Kleisli enriched

Walter Tholen[*]

Max Planck Institute for Mathematics, Bonn, Germany (current)
Dept. of Mathematics and Statistics, York University, Toronto, Canada (permanent)

Abstract

For a monad S on a category $\mathcal{K}$ whose Kleisli category is a quantaloid, we introduce the notion of modularity, in such a way that morphisms in the Kleisli category may be regarded as V-(bi)modules (= profunctors, distributors), for some quantale V. The assignment $S \longmapsto V$ is shown to belong to a global adjunction which, in the opposite direction, associates with every (commutative, unital) quantale V the prototypical example of a modular monad, namely the presheaf monad on V-**Cat**, the category of (small) V-categories. We discuss in particular the question whether the Hausdorff monad on V-**Cat** is modular.

Keywords: modular monad, Kleisli category, quantale, quantaloid, V-(bi)module, V-category, power-set monad, presheaf monad, Hausdorff monad.
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1. Introduction

For a monad S on a category $\mathcal{K}$, a morphism in the Kleisli category of S is given by a morphism of type $Y \longrightarrow SX$ in $\mathcal{K}$. As the carrier of the free Eilenberg-Moore algebra over X, naturally SX carries additional structure which may be inherited by the relevant hom-set of the Kleisli category. For example, when S is the power-set monad on **Set**, so that $SX = PX$ is the free sup-lattice over the set X, the Kleisli category is the (dual of the) category of sets and relations and, hence, a quantaloid, i.e., a **Sup**-enriched category. Less trivially, and more generally, taking presheaves over a (small) category X defines a monad P on **Cat** whose Kleisli category is the (bi)category of categories and bimodules (= profunctors, distributors), the rich structure of which is a fundamental tool for a substantial body of categorical research. We refer the reader in particular to [12], [6], [5] and [4], and the extensive lists of references in these papers which point the reader also to the origins of a theme that seems to have interested researchers for some forty years. Some of these papers consider the presheaf monad in the enriched context (see [8]), i.e., for V-**Cat**, where V is a symmetric monoidal-closed category (rather than the classical $V = $ **Set**), or even a bicategory (see [10], for example). The more manageable case when the bicategory is just a quantaloid has been considered by Stubbe [13] who exhibits the passage from V to the category of V-bimodules as a morphism of quantaloids.

In this paper we consider the further simplified case when V is a quantale, i.e., a one-object quantaloid which, based on Lawvere's treatment of metric spaces [9] and Barr's presentation of topological spaces [2], has been used to set up a common syntax for various categories of interest in analysis and topology; see, for example, [3], [11], [7], [14]. Specifically, with (the dual of) the Kleisli category of the presheaf monad P on V-**Cat** describing precisely the quantaloid V-**Mod** of V-categories and V-modules, we ask ourselves the question when the morphisms of the Kleisli category of an arbitrary monad S on a category may be treated as V-modules, for some quantale V. To this end we introduce the notion of a *modular monad* on an abstract category $\mathcal{K}$ which asks its Kleisli category to be a quantaloid (and, hence, a 2-category) in which $\mathcal{K}$-morphisms have adjoints. By means of a distinguished "unital" object E in $\mathcal{K}$ one may then associate with the monad S a quantale V and establish a fully faithful "comparison functor" from the (dual of the) Kleisli category to V-**Mod**. With some natural restrictions on both the objects and morphisms, this fully faithful functor plays the role of a unit of an adjunction between a (very large) category of modular monads and the category of (commutative unital) quantales, the counits of which are isomorphisms. In other words, assigning to every V the presheaf monad on V-**Cat** defines, up to categorical equivalence, a full reflective embedding of the category of quantales into a category of modular monads. In setting up this category, some care must be given to the definition of its morphisms since the assignments $S \longmapsto V$ and $V \longmapsto P$ behave surprisingly crudely with respect to the natural 2-categorical structures of these categories. We have therefore used 2-cells only to the minimal extent necessary to answer our original question.

The paper is written in a largely self-contained style; it therefore recalls some known facts, giving sufficient details in particular on our prototypical example of a modular monad, the presheaf monad on V-**Cat** (Section 2). Having already set up a "modular terminology" for the dual of the Kleisli category of a modular monad in that introductory section, in Section 3 we prove the comparison theorem with the modules of actual V-categories, for some suitable V. Section 4 contains a detailed discussion of the question to which extent the Hausdorff monad (see [1]) is modular. Finally, global correspondences between quantales and modular monads are established in Sections 5 and 6, first in terms of functors that are partially (pseudo-)inverse to each other, and then in terms of the somewhat suprising adjunction that exhibits the ordinary category of quantales as a very substantial part of a very large environment of (certain) modular monads.

2. Modular monads

Let $S = (S, \varepsilon, \nu)$ be a monad on a category $\mathcal{K}$. We denote the opposite of the Kleisli category of S by

$$S\text{-}\mathbf{Mod}.$$

Hence, its objects are the objects of $\mathcal{K}$, and a morphism $\varphi \colon X \longrightarrow Y$, also called S-*module* from X to Y, is given by a $\mathcal{K}$-morphism $\varphi \colon Y \longrightarrow SX$; composition with $\psi \colon Y \longrightarrow Z$ is defined by

$$\psi \circ \varphi := \nu_X \cdot S\varphi \cdot \psi,$$

and $1_X^* := \varepsilon_X \colon X \longrightarrow X$ is the identity morphism on X in S-**Mod**. Extending this notation, one has the right-adjoint functor

$$(-)^* \colon \mathcal{K}^{\mathrm{op}} \longrightarrow S\text{-}\mathbf{Mod}$$

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which sends $f\colon X \longrightarrow Y$ in $\mathscr{K}$ to $f^* := \varepsilon_Y \cdot f\colon Y \dashrightarrow X$ in $S\text{-}\mathbf{Mod}$. We denote its left adjoint by

$$\widehat{(-)}\colon S\text{-}\mathbf{Mod} \longrightarrow \mathscr{K}^{\mathrm{op}},$$

sending X to SX and $\varphi\colon X \dashrightarrow Y$ to $\hat{\varphi} := \nu_X \cdot S\varphi\colon SY \longrightarrow SX$ in $\mathscr{K}$. The adjunction produces two factorizations of φ, namely

$$\varphi = \hat{\varphi} \cdot \varepsilon_Y \quad \text{in } \mathscr{K} \qquad \text{and} \qquad \varphi = \varphi^* \circ \iota_X \quad \text{in } S\text{-}\mathbf{Mod},$$

with the morphisms ε_X serving as counits (in $\mathscr{K}^{\mathrm{op}}$), and the morphisms $\iota_X := 1_{SX}\colon X \dashrightarrow SX$ as units (in $S\text{-}\mathbf{Mod}$). Units and counits are connected by the triangular equalities (which are special cases of the factorization)

$$\widehat{\iota_X} \cdot \varepsilon_{SX} = 1_{SX} \quad \text{in } \mathscr{K} \qquad \text{and} \qquad \varepsilon_X^* \circ \iota_X = 1_X^* \quad \text{in } S\text{-}\mathbf{Mod}.$$

Definition 2.1. *We call the monad S on $\mathscr{K}$ **modular** if*

1. *$S\text{-}\mathbf{Mod}$ carries the structure of a quantaloid, that is: every hom-set carries the structure of a complete lattice, such that composition in $S\text{-}\mathbf{Mod}$ from either side preserves arbitrary suprema;*
2. *for every morphism $f\colon X \longrightarrow Y$ in $\mathscr{K}$, $f^*\colon Y \dashrightarrow X$ has a left adjoint in the 2-category $S\text{-}\mathbf{Mod}$, that is: there exists $f_*\colon X \dashrightarrow Y$ in $S\text{-}\mathbf{Mod}$ with $1_X^* \le f^* \circ f_*$ and $f_* \circ f^* \le 1_Y^*$;*
3. *there is an object E in $\mathscr{K}$ with $\mathscr{K}(E,E) = \{1_E\}$ and*

$$\bigvee_{x\colon E \to X} x_* \circ x^* = 1_X^*$$

   for all X in $\mathscr{K}$.

