KLEISLI OPERATIONS FOR TOPOLOGICAL SPACES

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Abstract. The axioms for a topology in terms of open sets follow necessarily from the intuitive relation of this concept with ultrafilter convergence. By contrast, the intuitive relations between neighbourhood systems or closure operations on the one hand and ultrafilter convergence on the other lead only to pretopologies. Kleisli operations, previously used in categorical algebra, greatly facilitate categorical descriptions of topological spaces, both in terms of neighbourhood systems and (ultra)filter convergence relations.

1. Introduction and main results

The development of the notion of topological space was intimately linked to the need of describing convergence in exact and sufficiently general terms. The first thesis of this paper is that the topology axioms for open sets (closure under finite intersection and arbitrary union) follow *necessarily* from the usual intuitive notion of convergence of ultrafilters.

More specifically, for a set X, let us on the one hand consider subsets $\tau \subseteq PX$ of the power set of X, without imposing any a-priori conditions on τ, but still thinking of its elements as of "open sets" of X. On the other hand we consider relations $a \subseteq UX \times X$ from the set UX of ultrafilters on X to X, again without any further condition, but thinking of $(\mathfrak{x}, x) \in a$ as of "$\mathfrak{x}$ converges to x" and therefore writing $\mathfrak{x} \xrightarrow{a} x$ instead. Given τ, it would then be natural to define $a = \psi(\tau)$ by

$$(1) \qquad \mathfrak{x} \xrightarrow{a} x \quad \iff \quad \forall\, A \in \tau\; (x \in A \Longrightarrow A \in \mathfrak{x})$$

2000 *Mathematics Subject Classification.* Primary Secondary.

Key words and phrases. (Ultra)filter, neighbourhood system, Kleisli operation, co-Kleisli operation, monad, lax algebra.

Partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT and NSERC is gratefully acknowledged.

("$\mathfrak{x}$ converges to x iff every open neighbourhood A of x lies in $\mathfrak{x}$"). Conversely, given a, one would naturally define $\tau = \varphi(a)$ by

$$A \in \tau \iff \forall \mathfrak{x} \xrightarrow{a} x \; (x \in A \Longrightarrow A \in \mathfrak{x}) \tag{2}$$

("A is open in X iff every ultrafilter converging to a point of A is actually an ultrafilter on A"). It is easy to see that ψ and φ are order-reversing maps (w.r.t. "$\subseteq$")

$$PPX \underset{\varphi}{\overset{\psi}{\rightleftarrows}} P(UX \times X) \tag{3}$$

which, in fact, constitute a Galois correspondence:

$$\tau \subseteq \varphi(\psi(\tau), \qquad a \subseteq \psi(\varphi(a))$$

for all $\tau \subseteq PX$, $a \subseteq UX \times X$. We prove in Section 2:

Theorem A. *The subsets $\tau \subseteq PX$ closed under the Galois correspondence* (3) *are exactly the topologies on X.*

In order to describe the relations $a \subseteq UX \times X$ closed under (3) most elegantly, in Section 2 we recall from [5] the *co-Kleisli composition $a * b$* for such structures, which is associative and has a right neutral element e_X^* (where $\mathfrak{x} \xrightarrow{e_X^*} x$ means that $\mathfrak{x}$ is the principal ultrafilter over x). Then topologies correspond bijectively to convergence structures satisfying a simple reflexivity/extensitivity and transitivity/idempotency condition:

Theorem B. *The relations $a \subseteq UX \times X$ closed under the Galois correspondence* (3) *are those satisfying*

$$e_X^* \subseteq a \quad \text{and} \quad a * a \subseteq a.$$

For finite X these conditions describe just reflexive and transitive relations on X, leading to the identification of topologies on X with preorders. For general X, these conditions are equivalent to those used by Barr [1] in order to represent topological spaces as lax algebras with respect to the ultrafilter monad, as we explain below (see Section 4). They have their roots in the iterated limit conditions first used by Kowalsky [8] and Kelley [7], which are nicely presented in [13]. The proof of Theorem B given in Section 2 is, however, quite different from the ones given by those authors.

To some extent the correspondence (3) is presented more easily if we, like Hausdorff [4] did, describe topologies in terms of neighbourhood systems. Hence, for every function $v : X \longrightarrow FX$ of a set X into the set of (proper) filters on X, let us define $a \subseteq UX \times X$ by

$$\mathfrak{x} \xrightarrow{a} x \iff v(x) \subseteq \mathfrak{x}. \tag{4}$$

Conversely, given a, define v by

$$A \in v(x) \iff \forall \mathfrak{x} \xrightarrow{a} x : A \in \mathfrak{x}. \tag{5}$$

These settings define a Galois correspondence

$$(FX)^X \underset{\chi}{\overset{\theta}{\rightleftarrows}} P(UX \times X) \tag{6}$$

where the set $(FX)^X$ of filter-valued functions on X is ordered pointwise by inclusion. Obviously, the correspondence (6) may be restricted to one between *pretopologies* v (satisfying $v \subseteq e_X$) and *pseudotopologies* a (satisfying $e_X^* \subseteq a$). In Section 3, we introduce a *Kleisli composition* $v * w$ for filter-valued functions (which is associative and has a neutral element $e_X : X \longrightarrow FX$) and prove the now somewhat surprising:

