A TOPOLOGIST'S VIEW OF CHU SPACES

ERALDO GIULI AND WALTER THOLEN†

Abstract. For a symmetric monoidal-closed category $\mathcal{X}$ and any object K, the category of K-Chu spaces is small-topological over $\mathcal{X}$ and small cotopological over $\mathcal{X}^{\mathrm{op}}$. Its full subcategory of $\mathcal{M}$-extensive K-Chu spaces is topological over $\mathcal{X}$ when $\mathcal{X}$ is $\mathcal{M}$-complete, for any morphism class $\mathcal{M}$. Often this subcategory may be presented as a full coreflective subcategory of Diers' category of affine K-spaces. Hence, in addition to their roots in the theory of pairs of topological vector spaces (Barr) and their connections with linear logic (Seely), the Dialectica categories (Hyland, de Paiva), and with the study of event structures for modeling concurrent processes (Pratt), Chu spaces seem to have a less explored link with algebraic geometry. We use the Zariski closure operator to describe the objects of the $*$-autonomous category of $\mathcal{M}$-extensive and $\mathcal{M}$-coextensive K-Chu spaces in terms of Zariski separation and to identify its important subcategory of complete objects.

1. Introduction

For a symmetric monoidal-closed category $\mathcal{X}$ and any object K of $\mathcal{X}$, in an appendix to Barr's [B1], Chu [C] described a $*$-autonomous category, whose objects became known as K-Chu spaces. The motivation and the historical roots of the construction from the theory of pairs of topological vector spaces are described in [B4], where the Chu construction is treated more generally in the non-symmetric case. The paper gives also the key references to the use of Boolean Chu spaces in theoretical computer science and in concrete duality theory, as developed primarily by Pratt [Pr]. For the connection of the Chu construction with linear logic and the Hyland–de Paiva Dialectica construction, we refer the reader to [B2], [Pa], [Pc].

This paper's first objective is to make explicit the topological nature of the Chu construction in the symmetric case, and to link it concretely to categories studied by Diers [D1, D2]. Hence, under mild hypotheses we show that the category $\mathsf{Chu}_K(\mathcal{X})$ "lives" over $\mathcal{X}$ and $\mathcal{X}^{\mathrm{op}}$ as a small-topological and small-cotopological category, respectively, a fact that, *inter alia*, facilitates the well-known computation of limits and colimits in $\mathsf{Chu}_K(\mathcal{X})$ (Section 2).

Relative to a well-behaved class $\mathcal{M}$ of morphisms in $\mathcal{X}$ we consider the subcategory $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ of $\mathcal{M}$-extensive K-Chu spaces (see [B2], [B5]), which turns out to actually be topological over $\mathcal{X}$ and which may be coreflectively embedded into Diers' topological

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Date: August 30, 2007.

1991 *Mathematics Subject Classification.* 18D10, 18D15, 18A35.

Key words and phrases. Chu space, extensive Chu space, affine object, separated Chu space, complete Chu space.

† Support by NSERC is gratefully acknowledged.

category $\mathsf{Aff}_K(\mathcal{X})$ of affine $\mathcal{X}$-objects over K. This embedding makes $\mathcal{M}$-extensive K-Chu spaces look like *internal* affine $\mathcal{X}$-objects over K (Sections 3 and 4). In Section 5 we give the "internal version" of our previous studies [G1], [G3] of Zariski separation and completion, showing in particular that a crucial tool for these notions in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, the Zariski closure, may actually be computed in $\mathsf{Aff}_K(\mathcal{X})$. We give a brief description of the monoidal structure of $\mathsf{Chu}_K(\mathcal{X})$ and its restriction to $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ in Section 6 and present the key part of this paper, namely a necessary and sufficient condition for the self-dual category of $\mathcal{M}$-extensive and $\mathcal{M}$-coextensive K-Chu spaces to coincide with the category of Zariski-separated $\mathcal{M}$-extensive K-Chu spaces.

We note that $\mathcal{M}$-coextensive Chu spaces were called $\mathcal{M}$-separated in [B3]. Luckily, our results show that, subject to some conditions, Barr's notion of separation coincides with that of Diers in the context of $\mathcal{M}$-extensive K-Chu spaces. Hence, the main part of this paper gives a new description of the category of Barr-separated $\mathcal{M}$-extensive K-Chu spaces which, under mild conditions was shown to be $*$-autonomous in [B2],[B5].

Although at the end of this paper we illustrate the general theory in terms of the classical Pontrjagin duality, a more elaborate list of examples will only be presented in the forthcoming paper [DGT]. We must also leave it to future work to make explicit the connections of this work with Sambin's formal topology (see [S]) which has many similar features, as well as with the studies on lax algebras (see [CT] and [Pa]).

2. Chu spaces over symmetrically tensored categories

2.1. **Definition.** *Let $\mathcal{X}$ be a category that comes equipped with a symmetric tensor product $\otimes$ and a fixed object K.* Hence, $\otimes : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ is a functor equipped with natural isomorphisms $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ satisfying $\sigma_{X,Y}^{-1} = \sigma_{Y,X}$. The category

$$\mathsf{Chu}_K(\mathcal{X})$$

of *K-Chu spaces over $\mathcal{X}$* has as its objects triples (X, r, A) with $\mathcal{X}$-objects X, A and an $\mathcal{X}$-morphism $r : X \otimes A \longrightarrow K$; its morphisms

$$(f, \varphi) : (X, r, A) \longrightarrow (Y, s, B)$$

are given by $\mathcal{X}$-morphisms $f : X \longrightarrow Y$ and $\varphi : B \longrightarrow A$ that make the diagram

(1)
$$\begin{tikzcd} X \otimes B \arrow[r, "f\otimes B"] \arrow[d, "X\otimes\varphi"'] & Y \otimes B \arrow[d, "s"] \\ X \otimes A \arrow[r, "r"] & K \end{tikzcd}$$

commutative. Since $\otimes$ is functorial, composition can be defined componentwise.

Symmetry of $\otimes$ makes $\mathsf{Chu}_K(\mathcal{X})$ self-dual: associating with $r : X \otimes A \longrightarrow K$ the morphism $r^\circ = (A \otimes X \xrightarrow{\sim} X \otimes A \longrightarrow K)$ defines an involution

$$(X, r, A) \longmapsto (A, r^\circ, X), \qquad (f, \varphi) \longmapsto (\varphi, f)$$

of $\mathsf{Chu}_K(\mathcal{X})$. We write $(X, r, A)^* = (A, r^\circ, X)$, $(f, \varphi)^* = (\varphi, f)$.

2.2. **Example.** The most basic example is $\mathcal{X} = \mathsf{Set}$, $\otimes = \times$ cartesian product, and $K = 2 = \{0, 1\}$, which gives the category $\mathsf{Chu}_2(\mathsf{Set})$ of *Boolean Chu spaces* (as considered in [Pr]). Here morphisms $r : X \times A \longrightarrow 2$ may be considered as relations $r : X \longrightarrow\!\!\!\!\!\!\!+\;\; A$ from X to A, or as 0-1-valued matrices whose rows and columns are labeled by X and A, respectively. Commutativity of (1) amounts to the condition

$$r(x, \varphi(b)) = s\,(f(x), b)$$

for all $x \in X$, $b \in B$ (so that the $(x, \varphi(b))$-entry of the matrix r coincides with $(f(x), b)$-entry of the matrix s), making f and φ look like "adjoint operators". Using relational composition, one may express the commutativity of (1) equivalently by the commutativity of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"', "+" marking] & Y \arrow[d, "s", "+" marking] \\ A \arrow[r, "\varphi^\circ", "+" marking] & B \end{tikzcd} \tag{2}$$

(Here we interpret maps as relations in the usual way.) The elements of X are often called *points* (or *objects*) of the Chu space (X, r, A), while the elements of A are referred to as *attributes*. We let this terminology prevail also in the general case (see Thm. 2.5).

2.3. **Example.** Let k be a commutative ring and $\mathcal{X}$ the category Mod_k of k-modules, equipped with its tensor product, and let K be some k-module. Then $\mathsf{Chu}_K(\mathsf{Mod}_k)$ is the (symmetric version of the) category considered by [B1] and many other authors, under various levels of generalization (see, for example, [B4], [K]).

We recall the following definition for the reader's convenience.

2.4. **Definition.** Let $U : \mathcal{A} \longrightarrow \mathcal{X}$ be a functor. A cone ([M], also natural source [AHS]) $\alpha : \Delta A \longrightarrow D$ with base $D : \mathcal{J} \longrightarrow \mathcal{A}$ in $\mathcal{A}$ is *U-cartesian* (or *U-initial*) if for every cone $\gamma : \Delta C \longrightarrow D$ in $\mathcal{A}$ and every $\mathcal{X}$ morphism $x : UC \longrightarrow UA$ there is exactly one $\mathcal{A}$-morphism $h : C \longrightarrow A$ with $Uh = u$ and $\alpha \cdot \Delta h = \gamma$. A *$U$-cartesian lifting* of a cone $\xi : \Delta X \longrightarrow UD$ in $\mathcal{X}$ with base D in $\mathcal{A}$ is a U-cartesian cone $\alpha : \Delta A \longrightarrow D$ with $UA = X$ and $U\alpha = \xi$. The functor U is *(small-)topological* if U-cartesian liftings exist for all cones $\xi : \Delta X \longrightarrow UD$ in $\mathcal{X}$ with (small) base D in $\mathcal{A}$. The dual notions are *U-cocartesian* (or *U-final*) *lifting* and *(small-)cotopological* functor. Note that the cases $\mathcal{J} = 1$ (when cones are just morphisms) and $\mathcal{J} = \emptyset$ (when cones are just objects) show that *a small-topological functor is a fibration with full and faithful right-adjoint functor.* Also note that topological functors are necessarily faithful and cotopological (see [AHS]), a statement that fails to be true for small-topological functors. That is why one may define topological functors using just discrete bases of arbitrary size. We tend to use the initial/final terminology predominantly when U is faithful.

2.5. **Theorem.** *Let $\mathcal{X}$ have small colimits preserved by $\otimes$ (in each variable). Then the domain and codomain functors*

$$\begin{aligned} \mathrm{pt} : \mathsf{Chu}_K(\mathcal{X}) &\longrightarrow \mathcal{X} \\ (X, r, A) &\longmapsto X \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \mathrm{at} : \mathsf{Chu}_K(\mathcal{X}) &\longrightarrow \mathcal{X}^{\mathrm{op}} \\ (X, r, A) &\longmapsto A \end{aligned}$$

are small-topological and small-cotopological, respectively.