For a modular monad S, we always fix the order that makes $S\text{-}\mathbf{Mod}$ a quantaloid and assume a fixed choice of the left adjoints f_* and the distinguished object E; in other words, modularity is not considered as a property of the monad, but as a structure on it.

Example 2.2. For the power-set monad $(P, \{-\}, \bigcup)$ on $\mathbf{Set}$, when one considers maps $\varphi\colon Y \longrightarrow PX$ as relations $\varphi\colon X \dashrightarrow Y$ (by writing $x\,\varphi\,y$ instead of $x \in \varphi(y)$), $P\text{-}\mathbf{Mod}$ is simply the category $\mathbf{Rel}$ of sets and relations. Its hom-sets inherit the inclusion order of power-sets, which makes $P\text{-}\mathbf{Mod}$ a quantaloid. One takes $f_*\colon Y \longrightarrow PX$ to be the inverse-image function of $f\colon X \longrightarrow Y$ in $\mathbf{Set}$, and with E a singleton set, condition 3 of 2.1 amounts to the trivial statement

$$(\exists x \in X : u = x \,\&\, x = v) \iff u = v$$

for all $u, v \in X$. Hence, P is modular.

Example 2.3. Replacing the 2-element chain in $PX = 2^X$ by an arbitrary frame V, one generalizes the previous example, as follows. For $f\colon X \longrightarrow Y$ in $\mathbf{Set}$, $P_V f\colon P_V X = V^X \longrightarrow P_V Y$ be left adjoint to $V^f\colon V^Y \longrightarrow V^X, \beta \longmapsto \beta \cdot f$; hence

$$(P_V f)(\alpha)(y) = \bigvee_{x \in f^{-1}(y)} \alpha(x),$$

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for all $\alpha\colon X \longrightarrow V$, $y \in Y$. The maps

$$x \xrightarrow{\ \delta_X\ } P_V X \qquad \text{and} \qquad P_V P_V X \xrightarrow{\ \nu_X\ } P_V X$$

with $\delta_X(x)(x') = \top$ (the top element in V) if $x = x'$ and $\bot$ (bottom) else, and with

$$\nu_X(\Sigma)(x) = \bigvee_{\alpha \in P_V X} \Sigma(\alpha) \wedge \alpha(x),$$

for all $\Sigma\colon V^X \longrightarrow V$, $x, x' \in X$, give P_V the structure of a monad. Using P instead of P_V, we can describe P-**Mod** equivalently as the quantaloid V-**Rel** of sets with V-valued relations $\varphi\colon X \longrightarrow Y$ as morphisms. Indeed, maps $\varphi\colon Y \longrightarrow PX = V^X$ correspond bijectively to maps $\tilde{\varphi}\colon X \times Y \longrightarrow V$, and composition in P-**Mod** becomes the ordinary composition of V-valued relations:

$$\widetilde{\psi \circ \varphi}(x, z) = (\tilde{\psi} \circ \tilde{\varphi})(x, z) = \bigvee_{y \in Y} \tilde{\psi}(y, z) \wedge \tilde{\varphi}(x, y),$$

for $\psi\colon Y \longrightarrow Z$, $x \in X$, $z \in Z$. The left adjoint f_* of f^* for $f\colon X \longrightarrow Y$ is obtained by interchanging variables:

$$f_*(y)(x) = f^*(x)(y) = \top$$

if $f(x) = y$, and $\bot$ else. Condition 3 of 2.1 is, as in the case $V = 2$, trivially satisfied also in general. Consequently, $P = P_V$ is a modular monad on **Set**.

Example 2.4. Let V be a unital quantale (= one-object quantaloid), i.e. a complete lattice with a binary associative operation $\otimes$ and a neutral element k such that $\otimes$ preserves suprema in each variable. (Every frame V as in 2.3 is a quantale, with $\otimes = \wedge$, $k = \top$.) The category V-**Cat** of (small) V-categories and V-functors has as objects sets X which come with a function $X \times X \longrightarrow V$ (whose value on (x, y) we denote by $X(x, y)$) such that

$$k \le X(x, x) \qquad \text{and} \qquad X(y, z) \otimes X(x, y) \le X(x, z);$$

morphisms $f\colon X \longrightarrow Y$ satisfy $X(x, y) \le Y(f(x), f(y))$, for all $x, y, z \in V$. The quantale V itself is a V-category, with the V-category structure $V(v, w) = v \multimap w$ given by its own "internal hom" defined by

$$z \le v \multimap w \iff z \otimes v \le w$$

for all $z, v, w \in V$. Moreover, V-**Cat** has an "internal hom" with

$$X \multimap Y = V\text{-}\mathbf{Cat}(X, Y) \qquad \text{and}$$
$$(X \multimap Y)(f, g) = \bigvee_{x \in X} Y(f(x), g(x)).$$

When V is commutative, V-**Cat** is symmetric monoidal closed, with $X \otimes Y = X \times Y$ and

$$(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y'),$$

and one can also form the opposite X^{op} of a V-category X, with $X^{\mathrm{op}}(x, y) = X(y, x)$. Now, the Yoneda embedding

$$\mathfrak{y}_X\colon X \longrightarrow P_V^\circ X := (X^{\mathrm{op}} \multimap V), \qquad x \longmapsto X(-, x),$$

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provides the unit of the presheaf monad $(P_V^\circ, \mathfrak{y}, \mathfrak{m})$ of $V\text{-}\mathbf{Cat}$, as follows. Writing P instead of P_V°, for $f\colon X \longrightarrow Y$, the V-functor $Pf\colon PX \longrightarrow PY$ is defined by

$$(Pf)(\alpha)(y) = \bigvee_{x \in X} Y(y, f(x)) \otimes \alpha(x)$$

for all $\alpha \in PX$, $y \in Y$, and the monad multiplication $\mathfrak{m}_X\colon PPX \longrightarrow PX$ is given by

$$\mathfrak{m}_X(\Sigma)(x) = \bigvee_{\alpha \in PX} \Sigma(\alpha) \otimes \alpha(x).$$

We claim that *the category $P\text{-}\mathbf{Mod}$ is precisely the category $V\text{-}\mathbf{Mod}$ whose objects are V-categories, and whose morphisms $\varphi\colon X \longrightarrow Y$ are V-(bi)modules, also called V-distributors or V-profunctors*, given by functions $\varphi\colon X \times Y \longrightarrow V$ satisfying

$$Y(y_1, y_2) \otimes \varphi(x_2, y_1) \otimes X(x_1, x_2) \le \varphi(x_1, y_2) \tag{$*$}$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$; composition with $\psi\colon Y \longrightarrow Z$ is defined by

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \otimes \varphi(x, y). \tag{$**$}$$

