in A \,:\, x\,r\,y,$$

for all $r : X \relbar\joinrel\mapsto Y$, $A \subseteq X$ and $B \subseteq Y$. With respect to this extension, the two axioms

$$\{x\}\,a\,x, \qquad (\mathfrak{A}\,(\hat{P}a)\,A \;\&\; A\,a\,x) \Rightarrow \bigcup \mathfrak{A}\,a\,x \qquad (\mathfrak{A} \subseteq PX, A \subseteq X, x \in X)$$

of a $(\mathbb{P},2)$-category (X,a) can be equivalently rewritten as the defining conditions of an abstract consequence relation $\vdash := a$ on the set X (of formulas):

(1) if $x \in A$, then $A \vdash x$,
(2) if $A \vdash x$ and $A \subseteq B$, then $B \vdash x$, and
(3) if $A \vdash y$ for all $y \in B$ and $B \vdash x$, then $A \vdash x$;

or we can think of a as a closure operator ($x \in \overline{A} :\iff A\,a\,x$) since in this notation the axioms above read as

(1) $A \subseteq \overline{A}$,
(2) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, and
(3) $\overline{\overline{A}} \subseteq \overline{A}$.

With this interpretation, the maps in $\mathcal{P}_{\mathrm{i}}$ are precisely the closure-preserving maps $f : X \to Y$ (that is, $\overline{f(A)} = f(\overline{A})$ for all $A \subseteq X$), while $f \in \mathcal{P}_{\mathrm{ii}}$ means equivalently that f^{-1} commutes with the closure (that is, $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$ for all $B \subseteq Y$) *and* that f is surjective, unless $X = \varnothing$. However, the maps in $f \in \mathcal{P}_{\mathrm{ii}}^{\circledast}$ are precisely the maps f for which f^{-1} commutes with the closure. A typical example of a morphism in $\mathcal{P}_{\mathrm{ii}}^{\circledast}$ is the inclusion $c_i : X_i \to X$ of X_i into the coproduct $X = \coprod_{i\in I} X_i$, where the closure on X is defined by (for $A \subseteq X$ and $x \in X$)

$$x \in \overline{A} :\iff x \in \overline{A \cap X_i} \qquad \text{where } x \in X_i.$$

5.4. **Topological spaces as lax algebras.** The principal result of [Barr, 1970] states that Top is isomorphic to $(\mathbb{\beta},2)$-Cat, where the ultrafilter monad $\mathbb{\beta} = (\beta, e, m)$ gets extended to Rel via

$$\mathfrak{x}(\hat{\beta}r)\mathfrak{y} :\iff \forall A \in \mathfrak{x},\, B \in \mathfrak{y}\,\exists x \in A,\, y \in B \,:\, x\,r\,y,$$

for all relations $r : X \relbar\joinrel\mapsto Y$ and ultrafilters $\mathfrak{x} \in \beta X$, $\mathfrak{y} \in \beta Y$. We recall that the ultrafilter functor $\beta : \mathsf{Set} \to \mathsf{Set}$ sends every set X to the set βX of its ultrafilters, and $\beta f(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$ for $f : X \to Y$ and $\mathfrak{x} \in \beta X$, equivalently, $\beta f(\mathfrak{x})$ is the filter generated by $\{f(A) \mid A \in \mathfrak{x}\}$. Furthermore, $e_X : X \to \beta X$ sends x to the principal filter $\{A \subseteq X \mid x \in A\}$ generated by x, and $m_X : \beta\beta X \to \beta X$ sends $\mathfrak{X} \in \beta\beta X$ to $m_X(\mathfrak{X}) = \{A \subseteq X \mid A^{\#} \in \mathfrak{X}\}$ where $A^{\#} = \{\mathfrak{x} \in \beta X \mid A \in \mathfrak{x}\}$.