Proof. For a small diagram $D : \mathcal{J} \longrightarrow \mathsf{Chu}_K(\mathcal{X})$ consider a cone $\Delta X \longrightarrow \mathrm{pt}\, D$ in $\mathcal{X}$, given by $\mathcal{X}$-morphisms $f_i : X \longrightarrow Y_i$ endowed with Chu-structures $Di = (Y_i, s_i, B_i)$ and Chu-morphisms $D\delta : (t_{i,j}, \tau_{i,j}) : Di \longrightarrow Dj$ for all $\delta : i \longrightarrow j$ in $\mathcal{J}$. We must find a Chu-space (X, r, A) and a cone $\alpha : \Delta(X, r, A) \longrightarrow D$ which is a pt-cartesian lifting of the given data. For that one forms

$$A = \mathrm{colim}\, \mathrm{pt}\, D \quad \text{with injections } \varphi_i : B_i \longrightarrow A \ (i \in \mathrm{ob}\, \mathcal{J})$$

and, using preservation of this colimit by $X \otimes (-)$, defines r by the commutative diagrams

$$\begin{tikzcd} X \otimes B_i \arrow[r, "f_i \otimes B_i"] \arrow[d, "X \otimes \varphi_i"'] & Y_i \otimes B_i \arrow[d, "s_i"] \\ X \otimes A \arrow[r, "r"] & K \end{tikzcd} \tag{3}$$

This is legitimate since the morphisms $s_i(f_i \otimes B_i)$ $(i \in \mathrm{ob}\, \mathcal{J})$ form a cocone over $X \otimes$ (at D): for $\delta : i \longrightarrow j$ in $\mathcal{J}$ one has

$$\begin{aligned} s_i(f_i \otimes B_i)(X \otimes \tau_{i,j}) &= s_i(Y_i \otimes \tau_{i,j})(f_i \otimes B_j) && \text{(functoriality of } \otimes) \\ &= s_j(t_{i,j} \otimes B_j)(f_i \otimes B_j) && (D\delta \text{ is a Chu-morphism}) \\ &= s_j(f_j \otimes B_j) && (\Delta X \longrightarrow \mathrm{pt}\, D \text{ is a cone}). \end{aligned}$$

It is clear that $\alpha_i = (f_i, \varphi_i)$ defines a cone. It remains to be shown that it is pt-cartesian. In fact, for any cone $\gamma : \Delta(Z, t, C) \longrightarrow D$ and any morphism $x : Z \longrightarrow X$ with $U\alpha \cdot \Delta x = \gamma$, writing $\gamma_i = (g_i, \psi_i)$ we may consider the cubes

$$\begin{tikzcd} & Z \otimes B_i \arrow[rr, "h \otimes B_i"] \arrow[dl, "1"'] \arrow[dd, "Z \otimes \varphi_i"] & & X \otimes B_i \arrow[dl, "f_i \otimes B_i"'] \arrow[dd, "X \otimes \varphi_i"] \\ Z \otimes B_i \arrow[rr, "g_i \otimes B_i"'] \arrow[dd, "Z \otimes \psi_i"'] & & Y_i \otimes B_i \arrow[dd, "s_i"'] & \\ & Z \otimes A \arrow[rr, "h \otimes A"] \arrow[dl, "Z \otimes \chi"] & & X \otimes A \arrow[dl, "r"] \\ Z \otimes C \arrow[rr, "t"'] & & K & \end{tikzcd} \tag{4}$$

Here $\chi : A \longrightarrow C$ is the $\mathcal{X}$-morphism with $\chi\varphi_i = \psi_i$ for all $i \in \mathrm{ob}\, \mathcal{J}$, hence, it makes the left faces of (4) commute. The right faces commute by (3), and the top- and back-faces do so by functoriality of $\otimes$. The front faces commute since each γ_i is a Chu-morphism. Since the family $(Z \otimes \varphi_i)_{i \in \mathrm{ob}\, \mathcal{J}}$ is (as a colimit cocone) epic, also the bottom face of (4) must commute, so that $(h, \chi) : (Z, t, C) \longrightarrow (X, r, A)$ is indeed a Chu-morphism. It is obviously the only morphism "over h" with $\alpha \cdot \Delta(h, \chi) = \gamma$.

The assertion about at follows from the self-duality of $\mathsf{Chu}_K(\mathcal{X})$, since

$$\begin{tikzcd} \mathsf{Chu}_K(\mathcal{X}) \arrow[rr, "(-)^*"] \arrow[dr, "\mathrm{pt}"'] & & \mathsf{Chu}_K(\mathcal{X})^{\mathrm{op}} \arrow[dl, "\mathrm{at}^{\mathrm{op}}"] \\ & \mathcal{X} & \end{tikzcd} \tag{5}$$

commutes. □

For any functor $U : \mathcal{A} \longrightarrow \mathcal{X}$, the U-cartesian lifting of a limit cone in $\mathcal{X}$ is a limit cone in $\mathcal{A}$. Consequently, the proof of 2.5 yields immediately:

2.6. **Corollary** ([B2]). *If $\mathcal{X}$ has all limits and colimits of a given type $\mathcal{J}$, and if the colimits are preserved by $\otimes$ (in each variable), then $\mathsf{Chu}_K(\mathcal{X})$ has also all limits and colimits of type $\mathcal{J}$. More precisely, the limit cone*

$$(X, r, A) \xrightarrow{(f_i, \varphi_i)} Di \qquad (i \in \mathrm{ob}\,\mathcal{J})$$

of a diagram $D : \mathcal{J} \longrightarrow \mathsf{Chu}_K(\mathcal{X})$ with $Di = (Y_i, s_i, B_i)$ is formed by $X = \lim Y_i$ with projections f_i in $\mathcal{X}$, $A = \operatorname{colim} B_i$ with injections φ_i in $\mathcal{X}$, and with r determined by diagram (3). Its colimit cocone

$$Di \xrightarrow{(k_i, \lambda_i)} (Z, t, C) \qquad (i \in \mathrm{ob}\,\mathcal{J})$$

is obtained as $Z = \operatorname{colim} Y_i$ with injections k_i in $\mathcal{X}$, $C = \lim B_i$ with projections λ_i in $\mathcal{X}$, and t determined by

$$t^\circ(C \otimes k_i) = s_i^\circ(\lambda_i \otimes Y_i)$$

for all $i \in \mathrm{ob}\,\mathcal{J}$. □

2.7. **Corollary.** *The functor* pt *of* 2.5 *is a fibration while* at *is a cofibration. A morphism $(f, \varphi) : (X, r, A) \longrightarrow (Y, s, B)$ in $\mathsf{Chu}_K(\mathcal{X})$ is* pt*-cartesian if, and only if, φ is an isomorphism in $\mathcal{X}$, and it is* at*-cocartesian if, and only if, f is an isomorphism in $\mathcal{X}$.* □

Finally, for $\mathcal{J} = \emptyset$ one obtains from the proof of 2.5:

2.8. **Corollary.** *Let $\mathcal{X}$ have an initial object 0, with $X \otimes 0 \cong 0$ for all $X \in \mathrm{ob}\,\mathcal{X}$. Then* pt *of* 2.5 *has a full and faithful right adjoint, while* at *has a full and faithful left adjoint. Via $X \longmapsto (X, 0 \to K, 0)$ $\mathcal{X}$ is reflectively embedded into $\mathsf{Chu}_K(\mathcal{X})$, while $A \longmapsto (0, 0 \to K, A)$ embeds $\mathcal{X}^{\mathrm{op}}$ coreflectively into $\mathsf{Chu}_K(\mathcal{X})$.* □

2.9. **Definition.** One calls the object K of $\mathcal{X}$ $\otimes$-*exponentiating* if, for all X in $\mathcal{X}$, the functor $X \otimes (-) : \mathcal{X} \to \mathcal{X}$ admits a couniversal arrow for K; that is, if there is an object K^X and a morphism

$$e_X : X \otimes K^X \to K$$

such that every morphism $r : X \otimes A \to K$ factors as

$$e_X(X \otimes \check{r}) = r$$

with a unique morphism $\check{r} : A \to K^X$. (Of course, when $\mathcal{X}$ is *closed* with respect to its tensor product, so that all objects are $\otimes$-exponentiating, in particular K has this property.) Since $\otimes$ is symmetric, we therefore have natural bijective correspondences

$$\frac{\dfrac{X \otimes A \xrightarrow{r} K}{A \xrightarrow{\check{r}} K^X}}{X \xrightarrow{\hat{r}} K^A.}$$

The defining diagram (1) for a Chu morphism $(f, \varphi) : (X, r, A) \to (Y, s, B)$ takes the equivalent forms

(6)
$$\begin{tikzcd} A \arrow[d, "\check{r}"'] & B \arrow[l, "\varphi"'] \arrow[d, "\check{s}"] \\ K^X & K^Y \arrow[l, "K^f"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\hat{r}"'] & Y \arrow[d, "\hat{s}"] \\ K^A \arrow[r, "K^\varphi"] & K^B \end{tikzcd}$$

2.10. **Proposition.** *The following conditions are equivalent:*

(i) K *is* $\otimes$*-exponentiating;*
(ii) pt *has a full and faithful left adjoint;*
(iii) at *has a full and faithful right adjoint.*

Proof. (i) $\Longrightarrow$ (ii): Given $f : X \to Y$ in $\mathcal{X}$ and (Y, s, B) in $\mathsf{Chu}_K(\mathcal{X})$, there is a unique morphism $\varphi : B \to K^X$ with

$$e_X(X \otimes \varphi) = s(f \otimes B)$$

Hence, $1_X : X \to \mathrm{pt}(X, e_X, K^X)$ is pt-universal for X in $\mathcal{X}$.
(ii) $\Longrightarrow$ (i) Without loss of generality, assume the pt-universal arrow for X in $\mathcal{X}$ to be 1_X, and exploit its universal property for $1_X : X \to \mathrm{pt}(X, r, A)$.
(i) $\Longleftrightarrow$ (iii) follows dually, with (5). $\square$

2.11. **Corollary.** *Let* K *be* $\otimes$*-exponentiating. Then* $\mathcal{X}$ *is coreflectively embedded into* $\mathsf{Chu}_K(\mathcal{X})$ *via* $X \mapsto (X, e_X, K^X)$*, while* $A \mapsto (K^A, \widetilde{e}_A, A)$ *(with* $\widetilde{e}_A : K^A \otimes A \xrightarrow{\sim} A \otimes K^A \xrightarrow{e_A} K$*) embeds* $\mathcal{X}^{\mathrm{op}}$ *reflectively into* $\mathsf{Chu}_K(\mathcal{X})$*.*

3. Extensive Chu spaces

3.1. **Definition.** In addition to $\mathcal{X}$, $\otimes$, K as in 2.1, we now *consider a class* $\mathcal{M}$ *of morphisms in* $\mathcal{X}$ *containing all isomorphisms and being closed under composition with them. We always assume* K *to be* $\otimes$*-exponentiating, and we keep the notation of* 2.9. A K-Chu space (X, r, A) is called $\mathcal{M}$*-extensive* if $(\check{r} : A \to K^X) \in \mathcal{M}$, and $\mathcal{M}$*-coextensive* if $(\hat{r} : X \to K^A) \in \mathcal{M}$. For $\mathcal{M} = \mathrm{Mono}\,\mathcal{X}$, the prefix $\mathcal{M}$ is omitted. Barr ([B2], [B5]) calls $\mathcal{M}$-coextensive K-Chu spaces $\mathcal{M}$-separated, a terminology that will receive a new justification in Section 6 below.