PROOF. Since $(*)$ is equivalent to $\varphi\colon X^{\mathsf{op}} \otimes Y \longrightarrow V$ being a V-functor we may as well think of φ as a V-functor $Y \longrightarrow PX$, writing $\varphi(x, y)$ as $\varphi(y)(x)$. Hence, all we need to verify is that the composition $(**)$ in $V\text{-}\mathbf{Mod}$ coincides with the Kleisli composition of $P\text{-}\mathbf{Mod}$, i.e.,

$$\bigvee_{y \in Y} \varphi(y)(x) \otimes \psi(z)(y) = (\mathfrak{m}_X \cdot P\varphi \cdot \psi)(z)(x) \tag{$***$}$$

for all $\varphi\colon Y \longrightarrow PX$, $\psi\colon Z \longrightarrow PY$ in $V\text{-}\mathbf{Cat}$, $x \in X$, $z \in Z$. By the Yoneda Lemma the left-hand side of $(***)$ may be rewritten and compared to the right-hand side, as follows:

$$\bigvee_{y \in Y} PX(\mathfrak{y}_X(x), \varphi(y)) \otimes \psi(z)(y) = (P\varphi)(\psi(z))(\mathfrak{y}_X(x))$$

$$\le (P\varphi)(\psi(z))(\mathfrak{y}_X(x)) \otimes \mathfrak{y}_X(x)(x)$$

$$\le \bigvee_{\alpha \in PX} (P\varphi)(\psi(z))(\alpha) \otimes \alpha(x)$$

$$= \mathfrak{m}_X\big((P\varphi)(\psi(z))\big)(x).$$

For "$\ge$", since

$$\left(\bigvee_{x' \in X} \alpha(x') \multimap \varphi(y)(x') \right) \otimes \alpha(x) \le (\alpha(x) \multimap \varphi(y)(x)) \otimes \alpha(x) \le \varphi(y)(x),$$

one has, for all $\alpha \in PX$,

$$(P\varphi)(\psi(z))(\alpha) \otimes \alpha(x) = \bigvee_{y \in Y} PX(\alpha, \varphi(y)) \otimes \psi(z)(y) \otimes \alpha(x)$$

$$\le \bigvee_{y \in Y} \varphi(y)(x) \otimes \psi(z)(y),$$

as desired. $\qquad\qquad\qquad\square$

With suprema formed pointwise (as in $V\text{-}\mathbf{Rel}$ of 2.3), $V\text{-}\mathbf{Mod}$ becomes a quantaloid, and it is easy to check the remaining conditions to confirm that P *is modular*, by putting $f_*(y)(x) := Y(f(x), y)$ for all $x \in X$, $y \in Y$, $f\colon X \longrightarrow Y$ in $V\text{-}\mathbf{Cat}$.
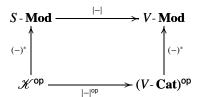
Note that Example 2.3 is a special case of 2.4, by restriction to *discrete V-categories*, i.e., sets.

In Section 4 we discuss an example showing that conditions 2, 3 of 2.1 do not follow from condition 1 of 2.1.

3. Exhibiting Kleisli morphisms as V-modules

The following theorem shows that Example 2.4 exhibits the prototypical modular monad, and it justifies the module terminology for the morphisms of its Kleisli category.

Theorem 3.1. *Let $S = (S, \varepsilon, \nu)$ be a modular monad on a category $\mathscr{K}$. Then there is a unital quantale V and a functor $|{-}|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ that can be lifted to a full and faithful homomorphism $|{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of quantaloids such that*

$$
\begin{array}{ccc}
S\text{-}\mathbf{Mod} & \xrightarrow{\;|-|\;} & V\text{-}\mathbf{Mod} \\
{\scriptstyle (-)^*}\Big\uparrow & & \Big\uparrow{\scriptstyle (-)^*} \\
\mathscr{K}^{\mathsf{op}} & \xrightarrow[\;|-|^{\mathsf{op}}\;]{} & (V\text{-}\mathbf{Cat})^{\mathsf{op}}
\end{array}
$$

commutes.

PROOF. The monoid structure of $V := S\text{-}\mathbf{Mod}(E, E) = \mathscr{K}(E, SE)$ makes V a unital quantale since $S\text{-}\mathbf{Mod}$ is a quantaloid. The hom-functor $|{-}| = \mathscr{K}(E, -)$ takes values in $V\text{-}\mathbf{Cat}$ if, for a $\mathscr{K}$ object X, we put

$$
|X|(x, y) := y^* \circ x_*
$$

for all $x, y \in |X|$. Indeed, the adjunctions $x_* \dashv x^*$ show the V-category laws for $|X|$, and for a morphism $f\colon X \longrightarrow Y$ in $\mathscr{K}$ one obtains the V-functoriality of $|f|$ from

$$
|X|(x, y) = y^* \circ 1_X^* \circ x_* \le y^* \circ f^* \circ f_* \circ x_* \le (f\cdot y)^* \circ (f\cdot x)_* = |Y|(|f|(x), |f|(x))
$$

for all $x, y \in |X|$; here we used the fact that the local adjunctions $f_* \dashv f^*$ compose, so that functoriality of $(-)^*$ produces a pseudofunctor $(-)_*\colon \mathscr{K} \longrightarrow S\text{-}\mathbf{Mod}$. For $\varphi\colon X \longrightarrow Y$ in $S\text{-}\mathbf{Mod}$ one defines $|\varphi|\colon |X| \longrightarrow |Y|$ in $V\text{-}\mathbf{Mod}$ by

$$
|\varphi|(x, y) = y^* \circ \varphi \circ x_*
$$

for $x \in |X|$, $y \in |Y|$; V-modularity follows immediately from the local adjunctions. Also, for $f\colon X \longrightarrow Y$ in $\mathscr{K}$, one has

$$
|f^*|(y, x) = x^* \circ f^* \circ y_* = (f\cdot x)^* \circ y_* = |Y|(y, |f|(x)) = |f|^*(y, x),
$$

for all $x \in |X|, y \in |Y|$. Next we show that $|-| \colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ preserves composition and suprema. In fact, for $\psi \colon Y \dashrightarrow Z$ and $x \in |X|, z \in |Z|$ one has:

$$
\begin{aligned}
|\psi \circ \varphi|(x, z) &= z^* \circ \psi \circ 1_Y^* \circ \varphi \circ x_* \\
&= z^* \circ \psi \circ \left(\bigvee_{y \in |Y|} y_* \circ y^* \right) \circ \varphi \circ x_* \\
&= \bigvee_{y \in |Y|} (z^* \circ \psi \circ y_*) \circ (y^* \circ \varphi \circ x_*) \\
&= (|\psi| \circ |\varphi|)(x, z);
\end{aligned}
$$

also, for $\varphi_i \colon X \dashrightarrow Y$ $(i \in I)$ and $x \in |X|, y \in |Y|$, one has:

$$
\begin{aligned}
\left| \bigvee_i \varphi_i \right|(x, y) &= y^* \circ \bigvee_i \varphi_i \circ x_* \\
&= \bigvee_i y^* \circ \varphi_i \circ x_* \\
&= \left(\bigvee_i |\varphi_i| \right)(x, y).
\end{aligned}
$$

Furthermore, since

$$
\varphi = 1_Y^* \circ \varphi \circ 1_X^* = \bigvee_{x \in |X|, y \in |Y|} y_* \circ y^* \circ \varphi \circ x_* \circ x^* = \bigvee_{x, y} y_* \circ |\varphi|(x, y) \circ x^*,
$$

φ is in fact determined by $|\varphi|$, so that $|-| \colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ is faithful. In order to show that $|-|$ is also full, given a V-module $\phi \colon |X| \longrightarrow |Y|$, one defines

$$
\varphi := \bigvee_{x, y} y_* \circ \phi(x, y) \circ x^*
$$

and obtains

$$
\begin{aligned}
|\varphi|(x', y') &= \bigvee_{x, y} (y')^* \circ y_* \circ \phi(x, y) \circ x^* \circ (x')_* \\
&= \bigvee_{x, y} |Y|(y, y') \circ \phi(x, y) \circ |X|(x', x) \\
&= \phi(x, y),
\end{aligned}
$$

with the last equality arising from the V-modularity of ϕ. □

Remark 3.2. The 2-functor $|-| \colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of 3.1 is not only full and faitful at the 1-cell level, but also at the 2-cell level, that is:

$$
|-| \colon S\text{-}\mathbf{Mod}(X, Y) \longrightarrow V\text{-}\mathbf{Mod}(|X|, |Y|)
$$

is an order-isomorphism, for all objects X, Y in $\mathscr{K}$. Indeed, if $|\varphi| \leq |\psi|$ for $\varphi, \psi \colon X \dashrightarrow Y$, then

$$
\varphi = \bigvee_{x, y} y_* \circ |\varphi|(x, y) \circ x^* \leq \bigvee_{x, y} y_* \circ |\psi|(x, y) \circ x^* = \psi.
$$

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Remark 3.3. For $\mathscr{K} = V\text{-}\mathbf{Cat}$ with a commutative unital quantale V and $S = P_V^\circ = P$ as in Example 2.4, the construction of Theorem 3.1 reproduces the given V as $P\text{-}\mathbf{Mod}(E, E) = V\text{-}\mathbf{Cat}(E, PE)$, with E the singleton V-category that is neutral w.r.t. the tensor product of $V\text{-}\mathbf{Cat}$. Indeed, V-functors $E \longrightarrow (E^{\mathrm{op}} \multimap V) \cong V$ correspond to elements of V, which produces an isomorphism $V\text{-}\mathbf{Mod}(E, E) \cong V$ of quantales. In fact, one also has (in the notation of 3.1) $|X| \cong X$ for every V-category X, in particular $|V| \cong V$ (as V-categories). Consequently, both horizontal functors in the diagram of 3.1 become equivalences of categories.