The isomorphism between Top and $(\mathbb{\beta},2)$-Cat is realised by thinking of a topological space X in terms of ultrafilter convergence: a relation $\mathfrak{x} \to x$ between ultrafilters and points of a set X is the convergence relation of a unique topology on X if and only if

$$e_X(x) \to x \qquad \text{and} \qquad (\mathfrak{X} \to \mathfrak{x} \;\&\; \mathfrak{x} \to x) \Rightarrow m_X(\mathfrak{X}) \to x,$$

for all $x \in X$, $\mathfrak{x} \in \beta X$ and $\mathfrak{X} \in \beta\beta X$; and a map $f : X \to Y$ between topological spaces is continuous precisely when f preserves convergence, i.e. $\mathfrak{x} \to x$ implies $\beta f(\mathfrak{x}) \to f(x)$. We also remark that the extension $\hat{\beta} : \mathsf{Rel} \to \mathsf{Rel}$ is flat (i.e. $\beta f = \hat{\beta} f$ for every map $f : X \to Y$) and preserves composition, that is, $\hat{\beta}$ is a functor. Moreover, $m : \hat{\beta}\hat{\beta} \to \hat{\beta}$ is a natural transformation rather then just op-lax, so that $\hat{\beta}$ fails to be a monad on Rel only because $e : 1 \to \hat{\beta}$ is not a natural transformation.

A continuous map $f : X \to Y$ belongs to $\mathcal{P}_{\mathrm{i}}$ precisely when, for every ultrafilter $\mathfrak{x} \in \beta X$ and every $y \in Y$ with $\beta f(\mathfrak{x}) \to y$, there is some $x \in X$ with $\mathfrak{x} \to x$ and $f(x) = y$. Then $X \to 1$ lies in $\mathcal{P}_{\mathrm{i}}$ if and only if every ultrafilter of X converges if and only if X is compact, and $\delta_X : X \to X \times X$ is in $\mathcal{P}_{\mathrm{i}}$ if and only if every ultrafilter of X has at most one limit point, that is, if X is Hausdorff.

It is known that a continuous map f lies in $\mathcal{P}_{\mathrm{i}}$ if and only if f is proper, that is, closed with compact fibres, or, equivalently, stably closed (see [Bourbaki, 1942], for instance). To explain this, we find it convenient to introduce the functor

$$M : \mathsf{Top} \to \mathsf{Ord}$$

which sends a topological space X to the ordered set $MX := \beta X$ where $\mathfrak{x} \leqslant \mathfrak{x}'$ whenever every closed set $A \in \mathfrak{x}$ belongs to $\mathfrak{x}'$, equivalently, every open set $A \in \mathfrak{x}'$ belongs to $\mathfrak{x}$. We note that this order relation on βX contains all information about the topology of X since $\mathfrak{x} \leqslant e_X(x)$ precisely when $\mathfrak{x} \to x$. With respect to this order, $Mf := \beta f$ becomes a monotone map $Mf : MX \to MY$. It is worthwhile to note that the order $\beta X \relbar\joinrel\mapsto \beta X$ on βX is given by $\hat{\beta}(\to) \cdot m_X^\circ$ for every topological space X with convergence $\to$. Using this fact, together with the functoriality of $\hat{\beta}$ and the naturality of m, one verifies that $Mf \in \mathcal{P}_{\mathrm{i}}$ (in the sense of Subsection 5.1) for every continuous map $f \in \mathcal{P}_{\mathrm{i}}$. In fact, one has:

Lemma 5.1. *Let $f : X \to Y$ in* Top. *Then $Mf \in \mathcal{P}_i$ if and only if f is closed.*

Proof. Assume first $Mf \in \mathcal{P}_{\mathrm{i}}$. Let $A \subseteq X$ be closed and $y \in \overline{f(A)}$. Hence, there is some $\mathfrak{y} \in \beta Y$ with $\mathfrak{y} \to y$ (which is equivalent to $\mathfrak{y} \leqslant e_Y(y)$) and $f(A) \in \mathfrak{y}$, and therefore there is some $\mathfrak{x} \in \beta X$ with $\beta f(\mathfrak{x}) = \mathfrak{y}$ and $A \in \mathfrak{x}$. By hypothesis, there exists some $\mathfrak{x}' \in \beta X$ with $\mathfrak{x} \leqslant \mathfrak{x}'$ and $\beta f(\mathfrak{x}') = e_Y(y)$. Therefore $f^{-1}(y) \in \mathfrak{x}'$ and $A \in \mathfrak{x}'$, hence $y = f(x)$ for some $x \in X$. Assume now that f is closed. Let $\mathfrak{x} \in \beta X$ and $\mathfrak{y} \in \beta Y$ with $\beta f(\mathfrak{x}) \leqslant \mathfrak{y}$. Hence, $f(A) \in \mathfrak{y}$ for every closed set $A \in \mathfrak{x}$, and therefore there exists some $\mathfrak{x}' \in \beta X$ wich contains all closed sets $A \in \mathfrak{x}$ and such that $\beta f(\mathfrak{x}') = \mathfrak{y}$. □