Theorem C. *All functions $v : X \longrightarrow FX$ are closed under the correspondence* (6) *while the relations $a \subseteq UX \times X$ closed under the correspondence* (6) *are those satisfying $e_X^* * a = a$. This equivalence can be restricted to pretopologies on X and those pseudotopologies a satisfying $e_X^* * a = a$, and further to the topological neighbourhood systems characterized by the conditions $v \subseteq e_X$ and $v \subseteq v * v$ and the relations $a \subseteq UX \times X$ described in Theorem B.*

For the sake of completeness, we also mention here the (covariant!) correspondence

$$(PX)^{PX} \underset{\lambda}{\overset{\kappa}{\rightleftarrows}} P(UX \times X) \tag{7}$$

described in [5], which assigns to any map $c : PX \longrightarrow PX$ (being thought of as a closure operation) the relation $a \subseteq UX \times X$ defined by

$$\mathfrak{x} \xrightarrow{a} x \iff \forall A \in \mathfrak{x} : x \in c(A); \tag{8}$$

conversely, given any $a \subseteq UX \times X$, one defines c by

$$x \in c(A) \iff \exists \mathfrak{x} \in UX : A \in \mathfrak{x} \text{ and } \mathfrak{x} \xrightarrow{a} x. \tag{9}$$

The order-preserving maps κ, λ satisfy

$$\lambda(\kappa(c)) \subseteq c, \qquad a \subseteq \kappa(\lambda(a))$$

for all $c \in (PX)^{PX}$ (ordered pointwise by inclusion) and $a \in P(UX \times X)$. Obviously, the correspondence (7) may be restricted to one between *extensive* functions c (satisfying $A \subseteq c(A)$ for all $A \subseteq X$) and pseudotopologies a (satisfying $e_X^* \subseteq a$).

Theorem D ([5])**.** *A function $c : PX \longrightarrow PX$ is closed under the correspondence* (7) *if and only if it is additive, and a relation $a \subseteq UX \times X$ is closed under* (7) *if and only if it satisfies $e_X^* * a = a$. Hence, when restricted to extensive functions c and pseudotopologies a, the fixed elements under* (7) *are, as for* (6)*, precisely the pretopologies on X. The mappings κ and λ then become homomorphisms with respect to ordinary composition of closure operations and the co-Kleisli operation for ultrafilter convergence structures.*

Theorems C and D make Theorems A and B even more surprising. Although all three correspondences (3), (6), (7) arise from the same natural intuition of convergence vis-a-vis the three standard descriptions of topological spaces, in terms of open sets, neighbourhoods, and closure operations, only the first one yields precisely the topologies as the fixed structures, whereas the others lead to structures encompassing even the much larger class of pretopologies (for which the closure operation is not required to be idempotent, i.e. to Čech closure operations).

We finally need to summarize the categorical context and implications of this work. The functor $U : \mathbf{Set} \longrightarrow \mathbf{Set}$ carries the structure of a *monad*, i.e. one has natural transformations $e : \mathrm{Id} \longrightarrow U$ and $m : UU \longrightarrow U$ satisfying $m(eU) = 1 = m(Ue)$ and $m(mU) = m(Um)$ (see [11]), the (strict) Eilenberg–Moore algebras of which had been identified as the compact Hausdorff spaces by Manes [10]. U allows for a natural extension to a functor $U : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ of the category of sets, with relations as morphisms, maintaining the status of m as a natural transformation but making the unit e only a so-called op-lax transformation. Still, for this lax monad $\mathbb{U}$ of $\mathbf{Rel}$, one may form the category $\mathbf{Alg}\,\mathbb{U}$ of lax $\mathbb{U}$-algebras. Theorems A and B are the backbone of the category equivalence $\mathbf{Alg}\,\mathbb{U} \simeq \mathbf{Top}$ first established by Barr [1] (see also [12]). The co-Kleisli operation is simply the Kleisli operation for the lax comonad $\mathbb{U}^*$ of the self-dual category $\mathbf{Rel}$, with $*$ referring to this self-duality. The question remains what the categorical meaning of Theorem C is. Although the maps $v : X \longrightarrow FX$ look more like coalgebra structures, we show in Section 4 how one can naturally regard them as algebra structures, for a suitable extension of the filter monad $\mathbb{F}$ from $\mathbf{Set}$ to (what we call) the *Lawvere category* $\mathbf{Law}$ of preordered sets, with ordered relations as morphisms [9].

In the same context, $\mathbb{F}$ can be used to describe $\mathbf{Top}$ categorically and constructively (without recourse to the Axiom of Choice) in terms of filter convergence, rather than ultrafilter convergence.

Categories of Eilenberg-Moore algebras with respect to monads of **Set** describe precisely the varieties of general algebras admitting free algebras (with no restriction on the arities or number of operations), with Kleisli operations playing a fundamental role in describing the syntax. Hence, our observations show that lax versions of these tools are perfectly suitable for general topology.

2. Open sets versus ultrafilter convergence

2.1. **Notation.** For a relation $r \subseteq X \times Y$ from a set X to a set Y we also write $r : X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$; often we consider r as a function $X \longrightarrow PY$, hence $r(x) = \{y \in Y \mid xry\}$ (writing xry instead of $(x, y) \in r$) for $x \in X$ and $r(A) = \bigcup_{x \in A} r(x)$ for $A \subseteq X$. The converse of r is denoted by $r^* : Y \longrightarrow\!\!\!\!\!\!\!\mapsto X$, and for $s : Y \longrightarrow\!\!\!\!\!\!\!\mapsto Z$ one has the composite $sr : X \longrightarrow\!\!\!\!\!\!\!\mapsto Z$ defined as usual by $(x(sr)z \iff \exists\, y : xry \text{ and } ysz)$.