Theorem 3.1 offers two ways of making a category $\mathscr{K}$ which carries a modular monad into an ordered category, by either declaring $|-|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ to be full and faithful on 2-cells, or $(-)^*\colon \mathscr{K}^{\mathrm{op}} \longrightarrow S\text{-}\mathbf{Mod}$; fortunately, the two options are equivalent.

Proposition 3.4. *With the assumptions and notations of Theorem* 3.1, *one has $|f| \leq |g|$ in $V\text{-}\mathbf{Cat}$ if, and only if, $f^* \leq g^*$ in $S\text{-}\mathbf{Mod}$, for all morphisms $f, g\colon X \longrightarrow Y$ in $\mathscr{K}$.*

PROOF. By the definition of the 2-categorical structure of $V\text{-}\mathbf{Cat}$, $|f| \leq |g|$ means

$$1_E^* \leq |Y|(|f|(x), |g|(x)) = x^* \circ g^* \circ f_* \circ x_*$$

for all $x \in |X|$. Consequently,

$$1_X^* = \bigvee_{x \in |X|} x_* \circ x^* \leq \bigvee_{x \in |X|} x_* \circ x^* \circ g^* \circ f_* \circ x_* \circ x^* \leq g^* \circ f_*$$

and, hence, $f^* \leq g^*$ by adjunction. The converse implication is obvious. $\qquad\square$

Corollary 3.5. *A category with a modular monad becomes a* 2-*category when one puts*

$$f \leq g :\Longleftrightarrow |f| \leq |g| \iff f^* \leq g^*.$$

This way all functors of the diagram of 3.1 *become full and faithful on* 2-*cells.*

4. The Hausdorff monad

For a commutative unital quantale V, let $(H, \{-\}, \bigcup)$ denote the Hausdorff monad on $V\text{-}\mathbf{Cat}$ which is a lifting of the power set monad 2.2 of $\mathbf{Set}$ along the forgetful functor $V\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set}$ (see [1]). Hence, $HX = PX$ as sets, and

$$HX(A, B) = \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y)$$

$$= \bigwedge_{x \in X} X(x, A) \multimap X(x, B),$$

where $X(x, B) = \bigvee_{y \in B} X(x, y)$, for all $A, B \subseteq X$, X in $V\text{-}\mathbf{Cat}$. An H-module $\varphi\colon X \longrightarrow Y$ is a V-functor $\varphi\colon Y \longrightarrow HX$, i.e., φ must satisfy

$$Y(y, y') \leq HX(\varphi(y), \varphi(y'))$$

for all $y, y' \in Y$, that is

$$Y(y, y') \otimes X(x, \varphi(y)) \leq X(x, \varphi(y'))$$

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for all $x \in X$. Composition of φ with $\psi \colon Y \longrightarrow Z$ is given by

$$(\psi \circ \varphi)(z) = \bigcup_{y \in \psi(z)} \varphi(y)$$

for all $z \in Z$. Finally, for $f \colon X \longrightarrow Y$ in $V\text{-}\mathbf{Cat}$, $f^* \colon Y \longrightarrow X$ is given by $f^*(x) = \{f(x)\}$ for all $x \in X$.

We discuss two options for making $H\text{-}\mathbf{Mod}$ into a quantaloid:

A. $\quad\quad\quad\quad\quad \varphi \leq \varphi' :\Longleftrightarrow \forall y \in Y : \quad \varphi(y) \subseteq \varphi'(y),$

B. $\quad\quad\quad\quad\quad \varphi \lesssim \varphi' :\Longleftrightarrow \forall y \in Y : \quad \varphi(y) \leq \varphi'(y) \quad$ (in the V-category HX),

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \Longleftrightarrow \forall y \in Y : \quad k \leq HX(\varphi(y), \varphi'(y)),$

for $\varphi, \varphi' \colon X \longrightarrow Y$ in $H\text{-}\mathbf{Mod}$.

Of course, "$\lesssim$" fails to be separated in general, but this is not essential, i.e., Definition 2.1 may be relaxed by dropping the antisymmetry requirement for the lattice structure of the hom-sets, without any detrimental effect on the subsequent theory; however, *see Remark* 4.3 *below*. With this proviso we may state:

Proposition 4.1. $H\text{-}\mathbf{Mod}$ *becomes a quantaloid under both orders, $\leq$ and $\lesssim$.*

PROOF. For $\varphi_i \colon X \longrightarrow Y$ ($i \in I$) in $H\text{-}\mathbf{Mod}$, $\varphi(y) := \bigcup_{i \in I} \varphi_i(y)$ defines a supremum under either order, which is easily seen to be preserved by composition in $H\text{-}\mathbf{Mod}$ from both sides. $\quad\quad\square$

The principal difference of the two structures is exhibited when we look at the order induced on $V\text{-}\mathbf{Cat}$ by $(-)^* \colon V\text{-}\mathbf{Cat} \longrightarrow H\text{-}\mathbf{Mod}$ (see 3.4): for $f, f' \colon X \longrightarrow Y$ in $V\text{-}\mathbf{Cat}$ one has:

A. $\quad\quad\quad f^* \leq g^* \iff \forall x \in X : \quad \{f(x)\} \subseteq \{g(x)\} \iff f = g,$

B. $\quad\quad\quad f^* \lesssim g^* \iff \forall x \in X : \quad k \leq X(f(x), g(x)) \iff f \leq g \quad$ (in $V\text{-}\mathbf{Cat}$).

Briefly, $(-)^*$ induces the discrete order on $V\text{-}\mathbf{Cat}$ under option A, and the "natural" order under option B.

Assume now that $f^* \colon Y \longrightarrow X$ has a left adjoint $f_* \colon X \longrightarrow Y$ in $H\text{-}\mathbf{Mod}$. Since

$$(f^* \circ f_*)(x) = f_*(f(x)), \quad\quad (f_* \circ f^*)(y) = f(f_*(y))$$

for all $x \in X, y \in Y$, under option A the adjointness conditions amount to

$$x \in f_*(f(x)) \quad\quad \text{and} \quad\quad f_*(y) \subseteq f^{-1}(y),$$

for all $x \in X, y \in Y$. In addition, V-functoriality of $f_* \colon Y \longrightarrow HX$ implies

$$Y(y, y') \leq \bigvee_{x' \in f_*(y')} X(x, x') \quad\quad\quad\quad\quad\quad\quad (*)$$

for all $y \in Y, x \in f_*(y)$. In case $V = 2$, so that $V\text{-}\mathbf{Cat} = \mathbf{Ord}$ is the category of (pre)ordered sets and monotone maps, these conditions force $f \colon X \longrightarrow Y$ to have an up-closed image: whenever $f(x) \leq y'$, then $y' = f(x')$ for some $x' \geq x$. But monotonicity of f does not guarantee its image to be up-closed.

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Under option B the adjointness conditions are equivalently described by

$$k \leq X\big(x, f_*(f(x))\big) \qquad \text{and} \qquad f_*(y) \subseteq \{x \in X \mid f(x) \leq y\}$$

for all $x \in X, y \in Y$. These conditions are trivially satisfied if, conversely, we now define f_* by

$$f_*(y) := \{x \in X \mid f(x) \leq y\} = \big\{x \in X \mid k \leq X(f(x), y)\big\}$$

for all $y \in Y$. In case $V = 2$, f_* satisfies also the (quite restrictive) V-functoriality condition (∗). In addition, condition 3 of 2.1 is trivially satisfied.