3.2. **Examples.**

(1) A Boolean Chu space (X, r, A), whose structure is given by a 0-1-valued $X \times A$-matrix, is (co)extensive if columns (rows) with distinct labels have distinct values.
(2) A k-Chu space (X, r, A) over Mod_k is extensive if

$$(\forall x \in X : r(x \otimes a) = 0) \implies a = 0$$

for all $a \in A$, and coextensive if

$$(\forall a \in A : r(x \otimes a) = 0) \implies x = 0$$

for all $x \in X$. For example, for $K = \mathbb{R}$, every Euclidean space (= $\mathbb{R}$-vector space that comes with a scalar product) can be naturally considered as an extensive and coextensive $\mathbb{R}$-Chu space.

We denote by $\mathcal{X}^{\mathbf{2}} = (\mathcal{X} \downarrow \mathcal{X})$ the category of morphisms of $\mathcal{X}$ and *consider* $\mathcal{M}$ *as a full subcategory of* $\mathcal{X}^{\mathbf{2}}$. Recall that $\mathcal{M}$ is part of an orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system (for morphisms of $\mathcal{X}$) if, and only if, $\mathcal{M}$ is reflective in $\mathcal{X}^{\mathbf{2}}$ and closed under composition in $\mathcal{X}$. In that case, Barr [B5] showed coreflectivity of $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ in $\mathsf{Chu}_K(\mathcal{X})$; his proof shows that, slightly more generally one has:

3.3. **Proposition.** *If $\mathcal{M}$ is reflective in $\mathcal{X}^{\mathbf{2}}$ (in particular, when $\mathcal{M}$ is part of an orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms of $\mathcal{X}$), then the $\mathcal{M}$-extensive K-Chu spaces form a full coreflective subcategory $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ of $\mathsf{Chu}_K(\mathcal{X})$.*

Proof. In order to construct the $\mathcal{M}$-extensive coreflection of a K-Chu space (X, r, A) over $\mathcal{X}$, we consider the reflection of $\check{r} : A \longrightarrow K^X$ (as an object of $\mathcal{X}^{\mathbf{2}}$) into $\mathcal{M}$, i.e., a locally orthogonal $\mathcal{M}$-factorization of $\check{r} = m \cdot e$, with $m : C \longrightarrow K^X$ in $\mathcal{M}$ (see [T]), which defines an $\mathcal{M}$-extensive K-Chu space (X, t, C) with $\check{t} = m$ and a Chu morphism

$$(1_X, e) : (X, t, C) \longrightarrow (X, r, A).$$

Any other morphism $(g, \psi) : (Y, s, B) \longrightarrow (X, r, A)$ with $(Y, s, B) \in \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ leads to the solid-line commutative diagram

(7)
$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "e"'] & B \arrow[dd, "\check{s} \in \mathcal{M}"] \\ C \arrow[ur, dashed, "\chi"'] \arrow[d, "m"'] & \\ K^X \arrow[r, "K^g"] & K^Y \end{tikzcd}$$

with the unique fill-in arrow χ. Hence, (g, ψ) factors uniquely as $(g, \psi) = (1_X, e)(g, \chi)$, with $(g, \chi) : (Y, s, B) \longrightarrow (X, t, C)$. □

3.4. **Corollary.** *If $\mathcal{M}$ is reflective in $\mathcal{X}^{\mathbf{2}}$, then the full subcategory $\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})$ of $\mathcal{M}$-coextensive K-Chu spaces is reflective in $\mathsf{Chu}_K(\mathcal{X})$.*

Proof. By 3.3, $\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})^{\mathrm{op}}$ is coreflective in $\mathsf{Chu}_K(\mathcal{X})^{\mathrm{op}}$, since the diagram

(8)
$$\begin{tikzcd} \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \arrow[r, "(-)^*", "\sim"'] \arrow[d, hook] & \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})^{\mathrm{op}} \arrow[d, hook] \\ \mathsf{Chu}_K(\mathcal{X}) \arrow[r, "(-)^*", "\sim"'] & \mathsf{Chu}_K(\mathcal{X})^{\mathrm{op}} \end{tikzcd}$$

commutes. □

3.5. **Definition.** Recall that $\mathcal{X}$ is called $\mathcal{M}$-*complete* (see [T]) if pullbacks of $\mathcal{M}$-morphisms (along arbitrary morphisms) exist and belong to $\mathcal{M}$, and if every family (of any size) of $\mathcal{M}$-morphisms with common codomain has an intersection (= generalized pullback) belonging to $\mathcal{M}$. $\mathcal{M}$-completeness of $\mathcal{X}$ means equivalently that sinks (= arbitrary families of morphisms with common codomain) have locally orthogonal $\mathcal{M}$-factorizations. Hence, $\mathcal{M}$-completeness implies reflectivity of $\mathcal{M}$ in $\mathcal{X}^{\mathbf{2}}$, i.e., the existence of locally orthogonal $\mathcal{M}$-factorizations for morphisms. Notice that $\mathcal{M}$-completeness implies $\mathcal{M} \subseteq \mathrm{Mono}\,\mathcal{X}$ (see [BT]). Furthermore, $\mathcal{M}$ is part of an orthogonal factorization system $(\mathbb{E}, \mathcal{M})$ for sinks if, and only if, $\mathcal{X}$ is $\mathcal{M}$-complete and $\mathcal{M}$ is closed under composition in $\mathcal{X}$.

3.6. **Theorem.** *If $\mathcal{X}$ is $\mathcal{M}$-complete, then the restrictions $\widetilde{\mathrm{pt}} : \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \longrightarrow \mathcal{X}$ and $\widetilde{\mathrm{at}} : \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}) \to \mathcal{X}^{\mathrm{op}}$ of* pt *and* at*, respectively, are topological functors, hence also cotopological. In particular, they are faithful, and both functors have full and faithful left- and right adjoints.*

Proof. We must find a $\widetilde{\mathrm{pt}}$-initial lifting for every family $f_i : X \longrightarrow Y_i$ of $\mathcal{X}$-morphisms with ($\mathcal{M}$-extensive) K-Chu structures (Y_i, s_i, B_i), $i \in I$. For that one considers a locally orthogonal $\mathcal{M}$-factorization of the family of morphisms $K^{f_i} \cdot \check{s}_i$:

(9)
$$\begin{tikzcd}
A \arrow[d, "\mathcal{M}\ni m"'] & B_i \arrow[l, "e_i"'] \arrow[d, "\check{s}_i"] \\
K^X & K^{Y_i} \arrow[l, "K^{f_i}"']
\end{tikzcd}$$

Hence, we obtain an $\mathcal{M}$-extensive K-Chu space (X, r, A) with $\check{r} = m$ and morphisms $(f_i, e_i) : (X, r, A) \longrightarrow (Y_i, s_i, B_i)$. Given any other family $(g_i, \psi_i) : (Z, t, C) \longrightarrow (Y_i, s_i, B_i)$ $(i \in I)$ with an $\mathcal{M}$-extensive K-Chu space (Z, t, C) and an $\mathcal{X}$-morphism $x : Z \longrightarrow X$ with $f_i x = g_i$ for all $i \in I$, we may consider the commutative solid-arrow diagram

(10)
$$\begin{tikzcd}
B_i \arrow[r, "\psi_i"] \arrow[d, "e_i"'] & C \arrow[dd, "\check{t}"] \\
A \arrow[ur, dashed, "\chi"'] \arrow[d, "m=\check{r}"'] & \\
K^X \arrow[r, "K^x"] & K^Z
\end{tikzcd}$$

and the unique fill-in arrow χ, yielding a unique Chu morphism $(x, \chi) : (Z, t, C) \longrightarrow (X, r, A)$ with the desired properties.

The assertion for at follows dually with diagram (8). □

3.7. **Remarks.**

(1) As we saw in 2.10, pt : $\mathsf{Chu}_K(\mathcal{X}) \longrightarrow \mathcal{X}$ has a full and faithful left adjoint $E : X \longmapsto (X, e_X, K^X)$, with $\check{e}_X = 1_{K^X}$. Since, without any hypothesis on $\mathcal{M}$, E takes its values in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, one has always that $\widetilde{\mathrm{pt}} : \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \longrightarrow \mathcal{X}$ has a full and faithful left adjoint. Likewise, $\widetilde{\mathrm{at}} : \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}) \longrightarrow \mathcal{X}^{\mathrm{op}}$ has always a full and faithful right adjoint.

(2) There are two distinct ways of constructing the limit of a diagram $D : \mathcal{J} \longrightarrow \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$. One way is to consider it as a diagram in $\mathsf{Chu}_K(\mathcal{X})$ and to form its pt-cartesian lifting (see 2.6) and then its coreflection into $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ (see 3.3), the needed sufficient conditions granted. The other is to form the $\widetilde{\mathrm{pt}}$-initial lifting according to Theorem 3.6. To form the colimit of D, one just considers D as a diagram in $\mathsf{Chu}_K(\mathcal{X})$ and forms its colimit, which must already lie in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$. Dual statements apply to $\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})$.

(3) Here is an important general case when $\mathcal{M}$ is *not* part of an orthogonal $(\mathcal{E},\mathcal{M})$-factorization system, but 3.3, 3.4 and 3.6 are still applicable: take $\mathcal{M}$ to be the class of regular monomorphisms, in a category $\mathcal{X}$ that has cokernel pairs and their equalizers. Then $\mathcal{M}$ is reflective in $\mathcal{X}^2$, and $\mathcal{X}$ is even $\mathcal{M}$-complete when $\mathcal{X}$ admits all intersections of (regular) monomorphisms. Still, $\mathcal{M}$ may fail to be closed under composition, as the categories of commutative rings (or k-algebras) and of all (small) categories show.

From the proof of 3.6 we also obtain:

3.8. **Corollary.** *If $\mathcal{M}$ is part of an orthogonal $(\mathcal{E},\mathcal{M})$-factorization system, then the $\widetilde{\mathrm{pt}}$-initial morphisms are those morphisms (f,φ) with φ in $\mathcal{E}$, and the $\widetilde{\mathrm{at}}$-cocartesian morphisms are those morphisms (f,φ) with $f\in\mathcal{E}$. If $\mathcal{M}$ is part of an orthogonal $(\mathbb{E},\mathcal{M})$-factorization system for sinks, then $\widetilde{\mathrm{pt}}$-initial cones and $\widetilde{\mathrm{at}}$-final cocones are characterized analogously.*

We note that 3.8 entails 2.7 since, for $\mathcal{M}=\mathrm{Mor}\,\mathcal{X}$, we have $\mathcal{E}=\mathrm{Iso}\,\mathcal{X}$, and

$$\mathsf{Chu}_K(\mathcal{X})=\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})=\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}).$$

It is not surprising that $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ (and, analogously, $\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})$) inherits a factorization system from $\mathcal{X}$. With

$$\overline{\mathcal{M}}=\{\,(f,\varphi)\mid f\in\mathcal{M},\varphi\in\mathcal{E}\,\}$$

denoting the class of $\widetilde{\mathrm{pt}}$-initial morphisms in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ with underlying $\mathcal{M}$-morphisms, one obtains:

3.9. **Corollary.** *If $\mathcal{M}$ is part of an orthogonal factorization system of $\mathcal{X}$, then $\overline{\mathcal{M}}$ is part of an orthogonal factorization system of $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$.*

Explicitly, the factorization

$$\begin{tikzcd}
 & (Z,t,C) \arrow[dr, "{(m,\eta)}"] & \\
(X,r,A) \arrow[ur, "{(e,\delta)}"] \arrow[rr, "{(f,\varphi)}"'] & & (Y,s,B)
\end{tikzcd} \tag{11}$$

is obtained by $(\mathcal{E},\mathcal{M})$-factoring $f=me$, by defining t such that $\check{t}\eta$ is an $(\mathcal{E},\mathcal{M})$-factorization of $K^m\cdot\check{s}$, and by letting δ be the induced morphism rendering the following diagram commutative:

$$\begin{tikzcd}
A \arrow[d, "\check{r}"'] & C \arrow[l, "\delta"] \arrow[d, "\check{t}"] & B \arrow[l, "\eta"] \arrow[d, "\check{s}"] \arrow[ll, bend right, "\varphi"'] \\
K^X & K^Z \arrow[l, "K^e"] & K^Y \arrow[l, "K^m"] \arrow[ll, bend left, "K^f"]
\end{tikzcd} \tag{12}$$

Hence, the companion $\widetilde{\mathcal{E}}$ of $\overline{\mathcal{M}}$ contains precisely those morphisms (f,φ) with f in $\mathcal{E}$, the companion of $\mathcal{M}$.