Recall that a *filter* $\mathfrak{a}$ on X is a subset $\mathfrak{a} \subseteq PX$ which (w.r.t. "$\subseteq$") is upwards closed and down-directed (so that finite subsets of $\mathfrak{a}$ have lower bounds in $\mathfrak{a}$); we also assume $\mathfrak{a}$ to be *proper* so that $\mathfrak{a} \neq PX$. More explicitly then, $\mathfrak{a}$ satisfies: $(A \in \mathfrak{a},\ A \subseteq B \subseteq X \implies B \in \mathfrak{a})$, $(A, B \in \mathfrak{a} \implies A \cap B \in \mathfrak{a})$, $X \in \mathfrak{a}$, $\emptyset \notin \mathfrak{a}$. Maximal filters (w.r.t. "$\subseteq$") are called *ultrafilters*. They are characterized by the additional property $(A \cup B \in \mathfrak{a} \implies A \in \mathfrak{a} \text{ or } B \in \mathfrak{a})$. We denote the set of all filters on X by FX, while UX is the set of all ultrafilters on X. For $x \in X$, the *principal filter* on X over x is denoted by $e_X(x) = \dot{x}$, i.e.

$$A \in e_X(x) \iff x \in A.$$

For $\mathfrak{A} \in FFX$, the *Kowalsky sum* $m_X(\mathfrak{A}) \in FX$ of $\mathfrak{A}$ is defined by

$$A \in m_X(\mathfrak{A}) \iff A^{\#} \in \mathfrak{A},$$

with $A^{\#}$ denoting the set of those filters on X inducing filters on A, i.e.

$$\mathfrak{a} \in A^{\#} \iff A \in \mathfrak{a}.$$

The maps $e_X : X \longrightarrow FX$ and $m_X : FFX \longrightarrow FX$ restrict to maps $e_X : X \longrightarrow UX$ and $m_X : UUX \longrightarrow UX$ if we replace filters by ultrafilters everywhere.

The lattice-theoretical notion dual to filter is *ideal*. We frequently use the well-known:

2.2. **Extension Lemma.** *For a filter $\mathfrak{a}$ and an ideal $\mathfrak{j}$ on X with $\mathfrak{a} \cap \mathfrak{j} = \emptyset$, there is an ultrafilter $\mathfrak{x} \supseteq \mathfrak{a}$ on X with $\mathfrak{x} \cap \mathfrak{j} = \emptyset$.*

Proof. A standard application of Zorn's Lemma produces a filter which is maximal amongst all filters $\mathfrak{x}$ on X satisfying $\mathfrak{x} \supseteq \mathfrak{a}$ and $\mathfrak{x} \cap \mathfrak{j} = \emptyset$. Such a filter turns out to be an ultrafilter. □

2.3. **Corollary.** *For any relation $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$, a filter $\mathfrak{a}$ on X and an ultrafilter $\mathfrak{y}$ on Y with $r[\mathfrak{a}] := \{r(A) \mid A \in \mathfrak{a}\} \subseteq \mathfrak{y}$, there is an ultrafilter $\mathfrak{x}$ on X with $\mathfrak{a} \subseteq \mathfrak{x}$ and $r[\mathfrak{x}] \subseteq \mathfrak{y}$.*

Proof. Apply 2.2 to the ideal $\mathfrak{j} := \{A \subseteq X \mid r(A) \notin \mathfrak{y}\}$. □

2.4. **The correspondence.** Definitions (1), (2) of the Introduction for the correspondence

$$PPX \underset{\varphi}{\overset{\psi}{\rightleftarrows}} P(UX \times X)$$

may be written as

$$\mathfrak{x} \xrightarrow{\psi(\tau)} x \iff \tau(x) \subseteq \mathfrak{x},$$
$$A \in \varphi(a) \iff a^*(A) \subseteq A^\#,$$

where $\tau(x) := \{A \in \tau \mid x \in A\}$ and $A^\# := \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$. One has

$$a^*\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} a^*(A_i), \qquad \bigcup_{i\in I} A_i^\# \subseteq \big(\bigcup_{i\in I} A_i\big)^\#,$$
$$a^*\big(\bigcap_{i\in I} A_i\big) \subseteq \bigcap_{i\in I} a^*(A_i), \qquad \bigcap_{i\in I} A_i^\# = \big(\bigcap_{i\in I} A_i\big)^\#,$$

with the last identity requiring finiteness of I. Considering $\tau = \varphi(a)$ we obtain easily:

2.5. **Corollary.** *Subsets $\tau \subseteq PX$ closed under the Galois correspondence are topologies (of open sets) on X.* □