These findings may be expressed in terms of the ordinary relational composition $\circ$, as follows:

Proposition 4.2. *The Hausdorff monad H on* **Ord** *(= 2-* **Cat***) becomes modular if one orders* H-**Mod**(X, Y) *by*

$$\varphi \lesssim \varphi' \iff \varphi \subseteq \varphi' \circ (\leq_X),$$

but not when one uses $(\varphi \leq \varphi' \iff \varphi \subseteq \varphi')$. *Here, a relation* φ *from X to Y is an H-module if, and only if,* $(\leq_Y \circ \varphi) \subseteq (\varphi \circ \leq_X)$. *With E a singleton set, the functor*

$$|-| : H\text{-}\mathbf{Mod} \longrightarrow 2\text{-}\mathbf{Mod}$$

of 3.1 *assigns to* φ *the relation* $\varphi \circ (\leq_X)$; *it makes the (pre)ordered sets* H-**Mod**(X, Y) *and* 2-**Mod**(X, Y) *equivalent (as categories), but not necessarily isomorphic.*

Here is the reason for this last statement:

Remark 4.3. Since $\lesssim$ fails to be antisymmetric, in general the functor $|-| : H\text{-}\mathbf{Mod} \longrightarrow 2\text{-}\mathbf{Mod}$ of 4.2 is *not* necessarily faithful (on 1-cells) but satisfies only

$$|\varphi| = |\varphi'| \iff \varphi \lesssim \varphi \ \& \ \varphi' \lesssim \varphi'.$$

However, $|-| : \mathbf{Ord} \longrightarrow \mathbf{Ord} = 2\text{-}\mathbf{Cat}$ of 3.1 is an equivalence of categories.

5. A functorial correspondence between modular monads and quantales

We describe the assigment $S \longmapsto V$ of Theorem 3.1 as a functor

$$\Delta : \mathbf{MODMON} \longrightarrow \mathbf{Quant}.$$

Here **Quant** has as objects unital quantales, and a morphism $\Phi : V \longrightarrow W$ must preserve suprema and the monoid structure given by the tensor product. The (very large) category **MODMON** has as objects categories $\mathscr{K}$ equipped with a monad $S = (S, \varepsilon, \nu)$, a distinguished object E in $\mathscr{K}$ and a fixed order that makes S-**Mod** a quantaloid and S modular. A morphism

$$(F, \alpha) : (\mathscr{K}, (S, \varepsilon, \nu), E) \longrightarrow (\mathscr{L}, (T, \eta, \mu), D)$$

of modular monads consists of a functor $F : \mathscr{K} \longrightarrow \mathscr{L}$ and a natural transformation $\alpha : FS \longrightarrow TF$ such that $FE \cong D$, $\alpha \cdot F\varepsilon = \eta F$, $\alpha \cdot F\nu = \mu F \cdot T\alpha \cdot \alpha S$, and the induced functor

$$\widetilde{(F, \alpha)} : S\text{-}\mathbf{Mod} \longrightarrow T\text{-}\mathbf{Mod}$$

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preserves suprema, i.e., is a morphism of quantaloids. Functoriality of

$$\widetilde{(F,\alpha)}: (X \xrightarrow{\;\varphi\;} Y) \longmapsto (FX \xrightarrow{\;\alpha_X \cdot F\varphi\;} FY)$$

is in fact guaranteed by the preceding conditions, while preservation of suprema amounts to the condition

$$\alpha_X \cdot F\left(\bigvee_i \varphi_i\right) = \bigvee_i (\alpha_X \cdot F\varphi_i),$$

for all $\varphi_i : X \longrightarrow Y$, $i \in I$. Now $\Delta(F,\alpha)$ is simply a hom-map of the functor $\widetilde{(F,\alpha)}$:

$$\Delta(F,\alpha) := \widetilde{(F,\alpha)}_{E,E} : V = S\text{-}\mathbf{Mod}(E,E) \longrightarrow W = T\text{-}\mathbf{Mod}(D,D),$$

which is indeed a morphism in **Quant**. Functoriality of Δ follows from the easily checked fact

$$\widetilde{(G,\beta)(F,\alpha)} = \widetilde{(G,\beta)}\,\widetilde{(F,\alpha)},$$

with $(G,\beta): (\mathscr{L},T,D) \longrightarrow (\mathscr{M},U,C)$ and

$$(G,\beta)(F,\alpha) = (GF, \beta F \cdot G\alpha)$$

in **MODMON**.

Calling a modular monad S on $\mathscr{K}$ *commutative* if the quantale $\Delta(\mathscr{K},S,E) = S\text{-}\mathbf{Mod}(E,E)$ is commutative, one has the restricted functor

$$\Delta: \mathbf{CMODMON} \longrightarrow \mathbf{CQuant}$$

of commutative objects on both sides. Next we will show that Example 2.4 provides the object function of a functor Γ in the opposite direction:

$$V \longmapsto \Gamma V = (V\text{-}\mathbf{Cat}, P_V^\circ, E_V),$$

with $E_V = E$ as in Remark 3.3. For a morphism $\Phi: V \longrightarrow W$ of commutative unital quantales, one defines the morphism $\Gamma\Phi$ in **CMODMON**, as follows. First of all, without change of notation we can regard Φ as a functor

$$V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}, \qquad X \longmapsto \Phi X = X,$$

keeping underlying sets fixed and mapping structures by Φ:

$$(\Phi X)(x,y) = \Phi(X(x,y)),$$

for all $x,y \in X$. In fact, Φ may be (more generally) regarded as a functor

$$V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod}, \qquad (X \xrightarrow{\;\varphi\;} Y) \longmapsto (\Phi X \xrightarrow{\;\Phi\varphi\;} \Phi Y),$$

with $(\Phi\varphi)(x,y) = \Phi(\varphi(x,y))$ for all $x \in X, y \in Y$. One then has

$$\Phi(f^*) = (\Phi f)^*, \qquad \Phi(f_*) = (\Phi f)_*$$

for all $f\colon X \longrightarrow Y$ in $V\text{-}\mathbf{Cat}$. We will show that the functor $V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod}$ above is in fact induced by a morphism

$$(\Phi, \pi)\colon (V\text{-}\mathbf{Cat}, P_V^\circ, E_V) \longrightarrow (W\text{-}\mathbf{Cat}, P_W^\circ, E_W),$$

as

$$\widetilde{(\Phi, \pi)}\colon V\text{-}\mathbf{Mod} = P_V^\circ\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod} = P_W^\circ\text{-}\mathbf{Mod},$$

where now Φ is regarded as a functor $V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}$. In order to define the natural transformation

$$\pi = \pi^\Phi\colon \Phi P_V^\circ \longrightarrow P_W^\circ \Phi,$$

for a V-category X we let

$$\pi_X\colon \Phi(X^{\mathrm{op}} \multimap V) \longrightarrow ((\Phi X)^{\mathrm{op}} \multimap W)$$

assign to every V-functor $\alpha\colon X^{\mathrm{op}} \longrightarrow V$ the map $(x \longmapsto \Phi(\alpha(x)))$. This map is indeed a W-functor since

$$(\Phi X)(x, y) = \Phi(X(x, y)) \le \Phi(\alpha(y) \multimap \alpha(x)) \le \Phi(\alpha(y)) \multimap \Phi(\alpha(x)),$$
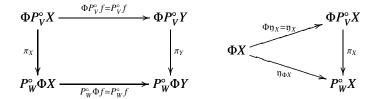
for all $x, y \in X$. Moreover, π_X is a W-functor since

$$\Phi\left(\bigwedge_{x \in X} \alpha(x) \multimap \beta(x)\right) \le \bigwedge_{x \in X} \Phi(\alpha(x) \multimap \beta(x)) \le \bigwedge_{x \in X} \Phi(\alpha(x)) \multimap \Phi(\beta(x)),$$

for all $\alpha, \beta \in P_V^\circ X$. While one immediately sees that $\widetilde{(\Phi, \pi)}$ is indeed the functor $V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod}$ described above, it is a bit more laborious to verify the remaining requirements for (Φ, π) being a morphism in $\mathbf{MODMON}$, namely:

Lemma 5.1. $\pi\colon \Phi P_V^\circ \longrightarrow P_W^\circ \Phi$ is a natural transformation with $\pi{\cdot}\Phi\mathfrak{y}^V = \mathfrak{y}^W\Phi$ and $\pi{\cdot}\Phi\mathfrak{m}^V = \mathfrak{m}^W\Phi{\cdot}P_W^\circ\pi{\cdot}\pi P_V^\circ$.