Proposition 5.2. *The following assertions are equivalent, for $f : X \to Y$ in* Top:

(i) *$f \in \mathcal{P}_i$;*
(ii) *f is stably closed;*
(iii) *f is closed with compact fibres.*

Proof. Certainly, every $f \in \mathcal{P}_{\mathrm{i}}$ is closed and hence stably closed. Furthermore, for $f : X \to Y$ stably closed and $y \in Y$, the projection $p_2 : f^{-1}(y) \times Z \to Z$ is closed, for every space Z, since it is the pullback of f along the constant map $c_y : Z \to Y, z \mapsto y$. Hence, by the Kuratowski-Mrówka Theorem (see [Clementino *et al.*, 2004a, Theorem 3.4] or [Escardó, 2004, Theorem 9.15]), $f^{-1}(y)$ is compact. Finally, assume that f is closed with compact fibres, and let $\mathfrak{x} \in \beta X$ and $y \in Y$ with $\mathfrak{x} \to y$. Since $\mathfrak{y} \leqslant e_Y(y)$, there exists some $\mathfrak{x}' \in \beta X$ with $\mathfrak{x} \leqslant \mathfrak{x}'$ and $f^{-1}(y) \in \mathfrak{x}'$, and compactness of $f^{-1}(y)$ implies that there is some $x \in f^{-1}(y)$ with $\mathfrak{x}' \to x$, hence $\mathfrak{x} \to x$. □

A continuous map $f : X \to Y$ lies in $\mathcal{P}_{\mathrm{ii}}$ if and only if, for all $x \in X$ and $\mathfrak{y} \in \beta Y$ with $\mathfrak{y} \to f(x)$, there exist some $\mathfrak{x} \in \beta X$ with $\beta f(\mathfrak{x}) = \mathfrak{y}$ and $\mathfrak{x} \to x$. Then every topological space X is $\mathcal{P}_{\mathrm{ii}}$-compact, whereby the $\mathcal{P}_{\mathrm{ii}}$-separated spaces are precisely the discrete spaces. Furthermore, one easily verifies:

Lemma 5.3. *Let $f : X \to Y$ in* Top*. Then the following assertions are equivalent:*

(i) $f \in \mathcal{P}_{ii}$;
(ii) $Mf \in \mathcal{P}_{ii}$ *(in the sense of Subsection 5.1);*
(iii) f *is open.*

Topological spaces can also be represented via filter convergence, and it is shown in [Seal, 2005] that $\mathsf{Top} \simeq (\mathbb{F}, 2)\text{-}\mathsf{Cat}$ with the extension of the filter functor $F : \mathsf{Set} \to \mathsf{Set}$ to Rel given by

$$\mathfrak{f}\,(\hat{F}r)\,\mathfrak{g} :\iff \forall B \in \mathfrak{g}\, \exists A \in \mathfrak{f}\, \forall x \in A\, \exists y \in B \,:\, x\,r\,y.$$