2.6. **Proof of Theorem A.** It remains to be shown that a topology τ on X is closed under the Galois correspondence 2.4, i.e. $\varphi(\psi(\tau)) \subseteq \tau$. Consider $A \in \varphi(a)$ with $a := \psi(\tau)$; it suffices to show that for every $x \in A$ there is $B \in \tau(x)$ with $B \subseteq A$. Assuming the opposite we would have, for some $x \in A$, $\tau(x) \cap PA = \emptyset$, so that 2.2 would give an ultrafilter $\mathfrak{x} \supseteq \tau(x)$ with $A \notin \mathfrak{x}$. Hence $\mathfrak{x} \xrightarrow{a} x$ which, with $A \in \varphi(a)$, would imply $A \in \mathfrak{x}$, a contradiction. □

2.7. **Co-Kleisli composition.** Every relation $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ gives a relation $Ur : UX \longrightarrow\!\!\!\!\!\!\mapsto UY$ defined by

$$\mathfrak{x}(Ur)\mathfrak{y} \quad :\iff \quad r^*[\mathfrak{y}] \subseteq \mathfrak{x} \quad \iff \quad r[\mathfrak{x}] \subseteq \mathfrak{y}.$$

In particular, any relation $a : UX \longrightarrow\!\!\!\!\!\!\mapsto X$ induces a relation $Ua : UUX \longrightarrow\!\!\!\!\!\!\mapsto UX$. Writing $\mathfrak{x} \xrightarrow{a} x$ for $\mathfrak{x}ax$, we usually write $\mathfrak{X} \xrightarrow{a} \mathfrak{x}$ instead of $\mathfrak{X}(Ua)\mathfrak{x}$. For $a : UX \longrightarrow\!\!\!\!\!\!\mapsto X$, $b : UX \longrightarrow\!\!\!\!\!\!\mapsto X$, the operation

$$a * b := a(Ub)m_X^* : UX \longrightarrow\!\!\!\!\!\!\mapsto X$$

is described by
(∗)
$$\mathfrak{x} \xrightarrow{a*b} x \iff \exists\, \mathfrak{y} \in UX, \mathfrak{X} \in UUX : m_X(\mathfrak{X}) = \mathfrak{x},\ \mathfrak{X} \xrightarrow{b} \mathfrak{y},\ \mathfrak{y} \xrightarrow{a} x.$$

We note that the operation is order-preserving in each variable, associative, and satisfies

$$a * e_X^* = a \quad \text{and} \quad a \subseteq e_X^* * a,$$

i.e. e_X^* is a strict left and lax right unit for the operation. More importantly for our purposes, ψ and φ become *lax homomorphisms*, as follows:

2.8. **Proposition.** *For $\tau, \sigma \subseteq PX$ and $a, b : UX \longrightarrow\!\!\!\!\!\!\!\mapsto X$ one has*

(1) $\psi(\tau) * \psi(\sigma) \subseteq \psi(\tau \cap \sigma), \quad \psi(PX) = e_X^*.$
(2) $\varphi(a) \cap \varphi(b) \subseteq \varphi(a * b), \quad \varphi(e_X^*) = PX.$

Proof. (1) Putting $a = \psi(\tau)$, $b = \psi(\sigma)$, for $\mathfrak{x} \xrightarrow{a*b} x$ we have the right-hand side of (∗) and must show $(\tau \cap \sigma)(x) \subseteq \mathfrak{x}$. But for $A \in \tau \cap \sigma$ with $x \in A$ one has $A \in \mathfrak{y}$ since $\mathfrak{y} \xrightarrow{a} x$, and then $b^*(A) \in \mathfrak{X}$ since $\mathfrak{X} \xrightarrow{b} \mathfrak{y}$. This implies $A^{\#} \in \mathfrak{X}$ and then $A \in \mathfrak{x} = m_X(\mathfrak{X})$ since from $A \in \sigma \subseteq \varphi(\psi(\sigma))$ one knows $b^*(A) \subseteq A^{\#}$. The identity $\psi(PX) = e_X^*$ is obvious. (2) follows similarly. □

2.9. **Corollary.** *Relations $a : UX \longrightarrow\!\!\!\!\!\!\!\mapsto X$ closed under the Galois correspondence* 2.4 *satisfy*

$$e_X^* \subseteq a \quad \textit{and} \quad a * a \subseteq a.$$

Proof. The inclusions follow with 2.8(1) from $\tau \subseteq PX$ and $\tau \cap \tau = \tau$. □

We are aiming at the converse proposition of 2.9. It is convenient to consider the *Zariski topology* on UX with respect to which $\mathcal{A} \subseteq UX$ is closed if any $\mathfrak{x} \in UX$ with $\bigcap \mathcal{A} \subseteq \mathfrak{x}$ lies in $\mathcal{A}$. Note that $\bigcap \mathcal{A} \subseteq \mathfrak{x}$ is equivalent to $\mathfrak{x} \subseteq \bigcup \mathcal{A}$. The relations $a : UX \longrightarrow\!\!\!\!\!\!\!\mapsto X$ for which e_X^* is left-neutral with respect to the co-Kleisli operation are now easily characterized:

2.10. **Lemma** [5]**.** *For any $a : UX \longrightarrow\!\!\!\!\!\!\!\mapsto X$, $e_X^* * a = a$ holds if and only if $a^*(x)$ is Zariski-closed for every $x \in X$.*