PROOF. The commutativity of the diagrams

$$
\begin{array}{ccc}
\Phi P_V^\circ X & \xrightarrow{\ \Phi P_V^\circ f = P_V^\circ f\ } & \Phi P_V^\circ Y \\
{\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\
P_W^\circ \Phi X & \xrightarrow[\ P_W^\circ \Phi f = P_W^\circ f\ ]{} & P_W^\circ \Phi Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & \Phi P_V^\circ X \\
 & \nearrow^{\Phi\mathfrak{y}_X = \mathfrak{y}_X} & \big\downarrow{\scriptstyle \pi_X} \\
\Phi X & & \\
 & \searrow_{\mathfrak{y}_{\Phi X}} & P_W^\circ X
\end{array}
$$

is immediate, for all $f\colon X \longrightarrow Y$ in $V\text{-}\mathbf{Cat}$:

$$
\begin{aligned}
P_W^\circ f(\pi_X(\alpha))(y) &= \bigvee_{x \in X} \Phi\big(Y(y, f(x))\big) \otimes \Phi(\alpha(x)) \\
&= \Phi\left(\bigvee_{x \in X} Y(y, f(x)) \otimes \alpha(x)\right) \\
&= \Phi(P_V^\circ f(\alpha)(y)) \\
&= \pi_Y(P_V^\circ f(\alpha))(y), \\
\pi_X(\mathfrak{y}_X(x))(x') &= \Phi(X(x', x)) = \mathfrak{y}_{\Phi X}(x)(x'),
\end{aligned}
$$

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for all $x, x' \in X$, $y \in Y$, $\alpha \in P_V^\circ X$. For the commutativity of

$$
\begin{array}{ccc}
\Phi P_V^\circ P_V^\circ X & \xrightarrow{\;\pi_{P_V^\circ X}\;} P_W^\circ \Phi P_V^\circ X \xrightarrow{\;P_W^\circ \pi_X\;} P_W^\circ P_W^\circ \Phi X \\[2pt]
\Phi\mathfrak{m}_X = \mathfrak{m}_X \Big\downarrow & \hspace{6cm} \Big\downarrow \mathfrak{m}_{\Phi X} \\[2pt]
\Phi P_V^\circ X & \xrightarrow{\hspace{5.5cm}} P_W^\circ \Phi X \\
& \hspace{2cm}{\scriptstyle \pi_X}
\end{array}
$$

let $\Sigma \in P_V^\circ P_V^\circ X$ and $x \in X$; then

$$
\pi_X(\mathfrak{m}_X(\Sigma)) = \Phi(\mathfrak{m}_X(\Sigma)(x))
$$

$$
= \Phi\Big(\bigvee_{\alpha \in P_V^\circ X} \Sigma(\alpha) \otimes \alpha(x) \Big)
$$

$$
= \bigvee_{\alpha \in P_V^\circ X} \Phi(\Sigma(\alpha)) \otimes \Phi(\alpha(x)).
$$

On the other hand,

$$
\mathfrak{m}_{\Phi X}\Big(P_W^\circ \pi_X(\pi_{P_V^\circ X}(\Sigma))\Big) = \bigvee_{\beta \in P_W^\circ \Phi X} P_W^\circ \pi_X(\pi_{P_V^\circ X}(\Sigma))(\beta) \otimes \beta(x)
$$

$$
= \bigvee_{\beta \in P_W^\circ(\Phi X)} \bigvee_{\alpha \in P_V^\circ X} P_W^\circ(\Phi X)(\beta, \pi_X(\alpha)) \otimes \pi_{P_V^\circ X}(\Sigma)(\alpha) \otimes \beta(x)
$$

$$
= \bigvee_{\alpha \in P_V^\circ X} \Phi(\Sigma(\alpha)) \otimes \Big(\bigvee_{\beta \in P_W^\circ(\Phi X)} P_W^\circ(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \Big).
$$

Hence, it suffices to show

$$
\bigvee_{\beta \in P_W^\circ(\Phi X)} P_W^\circ(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) = \Phi(\alpha(x));
$$

but

$$
P_W^\circ(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \leq \big(\beta(x) \multimap \Phi(\alpha(x)) \big) \otimes \beta(x) \leq \Phi(\alpha(x))
$$

for all $\beta \in P_W^\circ(\Phi X)$, and for $\beta := \Phi X(-, x)$ one has with the Yoneda Lemma

$$
P_W^\circ(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \geq \pi_X(\alpha)(x) \otimes \Phi X(x, x) \geq \Phi(\alpha(x)).
$$

$\square$

Since functoriality of Δ follows immediately from the definitions, we are now ready to summarize what we have proved so far:

Theorem 5.2. *There are functors*

$$
\mathbf{CQuant} \; \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \; \mathbf{CMODMON}
$$

with $\Delta\Gamma \cong 1$.

The question to "which extent" this pair of functors is adjoint is discussed in the next section.

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6. An adjunction between modular monads and quantales

For a commutative monad S on $\mathscr{K}$ with distiquished object E we first revisit the functors $|-|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ and $|-|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of 3.1 (with $V = S\text{-}\mathbf{Mod}(E, E)$) and show:

Proposition 6.1. *There is a natural transformation γ such that*

$$(|-|, \gamma)\colon (\mathscr{K}, S, E) \longrightarrow \Gamma V$$

is a morphism of monads with $\widetilde{(|-|, \gamma)} = |-|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$.

Proof. For X in $\mathscr{K}$ one defines

$$\gamma_X \colon |SX| \longrightarrow P_V^\circ|X| = (|X|^{\mathrm{op}} \multimap V)$$

by $\gamma_X(\varphi)(x) = \varphi \circ x_*$, for all $\varphi \in |SX| = \mathscr{K}(E, SX) = S\text{-}\mathbf{Mod}(X, E)$ and $x \in |X| = \mathscr{K}(E, X)$. With $y \in |X|$ one has

$$\gamma_X(\varphi)(y) \circ |X|(x, y) = \varphi \circ y_* \circ y^* \circ x_* \le \varphi \circ x_* = \gamma_X(\varphi)(x)$$

and then $|X|(x, y) \le \gamma_X(\varphi)(y) \multimap \gamma_X(\varphi)(x)$, so that $\gamma_X(\varphi)\colon |X|^{\mathrm{op}} \longrightarrow V$ is indeed a V-functor. In order to show that γ_X is a V-functor, we recall from Section 2 that the V-module φ may be written as $\varphi = \varphi^* \circ \iota_X$ and, with $\psi \in |SX|$, we obtain:

$$
\begin{aligned}
|SX|(\varphi, \psi) \circ \gamma_X(\varphi)(x) &= \psi^* \circ \varphi_* \circ \varphi \circ x_* \\
&= \psi^* \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_* \\
&\le \psi^* \circ \iota_X \circ x_* \\
&= \gamma_X(\psi(x))
\end{aligned}
$$

for all $x \in |X|$, hence $|SX|(\varphi, \psi) \le P_V^\circ|X|(\gamma_X(\varphi), \gamma_X(\psi))$. For $f\colon X \longrightarrow Y$ in $\mathscr{K}$, the diagram