3.10. **Remarks.**

(1) The presentation of $\widetilde{\mathcal{E}}$ may be somewhat surprising, since there is no restriction on φ when $(f,\varphi) \in \widetilde{\mathcal{E}}$. A more symmetric picture arises when we consider factorizations in $\mathsf{Chu}_K(\mathcal{X})$, rather than in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$: without proof we state that *when $(\mathcal{E},\mathcal{M})$ is an orthogonal factorization system of $\mathcal{X}$, then $(\overline{\mathcal{E}},\overline{\mathcal{M}})$ is an orthogonal factorization system of $\mathsf{Chu}_K(\mathcal{X})$ with*

$$\overline{\mathcal{E}} = \{(f,\varphi) \mid f \in \mathcal{E}, \varphi \in \mathcal{M}\},$$

provided that for all $e : X \to Y$ and $\eta : A \to B$ in $\mathcal{E}$ the diagrams

$$\begin{array}{ccc} X \otimes A & \xrightarrow{e\otimes A} & Y \otimes A \\ {\scriptstyle X\otimes\eta}\downarrow & & \downarrow{\scriptstyle Y\otimes\eta} \\ X \otimes B & \xrightarrow{e\otimes B} & Y \otimes B \end{array}$$

are pushout diagrams.

(2) Note that, without any supplementary conditions, 2.7 gives two further (but less interesting) orthogonal factorization systems for $\mathsf{Chu}_K(\mathcal{X})$: both, the pt-cartesian liftings of $\mathcal{M}$-morphisms and the at-cocartesian liftings of $\mathcal{E}$-morphisms are part of orthogonal factorization systems in $\mathsf{Chu}_K(\mathcal{X})$ when $(\mathcal{E},\mathcal{M})$ is an orthogonal factorization system of $\mathcal{X}$.

4. Extensive Chu spaces as affine $\mathcal{X}$-objects

4.1. **Definition.** Recall that, for any category $\mathcal{X}$ and an $\mathcal{X}$-object K, the category $\mathsf{Aff}_K(\mathcal{X})$ of affine $\mathcal{X}$-objects over K has as objects pairs $(X,\mathcal{A})$ with $X \in \mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}(X,K)$. A morphism $f : (X,\mathcal{A}) \longrightarrow (Y,\mathcal{B})$ is an $\mathcal{X}$-morphism with $bf \in \mathcal{A}$ for all $b \in \mathcal{B}$. Clearly, $\mathsf{Aff}_K(\mathcal{X})$ is a topological category over $\mathcal{X}$: given any family $f_i : X \longrightarrow Y_i$ with $\mathsf{Aff}_K(\mathcal{X})$-structures $\mathcal{B}_i$ on Y_i $(i \in I)$, the U-initial structure on $\mathcal{X}$ (with respect to $U : \mathsf{Aff}_K(\mathcal{X}) \longrightarrow \mathcal{X}$) is $\mathcal{A} = \{\, bf_i \mid i \in I,\, b \in \mathcal{B}_i \,\}$.

Let us now assume that, *with $\mathcal{M}$ closed under composition, $\mathcal{X}$ is an $\mathcal{M}$-complete, symmetric monoidal category, with tensor product $\otimes$ and $\otimes$-neutral element k, and let K be a $\otimes$-exponentiating object.* For an object A of $\mathcal{X}$, we consider the morphisms $k \longrightarrow A$ as *generalized elements* of A and define a functor

$$\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \xrightarrow{J} \mathsf{Aff}_K(\mathcal{X}),$$

as follows: $J(X,r,A) = (X, \{\, \alpha^\# \mid \alpha : k \longrightarrow A \,\})$, with $\alpha^\# : X \longrightarrow K$ corresponding to α via

$$\frac{\dfrac{k \xrightarrow{\alpha} A \xrightarrow{\check{r}} K^X}{X \otimes k \longrightarrow K}}{X \xrightarrow{\alpha^\#} K}$$

hence, $\alpha^\# = (X \xrightarrow{\sim} X \otimes k \xrightarrow{X\otimes\alpha} X \otimes A \xrightarrow{r} K)$. For $(f,\varphi) : (X,r,A) \longrightarrow (Y,s,B)$, $J(f,\varphi) = f$. We must show that f is indeed a morphism in $\mathsf{Aff}_K(\mathcal{X})$: for every $\beta : k \longrightarrow$

B, we have correspondences

$$\frac{\dfrac{k \xrightarrow{\beta} B \xrightarrow{\varphi} A \xrightarrow{\check{r}} K^X}{k \xrightarrow{\beta} B \xrightarrow{\check{s}} K^Y \xrightarrow{Kf} K^X}}{X \xrightarrow{f} Y \xrightarrow{\beta^\#} K,}$$

which shows $\beta^\# f = (\beta\varphi)^\#$ for all $\beta : k \longrightarrow B$. Note that J is faithful since $\mathcal{M} \subseteq \operatorname{Mono}\mathcal{X}$.

4.2. **Proposition.** *J has a right adjoint.*

Proof. Given $(X, \mathcal{A}) \in \mathsf{Aff}_K(\mathcal{X})$, for every $a : X \longrightarrow K$ in $\mathcal{A}$ let $\ulcorner a \urcorner$ correspond to a via

$$\frac{\dfrac{X \xrightarrow{a} K}{X \otimes k \longrightarrow K}}{k \xrightarrow{\ulcorner a \urcorner} K^X.}$$

Obtain A and r from an $(\mathbb{E}, \mathcal{M})$-factorization (see 3.5) of the family $(\ulcorner a \urcorner)_{a \in \mathcal{A}}$:

(13)
$$\begin{tikzcd} & A \arrow[dr, "\check{r}"] & \\ k \arrow[ur, "\eta_a"] \arrow[rr, "\ulcorner a \urcorner"] & & K^X \end{tikzcd}$$

Since, for all $a \in \mathcal{A}$, $(\eta_a)^\# = a$, we have a morphism

$$1_X : J(X, r, A) \longrightarrow (X, \mathcal{A})$$

in $\mathsf{Aff}_K(\mathcal{X})$, which acts as the co-unit of the adjunction, as we will show next. Indeed, given $g : J(Y, s, B) \longrightarrow (X, \mathcal{A})$ in $\mathsf{Aff}_K(\mathcal{X})$, for every $a : X \longrightarrow K$ in $\mathcal{A}$ we have $\beta_a : k \longrightarrow B$ with $\beta_a^\# = ag$. The $(\mathbb{E}, \mathcal{M})$-diagonalization property applied to the commutative diagrams

(14)
$$\begin{tikzcd} k \arrow[rr, "\beta_a"] \arrow[d, "\eta_a"'] & & B \arrow[d, "\check{s}"] \\ A \arrow[r, "\check{r}"] & K^X \arrow[r, "K^g"] & K^Y \end{tikzcd}$$

gives a unique morphism $\psi : A \longrightarrow B$ with $\check{s}\psi = K^g\check{r}$ (and $\psi\eta_a = \beta_a$ for all $a \in \mathcal{A}$), making $(g, \psi) : (Y, s, B) \longrightarrow (X, r, A)$ the unique $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$-morphism with $J(g, \psi) = g$. $\square$

Recall that an object G in $\mathcal{X}$ is an $\mathbb{E}$-*generator* of $\mathcal{X}$ if for all $X \in \mathcal{X}$, the family of all morphisms $G \longrightarrow X$ in $\mathcal{X}$ lies in $\mathbb{E}$. G is $\mathbb{E}$-*projective* if, for all $(e_i : X_i \longrightarrow Y)_{i \in I}$ in $\mathbb{E}$, every morphism $G \longrightarrow Y$ is of the form $e_i x$, for some $i \in I$ and $x : G \longrightarrow X_i$. Note that every $\mathbb{E}$-projective object is $\mathcal{E}$-projective, with $\mathcal{E} = \mathbb{E} \cap \operatorname{Mor}\mathcal{X}$ containing the singleton families in $\mathbb{E}$, but not conversely. (For example, in Set with $\mathbb{E} = \{$jointly epic families$\}$, a singleton set is $\mathbb{E}$-projective, while a two-element set is $\mathcal{E}$-projective but not $\mathbb{E}$-projective.)

4.3. **Theorem.** *If k is an $\mathbb{E}$-generator of $\mathcal{X}$, then $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is equivalent to a full coreflective subcategory of $\mathsf{Aff}_K(\mathcal{X})$ given by its image under J. If k is $\mathbb{E}$-projective, then J preserves initial structures, i.e., J maps $\widetilde{\mathrm{pt}}$-initial families to U-initial families.*

Proof. For the first part of the Theorem, it suffices to show that J is full when k is an $\mathbb{E}$-generator. Hence, consider $f : J(X, r, A) \longrightarrow J(Y, s, B)$ in $\mathsf{Aff}_K(\mathcal{X})$. For all $\beta : k \longrightarrow B$ one has $\alpha_\beta : k \longrightarrow A$ with $\beta^\# f = \alpha_\beta^\#$, that is, the solid arrow diagram

(15)
$$\begin{tikzcd} k \arrow[rr, "\alpha_\beta"] \arrow[d, "\beta"'] & & A \arrow[d, "\check{r}"] \\ B \arrow[r, "\check{s}"] \arrow[rru, dashed, "\varphi"] & K^Y \arrow[r, "K^f"] & K^X \end{tikzcd}$$

commutes for all β. By hypothesis, $\mathcal{X}(k, B) \in \mathbb{E}$, so that there is a unique diagonal fill-in morphism φ, rendering the desired $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$-morphism $(f, \varphi) : (X, r, A) \longrightarrow (Y, s, B)$.