With regard to this presentation of Top, maps in $\mathcal{P}_{\mathrm{i}}$ must be surjective since the "all-filter" converges to every point. The class $\mathcal{P}_{\mathrm{ii}}$ stays the same as for the ultrafilter presentation and consists precisely of the open maps; moreover, $\mathcal{P}_{\mathrm{ii}} = \mathcal{P}_{\mathrm{ii}}^{\circledast}$. A continuous map $f : X \to Y$ lies in $\mathcal{P}_{\mathrm{i}}^{\circledast}$ precisely when, for every filter $\mathfrak{f} \in FX$ and $y \in Y$, there exists some $x \in X$ with $\mathfrak{f} \to x$ and $x \leqslant y$ (where $x \leqslant y :\iff e_X(x) \to y$). Similar to what happened in Subsection 5.1, the morphisms in $\mathcal{P}_{\mathrm{i}}^{\circledast}$ are precisely the left adjoint morphisms in Top. To see this, take $y \in Y$ and let $\mathfrak{f} = \bigcap\{\mathfrak{g} \mid Ff(\mathfrak{g}) \to y\}$. Then for $x \in X$ one has ($\mathfrak{f} \to x$ & $f(x) \leqslant y$) if and only if

$$\mathfrak{g} \to x \iff Ff(\mathfrak{g}) \to y$$

for all $\mathfrak{g} \in FX$, which means that the map $g : Y \to X, y \mapsto x$, is continuous (since $g^* = f_*$, see Corollary 2.6) and indeed a right adjoint of f in Top.

5.5. **Metric spaces as lax algebras.** A *metric space* (in the generalised sense of [Lawvere, 1973]) is precisely a $(\mathbb{1}, [0,\infty])$-category, that is, a set X together with a distance function $a : X \times X \to [0,\infty]$ such that

$$0 \geqslant a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geqslant d(x,z),$$

for all $x, y, z \in X$. Here we consider $[0,\infty]$ as a quantale with order $\geqslant$ and operation $+$, which has 0 as neutral element. Note that the order is the opposite of the natural one, hence 0 is the top-element and ∞ the bottom element of $[0,\infty]$, and $\bigvee$ is given by inf. A $(\mathbb{1}, [0,\infty])$-functor $f : (X,a) \to (Y,b)$ is a map satisfying $a(x,x') \geqslant b(f(x), f(x'))$ for all $x, x' \in X$, that is, f is *non-expansive*, and we write Met for the category of metric spaces and non-expansive maps.

A non-expansive map $f : (X,a) \to (Y,b)$ belongs to $\mathcal{P}_{\mathrm{i}}$ precisely when

$$b(f(x), y) = \inf\{a(x,x') \mid x' \in X,\ f(x') = y\}$$

for all $x \in X$ and $y \in Y$. As in Subsection 5.1, every metric space is $\mathcal{P}_{\mathrm{i}}$-compact, while the $\mathcal{P}_{\mathrm{i}}$-separated metric spaces are precisely the discrete ones. Note that the hypothesis of Proposition 4.6 is satisfied here, hence the associated closure operator is idempotent and therefore $f : (X,a) \to (Y,b)$ is $(\mathcal{P}_{\mathrm{i}}, \mathcal{S})$-dense precisely when every $y \in Y$ is at finite distance $b(f(x), y) < \infty$ from some $f(x)$ with $x \in X$, and f is $(\mathcal{P}_{\mathrm{i}}, \mathcal{S})$-open if and only if f is pseudo-open.

5.6. **Approach spaces as lax algebras.** An approach space (see [Lowen, 1997]) is a pair (X, δ) consisting of a set X and an *approach distance* δ on X, that is, a function $\delta : X \times PX \to [0,\infty]$ satisfying

(1) $\delta(x, \{x\}) = 0$,

(2) $\delta(x,\varnothing)=\infty$,
(3) $\delta(x,A\cup B)=\min\{\delta(x,A),\delta(x,B)\}$,
(4) $\delta(x,A)\leqslant\delta(x,A^{(u)})+u$, where $A^{(u)}=\{x\in X\mid\delta(x,A)\leqslant u\}$,

for all $A,B\subseteq X$, $x\in X$ and $u\in[0,\infty]$. A map $f:X\to Y$ between approach spaces (X,δ) and (Y,δ') is called *non-expansive* if $\delta(x,A)\geqslant\delta'(f(x),f(A))$ for all $A\subseteq X$ and $x\in X$; equivalently, $f(A^{(u)})\subseteq f(A)^{(u)}$ for all $A\subseteq X$ and $u\in[0,\infty]$. The category of approach spaces and non-expansive maps is denoted by App.