$$
\begin{array}{ccc}
|SX| & \xrightarrow{\;|Sf|\;} & |SY| \\
{\scriptstyle \gamma_X}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_Y} \\
P_V^\circ|X| & \xrightarrow[\;P_V^\circ|f|\;]{} & P_V^\circ|Y|
\end{array}
$$

commutes since for all $\varphi \in |SX|$ and $y \in |Y|$ one has:

$$
\begin{aligned}
P_V^\circ|f|(\gamma_X(\varphi))(y) &= \bigvee_{x \in |X|} \gamma_X(\varphi)(x) \circ |Y|(y, |f|(x)) \\
&= \bigvee_{x \in |X|} \varphi^* \circ \iota_X \circ x_* \circ (f \cdot x)^* \circ y_* \\
&= \varphi^* \circ \iota_X \circ \left(\bigvee_{x \in |X|} x_* \circ x^* \right) \circ f^* \circ y_* \\
&= \varphi^* \circ \iota_X \circ f^* \circ y_* \\
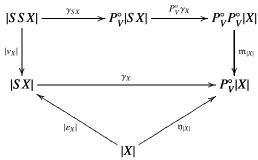&= \varphi^* \circ (Sf)^* \circ \iota_Y \circ y_* \\
&= (Sf \cdot \varphi)^* \circ \iota_Y \circ y_* \\
&= \gamma_Y(Sf \cdot \varphi)(y) \\
&= (\gamma_Y \cdot |Sf|)(\varphi)(y).
\end{aligned}
$$

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Hence, γ is a natural transformation. Finally, we must check that the following diagram commutes:

$$
\begin{array}{ccccc}
|SSX| & \xrightarrow{\;\gamma_{SX}\;} & P^\circ_V|SX| & \xrightarrow{\;P^\circ_V\gamma_X\;} & P^\circ_V P^\circ_V|X| \\
{\scriptstyle |\nu_X|}\downarrow & & & & \downarrow{\scriptstyle \mathfrak{m}_{|X|}} \\
|SX| & & \xrightarrow{\qquad\qquad\gamma_X\qquad\qquad} & & P^\circ_V|X|
\end{array}
$$

with lower maps $|\varepsilon_X|$ and $\mathfrak{y}_{|X|}$ down to $|X|$.

But for all $x, y \in |X|$ one has:

$$\gamma_X(|\varepsilon_X|(x))(y) = (\varepsilon_X\cdot x)^* \circ \iota_X \circ y_* = x^* \circ \varepsilon_X^* \circ \iota_X \circ y_* = x^* \circ y_* = \mathfrak{y}_{|X|}(x)(y).$$

Furthermore, for all $\theta \in |SSX|$ and $x \in X$ one obtains:

$$
\begin{aligned}
\gamma_X(|\nu_X|(\theta))(x) &= \gamma_X(\nu_X\cdot\theta)(x) \\
&= \theta^* \circ \nu_X^* \circ \iota_X \circ x_* \\
&= \theta^* \circ \iota_{SX} \circ \iota_X \circ x_* \\
&= \bigvee_{\varphi\in|SX|} \theta^* \circ \iota_{SX} \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_*.
\end{aligned}
$$

Using the Yoneda Lemma one may rewrite

$$\varphi^* \circ \iota_X \circ x_* = \gamma_X(\varphi)(x) = \bigvee_{\alpha\in P^\circ_V|X|} P^\circ_V|X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x),$$

which then gives

$$
\begin{aligned}
\gamma_X(|\nu_X|(\theta))(x) &= \bigvee_{\alpha\in P^\circ_V|X|}\bigvee_{\varphi\in SX} \theta^* \circ \iota_{SX} \circ \varphi_* \circ P^\circ_V|X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x) \\
&= \bigvee_{\alpha\in P^\circ_V|X|} P^\circ_V\gamma_X(\gamma_{SX}(\theta))(\alpha) \circ \alpha(x) \\
&= \mathfrak{m}_{|X|}\big(P^\circ_V\gamma_X(\gamma_{SX}(\theta))\big)(x).
\end{aligned}
$$

$\square$

For a commutative modular monad S on $\mathscr{K}$ with distiguished object E and a commutative unital quantale W, from 5.2 and 6.1 one obtains the map
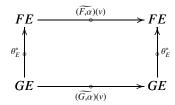
$$\mathbf{CQuant}(\Delta(\mathscr{K}, S, E), W) \longrightarrow \mathbf{CMODMON}((\mathscr{K}, S, E), \Gamma W)$$
$$\Phi \longmapsto \Gamma\Phi\cdot(|-|, \gamma)$$

which, however, cannot be expected to be surjective, not even "up to isomorphism": for a morphism $(F, \alpha)\colon (\mathscr{K}, S, E) \longrightarrow (W\text{-}\mathbf{Cat}, P^\circ_W, E_W)$ to be in its image, FX must have underlying set $\mathscr{K}(E, X)$, but one should allow for appropriate isomorphisms. Hence we must restrict the codomain of the above map appropriately.

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Definition 6.2. (1) *A morphism* $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ *of modular monads is **representable** if there is a natural isomorphism* $\tau\colon \mathscr{K}(E,-) \longrightarrow \mathscr{L}(D,F-)$.

(2) *A 2-**cell*** $\theta\colon (F,\alpha) \longrightarrow (G,\beta)$ *of morphisms* $(F,\alpha),(G,\beta)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ *of modular monads is a natural transformation* $\theta\colon F \longrightarrow G$ *with* $T\theta\cdot\alpha = \beta\cdot\theta S$ *;* (F,α) *and* (G,β) *are **isomorphic** if* θ *can be chosen to be an isomorphism, i.e., if all* θ_X *are isomorphisms.*

Remarks 6.3. (1) By the Yoneda Lemma, a natural transformation $\tau\colon \mathscr{K}(E,-) \longrightarrow \mathscr{L}(D,F-)$ is completely determined by a morphism $i\colon D \longrightarrow FE$ in $\mathscr{L}$, as $\tau_X(x) = Fx\cdot i$ for all $x \in \mathscr{K}(E,X)$. Since $\mathscr{K}(E,E) = \{1_E\}$ and $FE \cong D$ one sees that (F,α) *is representable if, and only if, the maps* $F_{E,X}\colon \mathscr{K}(E,X) \longrightarrow \mathscr{L}(FE,FX)$ *are bijective for all objects X in* $\mathscr{K}$.

(2) A 2-cell $\theta\colon (F,\alpha) \longrightarrow (G,\beta)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ induces a natural transformation $\theta^*\colon \widetilde{(G,\beta)} \longrightarrow \widetilde{(F,\alpha)}\colon S\text{-}\mathbf{Mod} \longrightarrow T\text{-}\mathbf{Mod}$ with $(\theta^*)_X = (\theta_X)^*$ for all object X in $\mathscr{K}$. Since $\mathscr{L}(D,D) \cong \mathscr{L}(D,FE) \cong \mathscr{L}(D,GE)$ are singleton sets, $\theta_E = j\cdot i^{-1}$ is an isomorphism, with the unique morphisms $i\colon D \longrightarrow FE$, $j\colon D \longrightarrow GE$ in $\mathscr{L}$. Consequently, for every $v \in V = S\text{-}\mathbf{Mod}(E,E)$ one has a commutative diagram

$$
\begin{array}{ccc}
FE & \xrightarrow{\;\;\widetilde{(F,\alpha)}(v)\;\;} & FE \\[2pt]
\Big\uparrow{\scriptstyle \theta_E^*} & & \Big\uparrow{\scriptstyle \theta_E^*} \\[2pt]
GE & \xrightarrow[\;\;\widetilde{(G,\alpha)}(v)\;\;]{} & GE
\end{array}
$$

with the horizontal arrows therefore determining the same element in $W = T\text{-}\mathbf{Mod}(D,D)$. Consequently: *if there is a 2-cell* $(F,\alpha) \longrightarrow (G,\beta)$ *of morphisms of modular monads, then* $\Delta(F,\alpha) = \Delta(G,\beta)$.