Proof. "*if*" We must show $e_X^* * a \subseteq a$. Now, $\mathfrak{x} \xrightarrow{e_X^* * a} x$ means $\mathfrak{X} \xrightarrow{a} e_X(x)$ for some $\mathfrak{X} \in UUX$ with $m_X(\mathfrak{X}) = \mathfrak{x}$. To conclude $\mathfrak{x} \xrightarrow{a} x$, since $a^*(x)$ is Zariski-closed, it suffices to show $\bigcap a^*(x) \subseteq \mathfrak{x}$. Hence, consider

$A \subseteq X$ with $A \in \mathfrak{y}$ whenever $\mathfrak{y} \xrightarrow{a} x$, hence $a^*(x) \subseteq \{\mathfrak{y} \in UX \mid A \in \mathfrak{y}\} = A^\#$. Since $\mathfrak{X} \xrightarrow{a} e_X(x)$, we have $a^*(x) \in \mathfrak{X}$ and therefore $A^\# \in \mathfrak{X}$, which means $A \in \mathfrak{x} = m_X(\mathfrak{X})$.

"*only if*" Let $\mathfrak{x} \subseteq \bigcup a^*(x)$. We need to show $\mathfrak{x} \xrightarrow{a} x$, and for that it suffices to confirm $\mathfrak{x} \xrightarrow{e_X^* * a} x$. Each $A \in \mathfrak{x}$ belongs to some $\mathfrak{y} \in a^*(x)$. Therefore $\{A^\# \mid A \in \mathfrak{x}\} \cup \{a^*(x)\}$ is a filter base on UX which, by 2.3, can be extended to an ultrafilter $\mathfrak{X} \in UUX$. It follows $\mathfrak{X} \xrightarrow{a} \dot{x}$ and $m_X(\mathfrak{X}) = \mathfrak{x}$, hence $\mathfrak{x} \xrightarrow{e_X^* * a} x$. □

2.11. **Proof of Theorem B.** Let $a : UX \to\!\!\!\!\!\to X$ satisfy $e_X^* \subseteq a$ and $a * a \subseteq a$, hence $e_X^* * a \subseteq a * a \subseteq a$ and therefore $e_X^* * a = a$, i.e. $a^*(x)$ is Zariski-closed by 2.10. With $\tau := \varphi(a)$, we must show $\psi(\tau) \subseteq a$. Let $\mathfrak{x} \xrightarrow{\psi(\tau)} x$, that is $\tau(x) \subseteq \mathfrak{x}$, where $(A \in \tau \iff a^*(A) \subseteq A^\#)$. In order to derive $\mathfrak{x} \xrightarrow{a} x$ it suffices to show $\bigcap a^*(x) \subseteq \mathfrak{x}$ since $a^*(x)$ is Zariski-closed, and for that it suffices to show

$$\bigcap a^*(x) \subseteq \uparrow \tau(x) := \{A \subseteq X \mid \exists\, B \in \tau(x) : B \subseteq A\}.$$

Hence, let $A \in \bigcap a^*(x)$ and consider

$$B := \{y \in X \mid a^*(y) \subseteq A^\#\}.$$

Then $x \in B$, and $B \subseteq A$ since $e_X^* \subseteq a$. Finally, to have $B \in \tau$ we must show $a^*(B) \subseteq B^\#$. Suppose $\mathfrak{y} \xrightarrow{a} y$ with $\mathfrak{y} \notin B^\#$, i.e. $B \notin \mathfrak{y}$. Hence $C \cap (X \setminus B) \neq \emptyset$ for all $C \in \mathfrak{y}$, so that there is $\mathfrak{z} \xrightarrow{a} z \in C$ with $A \notin \mathfrak{z}$. Hence $\{(X \setminus A)^\#\} \cup \{a^*(C) \mid C \in \mathfrak{y}\}$ is a filter base on UX. Now 2.3 gives $\mathfrak{X} \in UUX$ with $(X \setminus A)^\# \in \mathfrak{X}$ and $\mathfrak{X} \xrightarrow{a} \mathfrak{y}$. With $\mathfrak{y} \xrightarrow{a} y$ and $a * a \subseteq a$ this implies $m_X(\mathfrak{X}) \xrightarrow{a} y$. But since $A^\# \notin \mathfrak{X}$ we have $A \notin m_X(\mathfrak{X})$, hence $y \notin B$, as desired. □

2.12. **Remarks.**

(1) In 2.11, we have in fact $\bigcap a^*(x) = \uparrow \tau(x)$, and B is the τ-interior of A.

(2) Note that the condition $e_X^* \subseteq a$ describes pseudotopological (or Choquet [2]) spaces in terms of ultrafilter convergence, and if one adds to this the condition $e_X^* * a \subseteq a$ one obtains precisely pretopological spaces (see Theorems C and D of Section 1 and [6]).

3. Neighbourhood systems versus ultrafilter convergence

3.1. Kleisli composition. On the set

$$(FX)^X = \{v \mid v : X \longrightarrow FX\}$$

of filter-valued functions of a set X we introduce the operation

$$v * w := m_X(Fv)w : X \longrightarrow FX,$$

with $m_X : FFX \longrightarrow FX$ as in 2.1 (not to be confused with $m_X : UUX \longrightarrow UX$ as used in 2.7), and with $Fv : FX \longrightarrow FFX$ the usual functorial extension of F, so that

$$\mathcal{A} \in Fv(\mathfrak{x}) \iff \exists\, B \in \mathfrak{x} : v(B) \subseteq \mathcal{A}.$$

Hence, the elementwise description of $v * w$ is

$$\begin{aligned} A \in (v * w)(x) &\iff A^{\#} \in Fv(w(x)) \\ &\iff \exists\, B \in w(x) : v(B) \subseteq A^{\#} \\ &\iff \exists\, B \in w(x)\, \forall\, y \in B : A \in v(y) \\ &\iff \{y \in X \mid A \in v(y)\} \in w(x). \end{aligned}$$

With $(FX)^X$ ordered pointwise by inclusion, we obtain an operation that is order-preserving in each variable and associative and that satisfies

$$e_X * v = v \quad \text{and} \quad v * e_X = v,$$

i.e. that makes $(FX)^X$ a monoid.