In [Clementino and Hofmann, 2003] it is shown that $\mathsf{App}\simeq(\beta,[0,\infty])\text{-}\mathsf{Cat}$. Here a $(\beta,[0,\infty])$-category is a set X together with a function $a:\beta X\times X\to[0,\infty]$ satisfying

$$0\geqslant a(e_X(x),x)\qquad\text{and}\qquad\hat{\beta}a(\mathfrak{X},\mathfrak{x})+a(\mathfrak{x},x)\geqslant a(m_X(\mathfrak{X}),x),$$

where $\mathfrak{X}\in\beta\beta X$, $\mathfrak{x}\in\beta X$, $x\in X$ and

$$\hat{\beta}a(\mathfrak{X},\mathfrak{x})=\sup_{\mathcal{A}\in\mathfrak{X},A\in\mathfrak{x}}\ \inf_{\mathfrak{a}\in\mathcal{A},x\in A}a(\mathfrak{a},x).$$

The extension $\hat{\beta}:[0,\infty]\text{-}\mathsf{Rel}\to[0,\infty]\text{-}\mathsf{Rel}$ defined by the formula above is even a functor, and $m:\hat{\beta}\hat{\beta}\to\hat{\beta}$ is a natural transformation. Under the equivalence $\mathsf{App}\simeq(\beta,[0,\infty])\text{-}\mathsf{Cat}$, an approach distance $\delta:X\times PX\to[0,\infty]$ on X corresponds to

$$a:\beta X\times X\to[0,\infty],\ a(\mathfrak{x},x)=\sup_{A\in\mathfrak{x}}\delta(x,A),$$

and vice versa, every $a:\beta X\times X\to[0,\infty]$ corresponds to the approach distance

$$\delta:X\times PX\to[0,\infty],\ \delta(x,A)=\inf_{A\in\mathfrak{x}}a(\mathfrak{x},x).$$

Furthermore, $f:X\to Y$ is a non-expansive map $f:(X,\delta)\to(Y,\delta')$ if and only if $a(\mathfrak{x},x)\geqslant b(\beta f(\mathfrak{x}),f(x)$ (where $b:\beta Y\times Y\to[0,\infty]$ is induced by δ'), for all $\mathfrak{x}\in\beta X$ and $x\in X$.

By definition, a non-expansive map ($=(\beta,[0,\infty])$-functor) $f:(X,a)\to(Y,b)$ lies in $\mathcal{P}_{\mathrm{i}}$ if and only if, for all $\mathfrak{x}\in\beta X$ and $y\in Y$,

$$b(\beta f(\mathfrak{x}),y)\geqslant\inf\{a(\mathfrak{x},x)\mid x\in X,\ f(x)=y\}.$$

An approach space $X=(X,a)$ is $\mathcal{P}_{\mathrm{i}}$-compact if X is 0-*compact*, that is, if $\inf_{x\in X}a(\mathfrak{x},x)=0$ for all $\mathfrak{x}\in\beta X$. Furthermore, the diagonal $X\to X\times X$ lies in $\mathcal{P}_{\mathrm{i}}$ precisely when every ultrafilter $\mathfrak{x}\in\beta X$ has at most one "finite convergence point", that is, $a(\mathfrak{x},x)<\infty$ and $a(\mathfrak{x},x')<\infty$ imply $x=x'$. Consequently, $\mathcal{P}_{\mathrm{i}}$-$\mathsf{CompSep}$ is the category of compact Hausdorff spaces and continuous maps.

As for topological spaces, there is a tight connection between maps in $\mathcal{P}_{\mathrm{i}}$ and closed maps which is essentially shown in [Colebunders *et al.*, 2005]. Here a non-expansive map $f:(X,\delta)\to(Y,\delta')$ is called *closed* if

$$\delta'(y,f(A))\geqslant\inf\{\delta(x,A)\mid x\in X,\ f(x)=y\}$$

for all $A\subseteq X$ and $y\in Y$, which can equivalently be written as

$$f(A)^{(u)}\subseteq\bigcap_{v>u}f(A^{(v)})$$

for all $A \subseteq X$ and $u \in [0,\infty]$. For an approach space $X = (X, a)$, the $[0,\infty]$-relation $d := \hat{\beta}a \cdot m_X^\circ : \beta X \relbar\joinrel\mapsto \beta X$ is actually a metric on the set βX, which in terms of the approach distance δ of X can be written as

$$d(\mathfrak{x}, \mathfrak{x}') = \inf\{u \in [0,\infty] \mid \forall A \in \mathfrak{x} : A^{(u)} \in \mathfrak{x}'\}.$$