Next, consider a $\widetilde{\mathrm{pt}}$-initial family $(f_i, \varphi_i) : (X, r, A) \longrightarrow (Y_i, s_i, B_i)$ $(i \in I)$ in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$; hence, $(\varphi_i)_{i\in I} \in \mathbb{E}$. By hypothesis, every $\alpha : k \longrightarrow A$ is of the form $\varphi_i\beta$, for some $i \in I$ and $\beta : k \longrightarrow B_i$. Hence,

$$\{\, \alpha^\# \mid \alpha : k \longrightarrow A \,\} = \{\, \beta^\# f_i \mid i \in I,\ \beta : k \longrightarrow B_i \,\},$$

which shows that $(J(f_i, \varphi_i))_{i\in I}$ is U-initial. $\square$

4.4. **Remarks.**

(1) The K-affine structure $\mathcal{A}$ that J puts on X (for a given object $(X, r, A) \in \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$) may be equivalently described as the hom-set

$$\mathcal{A} = \mathsf{Chu}_K(\mathcal{X})\big((X, r, A), (K, \varrho_K, k)\big),$$

with the natural isomorphism $\varrho_K : K \otimes k \longrightarrow K$ given by the monoidal structure of $\mathcal{X}$.

(2) For $\mathcal{X} = \mathsf{Set}$ (with $\otimes = \times, K = 2$, as in 2.2) and $\mathcal{M} = \mathrm{Mono}\mathcal{X}$, the categories $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ and $\mathsf{Aff}_K(\mathcal{X})$ are obviously equivalent. Its objects are most conveniently described as "generalized spaces" $(X, \mathcal{A})$ with $\mathcal{A} \subseteq PX$, with "continuous" morphisms $f : (X, \mathcal{A}) \to (Y, \mathcal{B}) : f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. The papers [D1], [D2], [G1], [G2] consider a class Ω of operations $K^{n_\omega} \to K$ $(\omega \in \Omega)$ of operations (of arbitrary arity) on K and study the full subcategory of $\mathsf{Aff}_K(\mathcal{X})$ formed by those $(X, \mathcal{A})$ where $\mathcal{A}$ is a subalgebra of K^X (which may be done in any category $\mathcal{X}$ with powers of the object K). For example, in this way one retrieves the category Top of topological spaces as a hereditary coreflective subcategory of $\mathrm{Aff}_2(\mathsf{Set})$, considering the operations

$$\max_{i<n} a_i \text{ and } \min_{j<m} a_j$$

for arbitrary cardinals n and finite m, with the order taken in $2 = \{0 < 1\}$.

(3) In our more prototypical example $\mathcal{X} = \mathsf{Mod}_k$ (see 2.3) with $\mathcal{M} = \mathrm{Mono}\mathcal{X}$, $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is (equivalent to) the category whose objects (X, A) are given by a k-module X and a submodule A of $\mathrm{hom}_k(X, K)$, while objects $(X, \mathcal{A})$ in $\mathsf{Aff}_K(\mathcal{X})$ require $\mathcal{A}$ to be just a subset of $\mathrm{hom}_k(X, K)$. The right adjoint to the full embedding $J : \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \to \mathsf{Aff}_K(\mathcal{X})$ sends $(X, \mathcal{A})$ to (X, A) where A is the

submodule generated by $\mathcal{A} \subseteq \hom_k(X, K)$. We note that $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is not only coreflective but also hereditary in $\mathsf{Aff}_K(\mathcal{X})$. While J maps $\widetilde{\mathrm{pt}}$-initial morphisms to U-initial morphisms, this is no longer true for families. Hence, the requirement of (the second part of) 4.3 that k be $\mathbb{E}$-projective is essential. Indeed, while k is an $\mathcal{E}$-projective $\mathbb{E}$-generator, k trivially fails to be $\mathbb{E}$-projective: the fact that k^2 is spanned by the two principal axes does not prevent it from containing other one-dimensional submodules.

5. Separated Chu spaces and Zariski closure

In this section we assume $\mathcal{X}$ to be *a complete, symmetric monoidal category with a proper orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms, such that $\mathcal{X}$ has intersections of any-size families of $\mathcal{M}$-subobjects, and let K be $\otimes$-exponentiating.* Hence, $\mathcal{X}$ is $\mathcal{M}$-complete and has, in fact, an orthogonal $(\mathbb{E}, \mathcal{M})$-factorization system for sinks, with $\mathcal{E} = \mathbb{E} \cap \mathrm{Mor}\,\mathcal{X}$. We assume also that the K-Chu space (K, ϱ_K, k) (see 4.4) is $\mathcal{M}$-extensive; that is: *we assume that $\check{\varrho}_K : k \longrightarrow K^K$ lies in $\mathcal{M}$.* (In our prototypical example 2.3/4.4(3), for a field k the latter hypothesis prevents us only from making the bad choice $K = 0$. But for a ring k, such as $k = \mathbb{Z}$, the hypothesis is somewhat restrictive; for example, obviously K may not be chosen to be finite. However, the choice $K = k$ is always possible.)

5.1. **Definition.** An $\mathcal{M}$-extensive K-Chu space (X, r, A) is *(Zariski-) separated* if every $\widetilde{\mathrm{pt}}$-initial morphism $(f, \varphi) : (X, r, A) \longrightarrow (Y, s, B)$ in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ renders X as an $\mathcal{M}$-subobject of Y, that is: $f \in \mathcal{M}$ whenever $\varphi \in \mathcal{E}$.

5.2. **Proposition.** *If k is $\mathcal{E}$-projective, then $K = (K, \varrho_K, k)$ is separated. Moreover, every $\overline{\mathcal{M}}$-subobject of a power of K in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is separated.*

Proof. For every morphism $(g, \psi) : K \longrightarrow (Y, s, B)$ in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ with $\psi : B \longrightarrow k$ in $\mathcal{E}$, we obtain $\beta : k \longrightarrow B$ with $\psi\beta = 1_k$, since k is $\mathcal{E}$-projective. Now the commutative diagram below shows that g is split monic and, therefore, in $\mathcal{M}$ (since $\mathcal{E} \subseteq \mathrm{Epi}\,\mathcal{X}$):

$$\begin{tikzcd}
K \arrow[r, "g"] \arrow[d, "\varrho_K^{-1}"'] & Y \arrow[d, "\varrho_Y^{-1}"] \\
K \otimes k \arrow[r, "g\otimes k"] \arrow[d, "K\otimes\beta"'] & Y \otimes k \arrow[d, "Y\otimes\beta"] \\
K \otimes B \arrow[r, "g\otimes B"] \arrow[d, "K\otimes\psi"'] & Y \otimes B \arrow[d, "s"] \\
K \otimes k \arrow[r, "\sim", "\varrho_K"'] & K
\end{tikzcd} \tag{16}$$

Let now $(a_i, \alpha_i) : (X, r, A) \longrightarrow K$ $(i \in I)$ be $\widetilde{\mathrm{pt}}$-initial in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, such that the induced morphism $m = (a_i)_{i\in I} : X \longrightarrow \prod_{i\in I} K$ lies in $\mathcal{M}$, and let $(f, \varphi) : (X, r, A) \longrightarrow (Y, s, B)$ be pt-initial. $\mathcal{E}$-projectivity of k gives, for every $i \in I$, a morphism $\beta_i : k \longrightarrow B$ with $\varphi\beta_i = \alpha_i$. The morphisms $b_i = \beta_i^{\#} : Y \longrightarrow K$ satisfy $b_i f = a_i$ for all $i \in I$, so that f factors through m, which makes f lie in $\mathcal{M}$ as well (since $\mathcal{E} \subseteq \mathrm{Epi}\,\mathcal{X}$). $\square$

5.3. **Theorem.** *If k is an $\mathcal{E}$-projective $\mathbb{E}$-generator of $\mathcal{X}$, then the full subcategory $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ of separated objects is $\widetilde{\mathcal{E}}$-reflective in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ (with $\widetilde{\mathcal{E}}$ as in 3.9), $\overline{\mathcal{M}}$-cogenerated by K (so that every separated object is an $\overline{\mathcal{M}}$-subobject of a power of K). Moreover, K is $\overline{\mathcal{M}}$-injective in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, in particular in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$.*

Proof. For every $\mathcal{M}$-extensive K-Chu space (X, r, A), let

$$(g_X, \gamma_X) : (X, r, A) \longrightarrow \prod_{\alpha \in \mathcal{X}(k,A)} K$$

be the canonical morphism induced by all morphisms $(\alpha^{\#}, \alpha) : (X, r, A) \longrightarrow K$. The $\widetilde{\mathcal{E}}$-part of an $(\widetilde{\mathcal{E}}, \overline{\mathcal{M}})$-factorization of this morphism is easily seen to be the desired reflection of (X, r, A): this is an immediate consequence of the $(\widetilde{\mathcal{E}}, \overline{\mathcal{M}})$-diagonalization property, of the naturality of the morphisms (g_X, γ_X), and of the fact that, for (X, r, A) separated, (g_X, γ_X) lies in $\overline{\mathcal{M}}$. For the last claim one just observes that, since k is an $\mathbb{E}$-generator, the family of all morphisms $(\alpha^{\#}, \alpha) : (X, r, A) \longrightarrow K$ is $\widetilde{\mathrm{pt}}$-initial, so that the induced morphism (g_X, γ_X) is also $\widetilde{\mathrm{pt}}$-initial, and then in $\overline{\mathcal{M}}$, since (X, r, A) is separated.

To show $\overline{\mathcal{M}}$-injectivity for K, let $(m, \eta) : (X, r, A) \longrightarrow (Y, s, B)$ be in $\overline{\mathcal{M}}$, and consider any morphism $(\alpha^{\#}, \alpha) : (X, r, A) \longrightarrow K$. By the $\mathcal{E}$-projectivity of k, α factors as $\alpha = \eta\beta$, with $\beta : k \longrightarrow B$, which makes $(\alpha^{\#}, \alpha)$ factor as $(\alpha^{\#}, \alpha) = (\beta^{\#}, \beta)(m, \eta)$. □

For the remainder of this section, we *let k be an $\mathcal{E}$-projective $\mathbb{E}$-generator of $\mathcal{X}$.*

5.4. **Definition.** Recall that, for an $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $(X, \mathcal{A}) \in \mathsf{Aff}_K(\mathcal{X})$, one defines the Zariski closure of M in $(X, \mathcal{A})$ by

$$z_{(X,\mathcal{A})}(M) = \bigwedge \{ \mathrm{equ}(a, b) \mid a, b \in \mathcal{A},\ am = bm \}.$$

One easily shows (see [G3]) that z is an idempotent and hereditary closure operator of $\mathsf{Aff}_K(\mathcal{X})$ in the sense of [DG]. We may now restrict this closure operator to $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ (along the functor J of 4.1) and define the *Zariski closure* of M in $(X, r, A) \in \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ by

$$\zeta_{(X,r,A)}(M) = z_{J(X,r,A)}(M) = \bigwedge \{ \mathrm{equ}(\alpha^{\#}, \beta^{\#}) \mid \alpha, \beta \in \mathcal{X}(k, A),\ \alpha^{\#} m = \beta^{\#} m \}.$$

Since k is $\mathcal{E}$-projective, J maps $\widetilde{\mathrm{pt}}$-initial morphisms to U-initial morphisms (see 4.3), hence, J preserves the natural subobject structure. Therefore, ζ is, like z, idempotent and hereditary. In particular (see [DG], [DT]):

5.5. **Proposition.** *(ζ-dense morphisms, ζ-closed $\overline{\mathcal{M}}$-subobjects) form an orthogonal factorization system of $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$.*

Since $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ is $\overline{\mathcal{M}}$-cogenerated by K, ζ is simply the regular closure operator of $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$. Hence, the restriction of the factorization system 5.5 to $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ gives:

5.6. **Corollary.** *Every morphism in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ factors (epi, regular mono) = (ζ-dense, ζ-closed $\overline{\mathcal{M}}$-subobject).*

5.7. **Definition ([D1], [G3]).** A separated K-Chu space (X,r,A) is *complete* (or *absolutely ζ-closed*) if every morphism $(f,\varphi):(X,r,A)\longrightarrow(Y,s,B)$ in $\overline{\mathcal{M}}$ with codomain in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ is ζ-closed. We denote the full subcategory of complete objects in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ by $\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$ and put $\mathcal{D}=\overline{\mathcal{M}}\cap\{\,\zeta\text{-dense}\,\}=\overline{\mathcal{M}}\cap\mathrm{Epi}\,(\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X}))$.