From the calculation above we see immediately:

3.2. Proposition. *The neighbourhood systems describing topologies on a set X are exactly the functions $v : X \longrightarrow FX$ satisfying*

$$v \subseteq e_X \quad \textit{and} \quad v \subseteq v * v.$$

□

Let us now turn to the correspondence (6) and establish the counterpart of 2.8, exhibiting θ and χ as *lax* and *strict antihomomorphisms* respectively, under $*$:

3.3. Proposition. *For $v, w : X \longrightarrow FX$ and $a, b : UX \longrightarrow\!\!\!\!\!\!\rightarrow X$ one has*

(1) $\theta(v) * \theta(w) \subseteq \theta(w * v), \quad \theta(e_X) = e_X^*$.
(2) $\chi(a) * \chi(b) = \chi(b * a), \quad \chi(e_X^*) = e_X$.

Proof. (1) With $a = \theta(v)$, $b = \theta(w)$, assume $\mathfrak{x} \xrightarrow{a*b} x$, so that $\mathfrak{X} \xrightarrow{b} \mathfrak{y} \xrightarrow{a} x$ for some $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$ with $m_X(\mathfrak{X}) = \mathfrak{x}$, hence $v(x) \subseteq \mathfrak{y}$ and $b^*(B) \in \mathfrak{X}$ for all $B \in \mathfrak{y}$. We must show $(w * v)(x) \subseteq \mathfrak{x}$. Indeed, for every $A \in (w * v)(x)$ one has $B \in v(x)$ with $w(B) \subseteq A^{\#}$, which implies $b^*(B) \subseteq A^{\#} \in \mathfrak{X}$ and therefore $A \in m_X(\mathfrak{X}) = \mathfrak{x}$. Trivially, $\theta(e_X) = e_X^*$ and $\chi(e_X^*) = e_X$.

(2) The proof of "$\subseteq$" of the first identity is similar to (1). To see $\chi(b * a) \subseteq v * w$ with $v = \chi(a)$ and $w = \chi(b)$, assume $A \notin v * w(x)$. We conclude that $B := \{y \in X \mid A \in v(y)\} \notin w(x)$, that is: there is some $\mathfrak{x} \xrightarrow{b} x$ with $X \setminus B \in \mathfrak{x}$. For each $y \in X \setminus B$ there exists $\mathfrak{y} \xrightarrow{a} y$ such that $A \notin \mathfrak{y}$. Hence $\{(X \setminus A)^{\#}\} \cup \{a^*(C) \mid C \in \mathfrak{x}\}$ is a filter base, and from 2.3 we conclude the existence of $\mathfrak{X} \in UUX$ with $(X \setminus A)^{\#} \in \mathfrak{X}$ and $\mathfrak{X} \xrightarrow{a} \mathfrak{x}$. Therefore $m_X(\mathfrak{X}) \xrightarrow{b*a} x$ but $A \notin m_X(\mathfrak{X})$, which implies $A \notin \chi(b * a)(x)$. □

3.4. **Proof of Theorem C.** For any function $v : X \longrightarrow FX$, one has

$$\chi(\theta(v))(x) = \bigcap\{\mathfrak{a} \in UX \mid v(x) \subseteq \mathfrak{a}\} = v(x).$$

On the other hand, given a relation $a : UX \longrightarrow\!\!\!\!\!\!\mapsto X$, we have

$$\mathfrak{x} \in \theta(\chi(a))(x) \iff \bigcap a^*(x) \subseteq \mathfrak{x}.$$

Therefore $\theta(\chi(a))(x) = a(x)$ if and only if $a^*(x)$ is Zariski closed in UX, hence the characterization given by the Theorem follows from 2.10.

The second part of the statement follows immediately from Propositions 3.3 and 3.2. □

Finally, for the sake of completeness let us prove that the composition of the correspondences (3) and (6) yields the usual correspondence between topologies and the neighbourhood systems describing them.

3.5. **Proposition.**

(1) *For any* $v : X \longrightarrow FX$ *and* $\tau = \varphi\theta(v)$ *one has*

$$A \in \tau \iff \forall x \in A \, \exists B \in v(x) : B \subseteq A.$$

(2) *For a topology* $\tau \subseteq PX$ *and* $v = \chi\psi(\tau)$ *one has*

$$A \in v(x) \iff \exists B \in \tau : x \in B \subseteq A.$$

Proof. (1) $A \in \tau$ means by definition $A \in \mathfrak{x}$ whenever $v(x) \subseteq \mathfrak{x}$ with $x \in A$. Hence "$\Longleftarrow$" is trivial. Conversely, consider $x \in A$ and assume $B \not\subseteq A$ for all $B \in v(x)$. Then we can choose $\mathfrak{x} \in UX$ with $v(x) \subseteq \mathfrak{x}$, $X \setminus A \in \mathfrak{x}$. But $A \in \mathfrak{x}$ by hypothesis, a contradiction.