(3) For a 2-cell θ as above, from the naturality condition $\theta_X\cdot Fx = Gx\cdot\theta_E$ for all $x \in \mathscr{K}(E,X)$ and the fact that θ_E is the only morphism in $\mathscr{L}(FE,GE)$ one derives immediately: *if the family* $(Fx)_{x\in|X|}$ *is jointly epic, then there is at most one 2-cell* $(F,\alpha) \longrightarrow (G,\beta)$. When (F,α) is representable and $(\mathscr{L},T,D) = \Gamma W = (W\text{-}\mathbf{Cat}, P_W^\circ, E_W)$ for a commutative unitale quantale W, the epi-condition is certainly satisfied since there is a bijection

$$F_{E,X}\colon \mathscr{K}(E,X) \longrightarrow W\text{-}\mathbf{Cat}(FE,FX) \cong FX.$$

(4) For any morphism $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ of modular monads one has

$$\widetilde{(F,\alpha)}(f^*) = (Ff)^* \qquad \text{and} \qquad \widetilde{(F,\alpha)}(f_*) = (Ff)_*,$$

for all morphisms $f\colon X \longrightarrow Y$ in $\mathscr{K}$. Indeed,

$$\widetilde{(F,\alpha)}(f^*) = \alpha_Y\cdot F\varepsilon_Y\cdot Ff = \eta_{FY}\cdot Ff = (Ff)^*;$$

and since both, $\widetilde{(F,\alpha)}(f_*)$ and $(Ff)_*$ are left adjoint to $(Ff)^*$, also the second claim holds.

We can now set up the category **RCMODMON** whose objects are commutative modular monads (as in **CMODMON**) but whose morphisms are isomorphism classes of representable morphisms $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$; we denote the class of (F,α) by $[F,\alpha]$. By Remark 6.3(2), the functor

$$\Delta\colon \mathbf{RCMODMON} \longrightarrow \mathbf{CQuant}, \qquad [F,\alpha] \longmapsto \Delta(F,\alpha),$$

is well defined, and we can now state:

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Theorem 6.4. Δ *has a full and faithful right adjoint* Γ. *Hence,* **CQuant** *is equivalent to a full reflective subcategory of* **RCMODMON**.
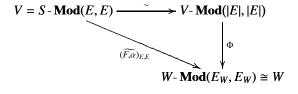
PROOF. With Γ defined by $\Phi \longmapsto [\Phi, \pi^\Phi]$ (see Section 5) we must prove that every morphism

$$[F, \alpha] \colon (\mathscr{K}, S, E) \longrightarrow \Gamma W = (W\text{-}\mathbf{Cat}, P_W^\circ, E_W)$$

in **RCMODMON** with a commutative unital quantale W factors as $[F, \alpha] = \Gamma\Phi\cdot[|-|, \gamma]$, with a uniquely determined morphism $\Phi \colon V = \Delta(\mathscr{K}, S, E) \longrightarrow W$ of quantales. By Remark 6.3(2), such Φ must necessarily satisfy

$$(\widetilde{\Phi, \pi^\phi})(\widetilde{|-|, \gamma}) \cong (\widetilde{F, \alpha});$$

in particular, the diagram

$$V = S\text{-}\mathbf{Mod}(E, E) \xrightarrow{\quad \sim \quad} V\text{-}\mathbf{Mod}(|E|, |E|)$$

$$(\widetilde{F, \alpha})_{E,E} \searrow \qquad \downarrow \Phi$$

$$W\text{-}\mathbf{Mod}(E_W, E_W) \cong W$$

must commute. In other words, up to trivial isomorphisms, Φ is neccessarily given by $(\widetilde{F, \alpha})_{E,E}$.

  Conversely, for proving its existence, let us define $\Phi \colon V \longrightarrow W$ by $\Phi(v) = (\widetilde{F, \alpha})_{E,E}(v)$ for all $v \in V$ (thus ignoring trivial bijections). Then Φ is certainly a morphism of quantales (see Section 5). Furthermore, denoting the underlying **Set**-functors of $V\text{-}\mathbf{Cat}$ and by U_V and U_W, respectively, from the representability of (F, α) we obtain a natural isomorphism

$$U_W F \xrightarrow{\sim} \mathscr{K}(E, -) = U_V|-| = U_W \Phi|-|,$$

with $|-| \colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ and $\Phi \colon V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}$ as in 3.1 and Section 5. We must now lift this **Set**-based isomorphism to a $W\text{-}\mathbf{Cat}$-based isomorphism $\theta \colon F \xrightarrow{\sim} \Phi|-|$. For ease of computation, we may without loss of generality assume that the **Set**-based isomorphism is actually an identity; hence, FX has underlying set $\mathscr{K}(E, X)$, for all objects X in $\mathscr{K}$, and we must show that FX and $\Phi|X|$ have the same W-category structure. But for all $x, y \in |X| = \mathscr{K}(E, X)$ one has with 6.3(4):

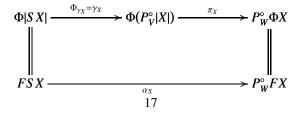$$(\Phi|X|)(x, y) = \Phi(|X|(x, y))$$
$$= \Phi(y^* \circ x_*)$$
$$= (\widetilde{F, \alpha})(y^*) \circ (\widetilde{F, \alpha})(x_*)$$
$$= (Fy)^* \circ (Fx)_*$$
$$= (FX)(x, y).$$

Finally, in order to confirm θ as a 2-cell $(F, \alpha) \xrightarrow{\sim} \Gamma\Phi\cdot(|-|, \gamma)$, under the assumption $\theta = 1$ we must show that the diagram

$$\Phi|SX| \xrightarrow{\Phi_{\gamma_X} = \gamma_X} \Phi(P_V^\circ|X|) \xrightarrow{\pi_X} P_W^\circ \Phi X$$
$$\| \qquad\qquad\qquad\qquad\qquad \|$$
$$FSX \xrightarrow{\qquad\qquad \alpha_X \qquad\qquad} P_W^\circ FX$$

commutes, for all objects X. To this end, let us first observe that α_X may be considered a W-module $FX \longrightarrow FSX$, and as such is represented as

$$\alpha_X = (\widetilde{F, \alpha})(\iota_X),$$

with $\iota_X \colon X \longrightarrow SX$ in S-**Mod** (see Section 2). Now, for all $\varphi \in |SX|$ and $x \in X$, we obtain:

$$
\begin{aligned}
\pi_X(\gamma_X(\varphi))(x) &= \Phi(\gamma_X(\varphi)(x)) \\
&= \Phi(\varphi \circ x_*) \\
&= \Phi(\varphi^* \circ \iota_X \circ x_*) \\
&= (\widetilde{F, \alpha})(\varphi^* \circ \iota_X \circ x_*) \\
&= (\widetilde{F, \alpha})(\varphi^*) \circ (\widetilde{F, \alpha})(\iota_X) \circ (\widetilde{F, \alpha})(x_*) \\
&= (F\varphi)^* \circ \alpha_X \circ (Fx)_* \\
&= \alpha_X(\varphi)(x).
\end{aligned}
$$

$\square$

Remark 6.5. While **MODMON** carries the structure of a 2-category (see 6.2(2)), the full extent of this structure is of limited interest for our purposes, since Δ maps every 2-cell to an identity morphism: see 6.3(2). Likewise, the natural 2-categorical structure of **CQuant**, given by ordering its hom-sets pointwise, is not helpful for our purposes: if $\Phi, \Psi \colon V \longrightarrow W$ are morphisms of commutative quantales with $\Phi(v) \le \Psi(v)$ for all $v \in V$, the natural transformation $\theta \colon \Phi \longrightarrow \Psi \colon V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}$ with $\theta_X = 1_X$ (at the **Set**-level) will in general *not* give a 2-cell $\Gamma\Phi \longrightarrow \Gamma\Psi$.

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