5.8. **Theorem.** *A separated K-Chu space is complete if, and only if, it is $\mathcal{D}$-injective. $\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$ is $\mathcal{D}$-reflective in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ and $\mathcal{D}$-cogenerated by K. Any morphism $(X,r,A)\longrightarrow(Y,s,B)\in\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$ in $\mathcal{D}$ serves as a reflection of (X,r,A) into $\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$.*

Proof. (Prop. 5 in [G3]) Let the separated K-Chu spce (X,r,A) be complete. As observed in the proof of 5.3, the canonical morphism (g_X,γ_X) lies in $\overline{\mathcal{M}}$, with its codomain $\prod_{\mathcal{X}(k,A)}K$ being separated. Hence, (X,r,A) is ζ-closed in a power of K which, like K, is $\mathcal{D}$-injective. Consequently, when we consider any $(Y,s,B)\hookrightarrow(Z,t,C)$ in $\mathcal{D}$, every morphism $(Y,s,B)\to(X,r,A)$ extends to a morphism $(Z,t,C)\to\prod_{\mathcal{X}(k,A)}K$, which must actually factor through (X,r,A) by the (ζ-dense, ζ-closed $\overline{\mathcal{M}}$)-diagonalization property. Hence, (X,r,A) is $\mathcal{D}$-injective. Conversely, assuming (X,r,A) to be $\mathcal{D}$-injective, we can factor a given $(X,r,A)\hookrightarrow(Y,s,B)$ in $\overline{\mathcal{M}}$ according to 5.6. The resulting morphism $(X,r,A)\to(Z,t,C)$ in $\mathcal{D}$ has a retraction. Being epic in $\mathsf{Sep}_K(\mathcal{X})$ it must actually be an isomorphism, which shows that (X,r,A) is ζ-closed in (Y,s,B). □

5.9. **Remarks.**

(1) Assume not only that k is an $\mathbb{E}$-projective $\mathbb{E}$-generator, but also that every morphism in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, with respect to which k is projective, lies in $\mathcal{E}$; hence,

$$\varphi:A\longrightarrow B\text{ in }\mathcal{E}\iff\forall\beta:k\longrightarrow B\,\exists\alpha:k\longrightarrow A\,(\varphi\alpha=\beta),$$

for every morphism φ in $\mathcal{X}$. Then J not only preserves initial morphisms (see 4.3), but also reflects them. As a consequence one obtains that *an object in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is separated (complete, respectively) if, and only if, its J-image is separated (complete, respectively) in $\mathsf{Aff}_K(\mathcal{X})$, as defined in* [G3].

(2) As far as the notions of separatedness and completeness are concerned, one may explore them in an arbitrary full, hereditary and coreflective subcategory of $\mathsf{Aff}_K(\mathcal{X})$, in lieu of $\mathsf{Ext}_K(\mathcal{X})$, or in lieu of a subcategory defined by algebraic operations on K (see 4.4(2)). Here we restrict ourselves to illustrating them in terms of the basic examples 4.4(2),(3)).

5.10. **Example.** The separated objects of $\mathsf{Chu}_2(\mathsf{Set})$ (with $\mathcal{M}=\text{Mono}$, see 4.4(2)) may be described as those 2-affine sets $(X,\mathcal{A}\subseteq PX)$ with

$$\forall x,y\in X(\mathcal{A}_x=\mathcal{A}_y\implies x=y)$$

where $\mathcal{A}_x=\{A\in\mathcal{A}\mid x\in A\}$. For an $\overline{\mathcal{M}}$-subobject $(X,\mathcal{A})\hookrightarrow(Y,\mathcal{B})$ one has $\mathcal{A}=\{B\cap X\mid B\in\mathcal{B}\}$, and a point $y\in Y$ lies in $\zeta_Y(X)=z_Y(X)$ precisely when

$$\forall B,C\in\mathcal{B}(B\cap X=C\cap X\implies B\cap\{y\}=C\cap\{y\}).$$

Let us prove in detail now (see also [G2]) that a separated Boolean Chu space $(X,\mathcal{A})$ *is complete if, and only if, every subset $\mathcal{C}\subseteq\mathcal{A}$ is of the form $\mathcal{C}=\mathcal{A}_x$ for some $x\in X$.*

To prove the sufficiency of the condition, consider $(X,\mathcal{A})\hookrightarrow(Y,\mathcal{B})$ in $\overline{\mathcal{M}}$ with $(Y,\mathcal{B})$ separated, and let $y\in\zeta_Y(X)$. Then $\mathcal{C}:=\{B\cap X\mid B\in\mathcal{B}_y\}$ may be written as $\mathcal{C}=\mathcal{A}_x$

for some $x \in X$. But then $\mathcal{B}_y = \mathcal{B}_x$: the inclusion "$\subseteq$" is trivial, and for "$\supseteq$" one observes that for $B \in \mathcal{B}_x$ one has $B \cap X \in \mathcal{A}_x = \mathcal{C}$, so that $B \cap X = C \cap X$ for some $C \in \mathcal{B}_y$. Since $y \in \zeta_Y(X)$, $y \in B$ follows, and then $B \in \mathcal{B}_y$. Separatedness of $(Y, \mathcal{B})$ gives $y = x \in X$, as desired.

Conversely, assume $(X, \mathcal{A})$ to be complete, but that there is $\mathcal{C} \subseteq \mathcal{A}$ not of the form $\mathcal{C} = \mathcal{A}_x$ for any $x \in X$. Then one considers $Y = X \cup \{*\}$ (with $*$ not in X) and let $\mathcal{B} = (\mathcal{A} \setminus \mathcal{C}) \cup \{A \cup \{*\} \mid A \in \mathcal{C}\}$. Then $\mathcal{A} = \{B \cap X \mid B \in \mathcal{B}\}$, so that $(X, \mathcal{A}) \hookrightarrow (Y, \mathcal{B})$ lies in $\overline{\mathcal{M}}$. Furthermore, for $x, y \in X$, $\mathcal{B}_x = \mathcal{B}_y$ implies $\mathcal{A}_x = \mathcal{A}_y$ and then $x = y$, since $(X, \mathcal{A})$ is separated. Furthermore, $x \in X$ can be separated from $*$: since $\mathcal{A}_x \neq \mathcal{C}$ there is $A \in \mathcal{A}$ with either $x \in A \not\in \mathcal{C}$ or $x \not\in A \in \mathcal{C}$; in the latter case one has $A \cup \{*\} \in \mathcal{B}_*$ with $x \not\in A \cup \{*\}$, while in the former case one has $A \in \mathcal{B}_x$ with $* \not\in A$. Consequently, $(Y, \mathcal{B})$ is separated, and $(X, \mathcal{A})$ must be ζ-closed in $(Y, \mathcal{B})$. However, when $B \cap X = C \cap X$ for $B, C \in \mathcal{B}$, then $B \setminus \{*\}, C \setminus \{*\}$ are either both in $\mathcal{C}$ or both in $\mathcal{A} \setminus \mathcal{C}$, so that $B \cap \{*\} = C \cap \{*\}$; hence $* \in \zeta_Y(X)$, a contradiction.

5.11. **Example.** An object (X, A) in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ with $\mathcal{X} = \mathsf{Mod}_k$ and $\mathcal{M} = \mathrm{Mono}\mathcal{X}$ (as in 4.4(3)) is separated if, and only if, $A \leq \hom(X, K)$ satisfies

$$\bigcap_{a \in A} \ker a = 0. \tag{17}$$

Indeed, (17) is clearly sufficient for separatedness since for any $(f, \varphi) : (X, A) \to (Y, B)$ with $A = \{b \cdot f \mid b \in B\}$ one has $\ker f \subseteq \bigcap_{a \in A} \ker a$, and for its necessity one considers the canonical morphism $g : X \to \prod_{a \in A} K =: Y$ with $\ker g = \bigcap_{a \in A} \ker a$ (as in the proof of 5.3); g is $\widetilde{\text{pt}}$-initial since (as 3.5 and 3.7(2) show) the structure B of Y is the submodule of $\hom(Y, K)$ generated by all projections $p_a : Y \to K$, and every $a \in A$ has the form $a = p_a \cdot g$.

In terms of Zariski closure, this means that *the separated objects are those* (X, A) *with* 0 *Zariski closed in* X.

For every $x \in X$ let us consider the submodule $A_x = \{a \in A \mid x \in \ker a\}$ of A. Then (17) can be reformulated as

$$\forall x \in X (A_x = A \implies x = 0),$$

and since $A_0 = A$, (17) is implied by

$$\forall x, y \in X (A_x \subseteq A_y \implies kx \supseteq ky). \tag{18}$$

We note in passing that, when k is a field and $K \neq 0$, then (18) is in turn implied by

$$A = \hom(X, K) \tag{19}$$

(since for $\{x, y\}$ linearly independent one can find $a \in \hom(X, K)$ whose kernel contains only one of x, y). Only in rare cases (for example, for $K = k$ a field and X finite-dimensional), (19) is also necessary for separatedness of (X, A); hence, in this case X carries a uniquely determined separated structure. For $K = k = \mathbb{Z}$, no finite $X \neq 0$ carries any separated structure (since $\hom(X, \mathbb{Z}) = 0$), showing the restrictiveness of the separation condition.

Returning to the case of a general K, to ask all objects $(X, \hom(X, K))$ to be separated is precisely to ask K to be a cogenerator in Mod_k. Hence, a good choice for K in case

$k = \mathbb{Z}$ would be $K = \mathbb{Q}/\mathbb{Z}$, $K = \hom_{\mathbb{Z}}(k, \mathbb{Q}/\mathbb{Z})$ for arbitrary k, and of course $K = k$ in case of a field k.

Let us denote by Sep the full subcategory of $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ formed by the objects (X, A) satisfying (18); hence, $\mathsf{Sep} \subseteq \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$. Its objects are quite easily (and more naturally) characterized as those (X, A) in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ for which every cyclic submodule is Zariski-closed in (X, A).