(2) For any $\tau \subseteq PX$ and $v = \chi\psi(\tau)$, $A \in v(x)$ means by definition $A \in \mathfrak{x}$ whenever $\tau(x) \subseteq \mathfrak{x}$. Again, "$\Longleftarrow$" is trivial, and for "$\Longrightarrow$"

suppose $B \not\subseteq A$ for all $B \in \tau(x)$. Then, if τ is a topology and therefore $\tau(x)$ a filterbase, we can find $\mathfrak{x} \in UX$ with $\tau(x) \subseteq \mathfrak{x}$, $X \setminus A \in \mathfrak{x}$, leading to a contradiction as in (1). □

4. Topological spaces as lax Eilenberg-Moore algebras

4.1. **Barr's presentation.** The ultrafilter monad $\mathbb{U} = (U, e, m)$ of **Set** (as defined in 2.1) allows for an extension to the category **Rel** of sets with relations as morphisms, as given in 2.7. U remains a functor and $m : UU \longrightarrow U$ a natural transformation, but $e : \mathrm{Id}_{\mathbf{Rel}} \longrightarrow U$ is only op-lax, that is: for $r : X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$ in **Rel**, in general the diagram

(10)
$$\begin{array}{ccc} X & \xrightarrow{e_X} & UX \\ {\scriptstyle r}\big\downarrow & \subseteq & \big\downarrow{\scriptstyle Ur} \\ Y & \xrightarrow[e_Y]{} & UY \end{array}$$

commutes only laxly, not strictly. $\mathbb{U}$-algebras (X, a) over **Rel** are defined by the lax commutativity conditions

(11)
$$\begin{array}{ccc} X & \xrightarrow{e_X} & UX \\ & {\scriptstyle 1_X}\searrow \;\subseteq & \big\downarrow{\scriptstyle a} \\ & & X \end{array} \qquad \begin{array}{ccc} UUX & \xrightarrow{Ua} & UX \\ {\scriptstyle m_X}\big\downarrow & \supseteq & \big\downarrow{\scriptstyle a} \\ UX & \xrightarrow[a]{} & X \end{array}$$

which may equivalently be displayed as $e_X^* \subseteq a$, $a * a \subseteq a$; the lax homomorphisms $f : (X, a) \longrightarrow (Y, b)$ are maps $f : X \longrightarrow Y$ satisfying

(12)
$$\begin{array}{ccc} UX & \xrightarrow{Uf} & UY \\ {\scriptstyle a}\big\downarrow & \subseteq & \big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We thus have the category $\mathbf{Alg}\,\mathbb{U}$.

With φ, ψ of (3) one obtains functors

(13)
$$\begin{array}{c} \mathbf{Top} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Alg}\,\mathbb{U}, \\ (X, \tau) \longmapsto (X, \psi(\tau)) \\ (X, \varphi(a)) \longleftarrow\!\shortmid\; (X, a) \end{array}$$

in fact a category equivalence, essentially by Theorems A and B.

4.2. **Ord and Law.** In Theorem C we use the order relation of FX in terms of inclusion of filters. We therefore extend the filter monad $\mathbb{F} = (F, e, m)$ of **Set** (see 2.1) to the category **Ord** of preordered sets (sets with a reflexive and transitive relation) and order-preserving maps. For a preordered set X (with the preorder normally denoted by $\leq$), FX is the set of filters of down(wards)-closed subsets, ordered geometrically by "$\supseteq$"; hence

$$\mathfrak{x} \leq \mathfrak{y} \iff \forall B \in \mathfrak{y}\, \exists A \in \mathfrak{x} : A \subseteq B \iff \mathfrak{x} \supseteq \mathfrak{y}.$$

Of course, when X is discrete, every subset of X is down-closed, and FX has the same meaning as before. A relation $r : X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$ of preordered sets is *monotone* (or a *bimodule*) if $r \subseteq X^* \times Y$ is up(wards)-closed (where X^* denotes the object obtained from X by reversing the preorder); explicitly,

$$x' \leq x r y \leq y' \implies x' r y'$$

for all $x, x' \in X$ and $y, y' \in Y$. Denoting by **Law** (in honour of Lawvere [9]) the category of preordered sets with monotone relations as morphisms, we can now extend $\mathbb{F}$ from **Set** (and **Ord**) to **Law** by defining $Fr : FX \longrightarrow\!\!\!\!\!\!\shortmid\;\; FY$ by

$$\begin{aligned} \mathfrak{x}(Fr)\mathfrak{y} &\iff r^*[\mathfrak{y}] \subseteq \mathfrak{x} \\ &\iff \forall B \in \mathfrak{y}\, \exists A \in \mathfrak{x}\, \forall x \in A\, \exists y \in B : xry. \end{aligned}$$