This construction defines a functor

$$M : \mathsf{App} \to \mathsf{Met},\ f : (X, a) \to (Y, b) \mapsto \beta f : (\beta X, d) \to (\beta Y, d')$$

(where d' denotes the metric $d' = \hat{\beta}b \cdot m_Y^\circ$ on βY). As in Subsection 5.4, M sends maps in $\mathcal{P}_\mathrm{i}$ to maps in $\mathcal{P}_\mathrm{i}$.

Lemma 5.4. *Let $f : X \to Y$ be in* App. *Then f is closed if and only if Mf lies in $\mathcal{P}_i$ (in the sense of Subsection 5.5).*

Proof. We write δ, δ' for the approach distance on X and Y respectively, $a : \beta X \times X \to [0,\infty]$ and $b : \beta Y \times Y \to [0,\infty]$ denote the corresponding ultrafilter convergence structures, and d is the metric $\hat{\beta}a \cdot m_X^\circ$ on βX and d' the metric $\hat{\beta}b \cdot m_Y^\circ$ on βY. Assume first that f is closed. Let $\mathfrak{x} \in \beta X$, $\mathfrak{y} \in \beta Y$ and $u \in [0,\infty]$ with $u > d'(\beta f(\mathfrak{x}), \mathfrak{y})$. Hence, for all $A \in \mathfrak{x}$, $f(A)^{(u)} \in \mathfrak{y}$ and consequently $f(A^{(v)}) \in \mathfrak{y}$ for all $v > u$. Therefore there exists some $\mathfrak{x}' \in \beta X$ with

$$\beta f(\mathfrak{x}') = \mathfrak{y} \qquad \text{and} \qquad \forall A \in \mathfrak{x} : A^{(v)} \in \mathfrak{x}',$$

hence $d(\mathfrak{x}, \mathfrak{x}') \leqslant v$. Assume now $Mf \in \mathcal{P}_\mathrm{i}$, and let $A \subseteq X$ and $u, v \in [0,\infty]$ with $v > u$. Let $y \in f(A)^{(u)}$. Then there is some $\mathfrak{y} \in \beta Y$ with $f(A) \in \mathfrak{y}$ and $v > b(\mathfrak{y}, y) = d'(\mathfrak{y}, e_Y(y))$. By hypothesis, there is some $\mathfrak{x}' \in \beta X$ with $\beta f(\mathfrak{x}') = e_Y(y)$ and $d(\mathfrak{x}, \mathfrak{x}') < v$, hence $f^{-1}(y) \cap A^{(v)} \neq \varnothing$. □

Proposition 5.5. *The following assertions are equivalent, for $f : X \to Y$ in* App*:*

(i) *$f \in \mathcal{P}_i$;*
(ii) *f is stably closed;*
(iii) *f is closed with* 0*-compact fibres.*

Proof. As in Proposition 5.2, where for the implication (ii)⇒(iii) now one uses the Kuratowski-Mrówka Theorem for approach spaces (see [Colebunders *et al.*, 2005], and [Hofmann, 2007] for a $(\mathbb{T}, \mathsf{V})$-version of this result). □

References

Barr, M. (1970), Relational algebras, in *Reports of the Midwest Category Seminar, IV*, pages 39–55, Lecture Notes in Mathematics, Vol. 137. Springer, Berlin.

Bourbaki, N. (1942), *Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres III et IV*, Actual. Sci. Ind., no. 916. Hermann & Cie., Paris, iv+158 pages.

Carboni, A., Lack, S. and Walters, R. F. C. (1993), Introduction to extensive and distributive categories, *J. Pure Appl. Algebra* **84** (2), 145–158.

Clementino, M. M., Giuli, E. and Tholen, W. (1996), Topology in a category: compactness, *Portugal. Math.* **53** (4), 397–433.