We now prove that $(X, A) \in \mathsf{Sep}$ *is absolutely closed in* Sep (so that whenever $(X, A) \leq (Y, B)$ with $(Y, B) \in \mathsf{Sep}$, (X, A) is actually Zariski-closed in (Y, B)), *provided that every submodule $C \leq A$ has the form $C = A_x$, for some $x \in X$.* For this one just mimics the argumentation of 5.10, as follows. Given $(X, A) \hookrightarrow (Y, B)$ and

$$y_0 = \zeta_Y(X) = \{y \in Y \mid \forall b \in B(X \subseteq \ker b \implies y \in \ker b)\}$$

consider $C := \{b|_X \mid b \in B_{y_0}\}$ and write $C = A_x$ with $x \in X$. Then $B_{y_0} = B_x$ ("$\subseteq$" is obvious, and for "$\supseteq$" one notices that every $b \in B_x$ gives some $c \in B_{y_0}$ with $c|_X = b|_X$, e.g. $(b - c)|_X = 0$ and then $b(y_0) = c(y_0) = 0$), and therefore $y_0 \in kx \subseteq X$, as desired.

We conjecture that the given sufficient condition is also necessary. Hence, in that case, assuming $\mathsf{Sep} = \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$, a separated object would be complete if, and only if, every submodule $C \leq A$ has the form $C = A_x$, for some x. We will interpret this condition more naturally in 6.12 below.

6. Enrichment and self-duality

6.1. **Theorem ([C]).** *For $\mathcal{X}$ symmetric monoidal closed with pullbacks, and for any object K, $\mathsf{Chu}_K(\mathcal{X})$ is also symmetric monoidal closed, in fact $*$-autonomous (in the sense of* [B1]*) with its natural involution $(-)^*$.*

Proof. We just recall the basic constructions from [C] (where the reader finds all verifications). The tensor product

$$(X, r, A) \otimes (Y, s, B) = (X \otimes Y, t, P)$$

in $\mathsf{Chu}_K(\mathcal{X})$ is defined by the pullback diagram

(20)
$$\begin{tikzcd} P \arrow[rr] \arrow[dd] & & A^Y \arrow[d, "(\check{r})^Y"] \\ & & (K^X)^Y \arrow[d, "\cong"] \\ B^X \arrow[r, "(\check{s})^X"] & (K^Y)^X \arrow[r, "\cong"] & K^{X\otimes Y} \end{tikzcd}$$

with $t : (X \otimes Y) \otimes P \longrightarrow K$ the mate of the diagonal morphism $P \longrightarrow K^{X\otimes Y}$ of the pullback. The unit of the tensor product is

$$(k, \lambda_K : k \otimes K \xrightarrow{\sim} K, K) = (K, \varrho_K, k)^*.$$

The internal hom of $\mathsf{Chu}_K(\mathcal{X})$ can be defined by

$$(Y, s, B)^{(X,r,A)} = \big((X, r, A) \otimes (Y, s, B)^*\big)^* = (Q, h, X \otimes B)$$

where h is the mate of the diagonal morphism in the pullback diagram

(21)
$$\begin{tikzcd} Q \arrow[rr] \arrow[dd] & & A^B \arrow[d, "(\check{r})^B"] \\ & & (K^X)^B \arrow[d, "\cong"] \\ Y^X \arrow[r, "(\hat{s})^X"] & (K^B)^X \arrow[r, "\cong"] & K^{X\otimes B} \end{tikzcd}$$

When $(Y, s, B) = (K, \varrho_K, k) = K$, then $\hat{s} = \hat{\varrho}_K : K \longrightarrow K^k$ is an isomorphism, and $(\check{r})^B = (\check{r})^k \cong \check{r}$ (in $\mathcal{X}^{\mathbf{2}}$), so that, up to natural isomorphism, (21) coincides with

(22)
$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\check{r}"'] & A \arrow[d, "\check{r}"] \\ K^X \arrow[r, "1_{K^X}"] & K^X \end{tikzcd}$$

This shows
$$(X, r, A)^* \cong K^{(X,r,A)},$$
naturally. □

6.2. **Corollary.** *The functor $(-)^* : \mathsf{Chu}_K(\mathcal{X})^{\mathrm{op}} \longrightarrow \mathsf{Chu}_K(\mathcal{X})$ is naturally isomorphic to $K^{(-)}$, making K the dualizing object of $\mathsf{Chu}_K(\mathcal{X})$.*

6.3. **Example.** In $\mathsf{Chu}_2(\mathsf{Set})$, keeping the notation of (20) and (21), one has
$$\begin{aligned} P &= \{\, (f : X \longrightarrow B, \varphi : Y \longrightarrow A) \mid \forall x \in X\, \forall y \in Y\, (s(y, f(x)) = r(x, \varphi(y))) \,\} \\ &= \mathsf{Chu}_2(\mathsf{Set})\big((X, r, A), (B, s^\circ, Y)\big), \\ Q &= \{\, (f : X \longrightarrow Y, \varphi : B \longrightarrow A) \mid \forall x \in X\, \forall b \in B\, (s(f(x), b) = r(x, \varphi(b))) \,\} \\ &= \mathsf{Chu}_2(\mathsf{Set})\big((X, r, A), (Y, s, B)\big). \end{aligned}$$

6.4. **Remarks.**

(1) If $\mathcal{X}$ is already $*$-autonomous, with dualizing object K, then $\mathsf{Chu}_K(\mathcal{X})$ is equivalent to an easily described comma category, namely
$$\mathsf{Chu}_K(\mathcal{X}) \simeq (\mathrm{Id}_{\mathcal{X}} \downarrow (-)^*).$$
In particular, for every symmetric monoidal-closed category $\mathcal{X}$ and $K \in \mathrm{ob}\,\mathcal{X}$ one has
$$\mathsf{Chu}_{(K,\varrho_K,k)}(\mathsf{Chu}_K(\mathcal{X})) \simeq (\mathrm{Id}_{\mathsf{Chu}_K(\mathcal{X})} \downarrow (-)^*).$$
This observation ultimately leads to Pavlovic's [Pc] sophisticated characterization of the Chu construction in terms of a universal property.

(2) If the class $\mathcal{M}$ is stable under binary intersection (in particular, if $\mathcal{M}$ is stable under pullback and closed under composition), and if $\mathcal{M}$ satisfies the condition

A. *for all $m : X \to Y$ in $\mathcal{M}$ and objects A, $m^A : X^A \to Y^A$ is also in $\mathcal{M}$,* then the tensor product of $\mathsf{Chu}_K(\mathcal{X})$ restricts to $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, as an easy examination of diagram (20) reveals. Similarly, under the same conditions diagram (21) shows that, for (X, r, A) $\mathcal{M}$-extensive and (Y, s, B) $\mathcal{M}$-coextensive, the hom-object $(Y, x, B)^{(X,r,A)}$ is $\mathcal{M}$-coextensive.

(3) The status of condition A is examined in Prop. 4.1 of [B5]. Although the formulation of that proposition is not correct, its proof shows that A is equivalent to each of B, C below, provided that $(\mathcal{E}, \mathcal{M})$ is a prefactorization system of $\mathcal{X}$ in the sense of [FK], in particular when $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorization system:

B. *for all* $\varphi : A \to B$ *in* $\mathcal{E}$ *and objects* X, $\varphi \otimes X : A \otimes X \to B \otimes X$ *is also in* $\mathcal{E}$,
C. *for all* $m : X \to Y$ *in* $\mathcal{M}$ *and* $\varphi : A \to B$ *in* $\mathcal{E}$, *the diagram*

$$\begin{tikzcd} X^B \arrow[r, "X^\varphi"] \arrow[d, "m^B"'] & X^A \arrow[d, "m^A"] \\ Y^B \arrow[r, "Y^\varphi"] & Y^A \end{tikzcd}$$

is a pullback.

In the notation of 2.9, since $\check{r} = (r^\circ)\hat{}$ and $\hat{r} = (r^\circ)\check{}$, the involution $(-)^*$ preserves the full subcategory

$$\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \cap \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}),$$

so that this subcategory is self-dual, for any $\mathcal{M}$. The point now is to characterize its objects more conveniently. For that we *assume* $\mathcal{M}$ *to be closed under composition and left-cancellable* (so that $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$), a hypothesis granted automatically when $\mathcal{M}$ is part of an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms with $\mathcal{E} \subseteq \mathrm{Epi}\mathcal{X}$. K *remains* $\otimes$*-exponentiating.*

6.5. **Proposition.** *A* K*-Chu space* (X, r, A) *is* $\mathcal{M}$*-coextensive if, and only if, every* pt*-cartesian morphism* $(f, \varphi) : (X, r, A) \to (Y, s, B)$ *has* $f \in \mathcal{M}$, *e.g.* $f \in \mathcal{M}$ *whenever* φ *is an isomorphism (see* 2.7*).*

Proof. Showing "only if", the right square of (6) (see 2.9) has $\hat{r} \in \mathcal{M}$ and K^φ iso, by hypothesis, so that $\hat{s}f = K^\varphi \hat{r} \in \mathcal{M}$. Now $f \in \mathcal{M}$ follows with the left-cancellability of $\mathcal{M}$. Conversely, one considers the pt-cartesian morphism

$$(\hat{r}, 1_A) : (X, r, A) \to (A, e_A, K^A)^* = (K^A, e_A^\circ, A)$$

to obtain $\hat{r} \in \mathcal{M}$ from the hypothesis. □

In order for us to use the latter argumentation also in the subcategory $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, we need to have $(A, e_A, K^A)^* \in \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$.

6.6. **Lemma.** $(A, e_A, K^A)^*$ *is* $\mathcal{M}$*-extensive if, and only if, there is any* $\mathcal{M}$*-extensive* K*-Chu space* (X, r, A)*.*

Proof. We need to show that the natural morphism

$$i_A := (e_A^\circ)\check{} : A \to K^{K^A}$$

is in $\mathcal{M}$ when (X, r, A) is $\mathcal{M}$-extensive. Since $\mathcal{M}$ is left-cancellable and $\check{r} \in \mathcal{M}$, it suffices to show $\check{r} = K^{\hat{r}} i_A$. But that equation follows from the universality of e_X, since all faces but the upper triangle in the following diagram are known to commute:

$$\begin{tikzcd}
 & X \otimes K^{K^A} \arrow[dr, "X\otimes K^{\hat{r}}"] \arrow[dd, "\hat{r}\otimes K^{K^A}"'] & \\
X \otimes A \arrow[ur, "X\otimes i_A"] \arrow[rr, "X\otimes\check{r}"] \arrow[dd, "\hat{r}\otimes A"'] & & X \otimes K^X \arrow[dd, "e_X"] \\
 & K^A \otimes K^{K^A} \arrow[dr, "e_{K^A}"] & \\
K^A \otimes A \arrow[ur, "K^A\otimes i_A"] \arrow[rr, "e_A^\circ"'] & & K
\end{tikzcd}$$

□

6.7. **Corollary.** *Every separated $\mathcal{M}$-extensive K-Chu space is $\mathcal{M}$-coextensive.*

Proof. By the Lemma, the argumentation used in 6.5 may be restricted to $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$. □

6.8. **Theorem.** *Let $(\mathcal{E}, \mathcal{M})$ be an orthogonal factorization system for morphisms of $\mathcal{X}$ with $\mathcal{E} \subseteq \mathrm{Epi}\mathcal{X}$. If the functor $K^{(-)} : \mathcal{X}^{\mathrm{op}} \to \mathcal{X}$ maps $\mathcal{E}$ into $\mathcal{M}$, then the identity*

$$\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \cap \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}) = \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$$

holds; in particular, this is then a self-dual subcategory of $\mathsf{Chu}_K(\mathcal{X})$. $K^{\mathcal{E}} \subseteq \mathcal{M}$ is also a necessary condition for this identity to hold, provided that $(A, e_A, K^A)^*$ is $\mathcal{M}$-extensive for all $A \in \mathcal{X}$.