F remains a functor and $m : FF \longrightarrow F$ a natural transformation, but (as for $\mathbb{U}$) $e : \mathrm{Id}_{\mathbf{Law}} \longrightarrow F$ is only op-lax. But we must be careful about how to regard e and m as monotone relations. There are in fact two natural embeddings

$$\mathbf{Ord} \underset{-^*}{\overset{-_*}{\rightrightarrows}} \mathbf{Law}.$$

Both map objects identically, and for a monotone map $f : X \longrightarrow Y$ one defines monotone relations

$$\begin{aligned} f_* &: X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y \quad \text{by} \quad x f_* y \iff f(x) \leq y, \\ f^* &: Y \longrightarrow\!\!\!\!\!\!\shortmid\;\; X \quad \text{by} \quad y f^* x \iff y \leq f(x); \end{aligned}$$

hence $-_*$ is covariant and $-^*$ contravariant. But this is not the whole story: with the pointwise order of $\mathbf{Ord}(X, Y)$ and with $\mathbf{Law}(X, Y)$ ordered by inclusion, $-_*$ gives a contravariant full embedding $\mathbf{Ord}(X, Y) \longrightarrow \mathbf{Law}(X, Y)$ and $-^*$ a covariant full embedding $\mathbf{Ord}(X, Y) \longrightarrow \mathbf{Law}(X, Y)$. Briefly, in 2-categorical language, $-_*$ is covariant on 1-cells but contravariant on 2-cells, and the converse is true for $-^*$. Consequently, we obtain a lax monad $\mathbb{F}_* = (F, e_*, m_*)$ and a lax comonad $\mathbb{F}^* = (F, e^*, m^*)$ of **Law**. In what follows, we shall however use only $\mathbb{F}_*$.

4.3. **Lax $\mathbb{F}_*$-algebras.** One defines the category $\mathbf{Alg}\,\mathbb{F}_*$ to have as objects sets X (considered as discrete preordered sets) with a monotone relation $a : FX \longrightarrow\!\!\!\!\!\!\!\!\;\;\;\;\;\mapsto X$ satisfying the conditions

$$1_X \le a(e_X)_* \quad \text{and} \quad a(Fa) \le a(m_X)_* \tag{14}$$

which, by left-adjointness of f_* to f^* in the 2-category **Law**, are equivalently expressed by

$$e_X^* \le a \quad \text{and} \quad a(Fa)m_X^* \le a. \tag{15}$$

Hence, putting $a * a = a(Fa)m_X^*$, we have the same conditions as in Theorem B, with ultrafilters replaced by filters. Morphisms $f : (X,a) \longrightarrow (Y,b)$ are mappings satisfying the continuity condition $f_*a \le bFf_*$. (Note that one has $F(f_*) = (Ff)_*$.) Hence, diagrammatically $\mathbf{Alg}\,\mathbb{F}_*$ is defined by

$$\begin{array}{ccc} X & \xrightarrow{(e_X)_*} & FX \\ & \searrow^{1_X} \;\subseteq & \downarrow a \\ & & X \end{array} \qquad \begin{array}{ccc} FFX & \xrightarrow{Fa} & FX \\ (m_X)_* \downarrow & \supseteq & \downarrow a \\ FX & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{Ff_*} & FY \\ a \downarrow & \subseteq & \downarrow b \\ X & \xrightarrow[f_*]{} & Y \end{array}$$

4.4. **Theorem.** $\mathbf{Alg}\,\mathbb{F}_*$ *is equivalent to the category* **Top**.

Proof. (15) amounts to the convergence conditions

$$e_X(x) \xrightarrow{a} x, \qquad \big(\mathfrak{A} \xrightarrow{a} \mathfrak{b} \xrightarrow{a} x \implies m_X(\mathfrak{A}) \xrightarrow{a} x\big)$$

for all $x \in X$, $\mathfrak{b} \in FX$, $\mathfrak{A} \in FFX$. Monotonicity of a amounts to

$$\mathfrak{a}' \supseteq \mathfrak{a} \xrightarrow{a} x \implies \mathfrak{a}' \xrightarrow{a} x.$$

These are precisely the conditions which describe topological spaces in terms of filter convergence (see [12], [13]). This fact may be seen also directly using the proofs given in Section 2, by following the principle that 'up-directed sets of filters behave like sets of ultrafilters'. Specifically, for ultrafilters one has $A \notin \mathfrak{x} \implies X \setminus A \in \mathfrak{x}$, whereby for filters holds $A \notin \mathfrak{a} \implies X \setminus A \in \mathfrak{b}$ for some finer filter $\mathfrak{b}$. From that we conlcude that, for a filter $\mathfrak{a}$ and $\mathcal{A} \subseteq FX$ up-directed, $\mathfrak{a} \supseteq \bigcap \mathcal{A} \iff \mathfrak{a} \subseteq \bigcup \mathcal{A}$. □

4.5. **Functional description of lax algebras.** For an object (X,a) of $\mathbf{Alg}\,\mathbb{F}_*$, one has $a = v^*$ with a mapping $v : X \to FX$. In fact, one takes $v(x) := \bigcap a^*(x)$ and obtains $a = v*$ with the filter version of 2.10. Condition (15) translates to

$$e_X \le v \quad \text{and} \quad m_X(Fv)v \le v \tag{16}$$

in **Ord**, which are precisely the conditions appearing in Theorem C. Since the continuity condition $f_* a \leq bFf_*$ translates into $(Ff)v \leq wf$ in **Ord**, we obtain:

4.6. **Corollary.** *The relational description* (15) *of Top is functional (in the sense of* (16)*).* □

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