Clementino, M. M., Giuli, E. and Tholen, W. (2004a), A functional approach to general topology, in *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 103–163, Cambridge Univ. Press, Cambridge.

Clementino, M. M. and Hofmann, D. (2002), Triquotient maps via ultrafilter convergence, *Proc. Amer. Math. Soc.* **130** (11), 3423–3431.

Clementino, M. M. and Hofmann, D. (2003), Topological features of lax algebras, *Appl. Categ. Structures* **11** (3), 267–286.

Clementino, M. M., Hofmann, D. and Janelidze, G. (2005), Local homeomorphisms via ultrafilter convergence, *Proc. Amer. Math. Soc.* **133** (3), 917–922.

Clementino, M. M., Hofmann, D. and Tholen, W. (2004b), One setting for all: metric, topology, uniformity, approach structure, *Appl. Categ. Structures* **12** (2), 127–154.

Clementino, M. M. and Tholen, W. (1996), Tychonoff's theorem in a category, *Proc. Amer. Math. Soc.* **124** (11), 3311–3314.

Clementino, M. M. and Tholen, W. (2003), Metric, topology and multicategory—a common approach, *J. Pure Appl. Algebra* **179** (1-2), 13–47.

Colebunders, E., Lowen, R. and Wuyts, P. (2005), A Kuratowski-Mrówka theorem in approach theory, *Topology Appl.* **153** (5-6), 756–766.

Dikranjan, D. and Giuli, E. (1987), Closure operators. I, *Topology Appl.* **27** (2), 129–143.

Dikranjan, D. and Giuli, E. (1989), Compactness, minimality and closedness with respect to a closure operator, in *Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988)*, pages 284–296, World Sci. Publ., Teaneck, NJ.

Escardó, M. H. (2004), Synthetic topology of data types and classical spaces, *Electronic Notes in Theoretical Computer Science* **87**, 21–156.

Herrlich, H., Salicrup, G. and Strecker, G. E. (1987), Factorizations, denseness, separation, and relatively compact objects, *Topology Appl.* **27** (2), 157–169.

Hofmann, D. (2007), Topological theories and closed objects, *Adv. Math.* **215** (2), 789–824.

Janelidze, G. and Sobral, M. (2002), Finite preorders and topological descent. I, *J. Pure Appl. Algebra* **175** (1-3), 187–205, special volume celebrating the 70th birthday of Professor Max Kelly.

Lawvere, F. W. (1973), Metric spaces, generalized logic, and closed categories, *Rend. Sem. Mat. Fis. Milano* **43**, 135–166 (1974), also in: Repr. Theory Appl. Categ. No. 1 (2002), 1–37.

Lowen, R. (1997), *Approach spaces*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, x+253 pages, the missing link in the topology-uniformity-metric triad, Oxford Science Publications.

Manes, E. G. (1969), A triple theoretic construction of compact algebras., Sem. on Triples and Categorical Homology Theory, ETH Zürich 1966/67, Lect. Notes Math. 80, 91-118 (1969).

Manes, E. G. (1974), Compact Hausdorff objects, *General Topology and Appl.* **4**, 341–360.

Manes, E. G. (2010), Monads in topology, *Topology Appl.* **157** (5), 961–989.

Penon, J. (1972), Sur les objets séparés et les objets compacts, in *Esquisses mathématiques, No. 17*, pages iv+111, Fac. Sci. Univ. Paris VII, Paris, thèse 3ème cycle, Specialité: Mathématiques, Université de Paris

VII.

Seal, G. J. (2005), Canonical and op-canonical lax algebras, *Theory Appl. Categ.* **14**, 221–243.

Tholen, W. (1999), A categorical guide to separation, compactness and perfectness, *Homology Homotopy Appl.* **1**, 147–161.

Tholen, W. (2009), Ordered topological structures, *Topology Appl.* **156** (12), 2148–2157.

Wood, R. J. (1982), Abstract proarrows. I, *Cahiers Topologie Géom. Différentielle* **23** (3), 279–290.

Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal
E-mail address: dirk@ua.pt

Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada, M3J 1P3
E-mail address: tholen@mathstat.yorku.ca