Proof. In order to show the inclusion

$$\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \cap \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}) \subseteq \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$$

using closure of $\mathcal{M}$ under composition one can argue as in the first part of the proof of 6.5, since it suffices to have $K^{\varphi} : K^A \to K^B$ in $\mathcal{M}$ whenever $\varphi : A \to B$ in $\mathcal{E}$. The reverse inclusion has already been stated in 6.7.

For the necessity of $K^{\mathcal{E}} \subseteq \mathcal{M}$, let $\varphi : B \to A$ be in $\mathcal{E}$ and consider the K-Chu morphism

$$(K^{\varphi}, \varphi) : (K^A, e_A^\circ, A) \to (K^B, e_B^\circ, B).$$

It trivially lies in $\mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})$ (since $(e_A^\circ)^A = 1_{K^A}$), but by hypothesis also in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ and therefore in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$. Hence, $\varphi \in \mathcal{E}$ implies $K^{\varphi} \in \mathcal{M}$, as desired. □

6.9. **Remarks.**

(1) $\mathcal{M}$-extensitivity of $(A, e_A, K^A)^*$ simply means that $i_A = (\check{e_A^\circ}) : A \to K^{K^A}$ lies in $\mathcal{M}$. By Lemma 6.6 it is sufficient that A appears in *some* $\mathcal{M}$-extensive K-Chu space (X, r, A). For $A = k$, $\mathcal{M}$-extensitivity of

$$(A, e_A, K^A)^* \cong (K, \rho_K, k)$$

was already used in Section 4. Under the (rather restrictive) hypothesis that, in the presence of products in $\mathcal{X}$, k be an $\mathcal{M}$-cogenerator in $\mathcal{X}$, $\mathcal{M}$-extensitivity of (K, ρ_K, k) implies the same property for all objects $(A, e_A, K^A)^*$, as the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "m_A"'] & K^{K^A} \arrow[d, "<K^{K^\xi}>_{\xi\in\mathcal{X}(A,k)}"] \\ \prod_{\mathcal{X}(A,k)} k \arrow[r, "\prod_{\mathcal{X}(A,k)} i_k"'] & \prod_{\mathcal{X}(A,k)} K^{K^k} \end{tikzcd}$$

shows. Indeed, $m_A \in \mathcal{M}$, by hypothesis, as well as $\prod_{\mathcal{X}(A,k)} i_k$, since $i_k \cong (\check{\rho_K} : k \to K^K) \in \mathcal{M}$, so that the composition–cancellation argumentation becomes applicable.

(2) *If k is a generator of $\mathcal{X}$ and $\mathcal{M} = \mathrm{Mono}\mathcal{X}$, then the condition $K^{\mathcal{E}} \subseteq \mathcal{M}$ holds.* Indeed, assuming $K^\varphi_{\gamma_1} = K^\varphi_{\gamma_2}$ with $\gamma_i : k \to K^A$, the commutative diagram

$$\begin{tikzcd} B \arrow[r, "\cong"] \arrow[d, "\varphi"'] & B\otimes k \arrow[r, "B\otimes\gamma_i"] \arrow[d, "\varphi\otimes k"'] & B\otimes K^A \arrow[r, "B\otimes K^\varphi"] \arrow[d, "\varphi\otimes K^A"] & B\otimes K^B \arrow[d, "e_B"] \\ A \arrow[r, "\cong"] & A\otimes k \arrow[r, "A\otimes\gamma_i"] & A\otimes K^A \arrow[r, "e_A"] & K \end{tikzcd}$$

gives $e_A(A\otimes\gamma_1) = e_A(A\otimes\gamma_2)$ when φ is epic, and then $\gamma_1 = \gamma_2$.

6.10. **Corollary.** *Let $\mathcal{X}$ have* (StrongEpi, Mono)*-factorizations, and let the $\otimes$-unit k be a generator in $\mathcal{X}$. Then, for every object K of $\mathcal{X}$, the category of separated K-Chu spaces is selfdual.* □

We are left with the question when $\mathcal{A} := \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \cap \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X})$ is actually $*$-autonomous, which has been answered in [B5]. We already saw in 6.4(3) that $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ is $\otimes$-closed in $\mathsf{Chu}_K(\mathcal{X})$ when the class $\mathcal{M}$ is pullback stable, closed under composition and satisfies condition A. Since the unit object (k, λ_K, K) lies trivially in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, this category is symmetric monoidal, inheriting its structure from $\mathsf{Chu}_K(\mathcal{X})$, whenever $\mathcal{X}$ is symmetric monoidal closed with pullbacks. It was proved in [B5] that, in order to make $\mathcal{A}$ $*$-autonomous, it suffices to show that the $\mathcal{M}$-coextensional reflection (Y, t, A) of an object (X, r, A) in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$ stays $\mathcal{M}$-coextensional. But by 3.3, (Y, t, A) may be obtained from an orthogonal $(\mathcal{E}, \mathcal{M})$-factorization

$$\begin{tikzcd} & Y \arrow[dr, "m=\hat{t}"] & \\ X \arrow[ur, "e"] \arrow[rr, "\hat{r}"'] & & K^A \end{tikzcd}$$

Transposition gives $\check{r} = K^e \cdot \check{t}$. Since $\check{r} \in \mathcal{M}$ by hypothesis, there are two ways of concluding $\check{t} \in \mathcal{M}$: when $\mathcal{M}$ is left cancellable (hence, in particular when $\mathcal{E} \subseteq \mathrm{Epi}\mathcal{X}$), or when $K^{\mathcal{E}} \subseteq \mathcal{M}$.

In summary one obtains:

6.11. **Corollary.** *Under the hypothesis of* 6.8, $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \cap \mathrm{Cxt}_{K,\mathcal{M}}(\mathcal{X}) = \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ *is* $*$*-autonomous.* □

6.12. **Example.** We return to Example 5.10 which trivially satisfies hypotheses of 6.8. The formal dual $(X,A)^* = (A,X)$ of an object (X,A) in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ is concretely given by the identification of points $x \in X$ with their evaluation maps

$$\omega_X : A \to K, a \mapsto a(x).$$

Now, in the notation of 5.10 $\ker \omega_x = A_x$. Hence, assuming $\mathsf{Sep} = \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$, in 5.10 we proved that *a sufficient* (and, as we conjecture, also necessary) *condition for* (X,A) *to be complete is that in* $(X,A)^*$ *every subobject is the kernel of a "character", i.e., of an element of its structure.*

It is well known (and quite easy to see) how to retrieve some classical dualities from the self-dual category $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$, see particularly [Pr], [G2]. For example, the Stone duality may be obtained by suitably extending the characterization of complete objects as given in 5.10, using the presentation of topological spaces given in 4.4(2). Here we restrict ourselves to giving a brief account of the Pontrjagin duality within the setting of this paper.

6.13. **Example.** We consider the category $\mathcal{X} = \mathsf{AbGrp} = \mathsf{Mod}_{\mathbb{Z}}$ with $K = \mathbb{R}/\mathbb{Z}$ the torus and $\mathcal{M} = \mathrm{Mono}\mathcal{X}$. For (X,A) in $\mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X})$, provide X with the initial topology with respect to $A \leq \hom(X,K)$. This defines a functor

$$T : \mathrm{Ext}_{K,\mathcal{M}}(\mathcal{X}) \to \mathsf{TAG}$$

into the category of topological abelian groups. T is easily seen to be right adjoint: its left adjoint S assigns to a topological abelian group X the object (X,A) with $A = \mathsf{TAG}(X,K)$ the group of continuous characters on X. The full image of T is given by the subcategory of *totally bounded* abelian groups (e.g. the TAG-objects with compact completion), and the image of $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ under T is precisely given by those totally bounded groups that are Hausdorff, that is: by the category PTAG of precompact topological abelian groups (see 4.26 and 6.5 of [D]; those assertions are not trivial and go essentially back to Comfort and Ross [CR]). Hence, $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ *is equivalent to* PTAG.

Let us now see how the self-duality $(-)^*$ of $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ presents itself in the PTAG-language. For a topological abelian group X,

$$\widehat{X} := T(SX)^* = \mathsf{TAG}(X,K)$$

is the group of continuous characters on X, provided with the initial topology with respect to the evaluation maps

$$\omega_x : \widehat{X} \to K, \quad a \mapsto a(x),$$

$x \in X$; for X precompact, $\widehat{X}$ is also precompact, and we have $\widehat{\widehat{X}} \cong X$. In other words, *the rather formal self-duality of* $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ *is in fact the highly non-trivial self-duality of* PTAG, as described by Raczkowski and Trigos-Arrieta [RT] and, in the more general setting of topological modules, by Menni and Orsatti [MO].

For the characterization of $\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$, let us first note that the Zariski closure of (X, A) in (Y, B) of $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X}) = \mathsf{PTAG}$ coincides with the Kuratowski closure:

$$\zeta_Y(X) = \bigcap\{\ker b \mid b \in B, X \subseteq \ker b\} = \overline{X}.$$

Hence, $\mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X})$ is equivalent to the full reflective subcategory of PTAG given by the absolutely-closed objects, e.g. by the compact Hausdorff abelian groups. On the other hand, the category $\mathcal{X} = \mathsf{AbGrp}$ is coreflectively embedded in $\mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})$ via $X \mapsto (X, \hom(X, K))$ (see 2.11) and becomes the subcategory of discrete groups in PTAG. Now the classical Pontrjagin duality between compact and discrete abelian groups comes about by restricting the self-duality of PTAG and is therefore formally described by:

$$\begin{tikzcd} (X, \hom(X,K)) & \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X})^{\mathrm{op}} \arrow[r, "\sim"] & \mathsf{Sep}_{K,\mathcal{M}}(\mathcal{X}) \\ X \arrow[u, mapsto] & \mathcal{X}^{\mathrm{op}} \arrow[u] \arrow[r, "\sim"] & \mathsf{Cpl}_{K,\mathcal{M}}(\mathcal{X}) \arrow[u] \end{tikzcd}$$

Acknowledgement: The authors are grateful to Dikran Dikranjan for very helpful comments and suggestions, especially with respect to Examples 5.11, 6.12 and 6.13. They also thank Valeria de Paiva for stimulating comments on some aspects of the paper.

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