Closure operators and their middle-interchange law

Dedicated to my friend Eraldo Giuli

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Abstract

The paper presents a new definition of closure operator which encompasses the standard Dikranjan-Giuli notion, as well as the Bourn-Gran notion of normal closure operator. As is well known, any two closure operators C, D in a category may be composed in two ways: For a subobject $M \to X$ one may consider $D_X(C_X M)$ or $D_{C_X(M)}(M)$ as the value at M of a new closure operator $D \cdot C$ or $D * C$, respectively. The two binary operations are linked by a lax middle-interchange law. This paper explores situations in which the law holds strictly.

Key words: closure operator, composite, cocomposite, idempotent closure operator, weakly hereditary closure operator, middle-interchange law.
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1 Introduction

The most important aspects of the categorical theory of closure operators were presented by Dikran Dikranjan and Eraldo Giuli in their fundamental article [DG]. The important role of closure operators in various branches of mathematics has subsequently been studied in two monographs ([DT], [Ca]) and in a large array of research articles. We mention here in particular their role in characterizing epimorphisms in many full subcategories of topological spaces and in settling the question of cowellpoweredness of such subcategories, as well as their ability to provide a topological intuition for problems of algebra, especially in torsion and radical theory. A closure operator may also provide enough structure for an abstract category to be able to regard its objects as spaces and establish a general theory of separation, compactness, and perfectness: See [CGT1], [CGT2].

The book [DT] utilizes quite systematically the two ways in which one may compose a closure operator C with an operator D, called composition $(D \cdot C)$ and cocomposition

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$(D * C)$, and it mentions *en passant* that a *lax middle-interchange law* holds for closure operators:

$$(C * D) \cdot (E * F) \leqslant (C \cdot E) * (D \cdot F) \qquad \text{(mil)}$$

(Exercise 4.A in [DT]). The primary purpose of this paper is to present situations when this law holds strictly. Roughly, we prove that when one of the four operators is idempotent and weakly hereditary and two others are comparable by order to the first operator, then (mil) holds strictly, and this fact is characteristic for the idempotency and weak heredity of the first operator.

The (pre)order of closure operators and its smooth interaction with the two binary operations make the proofs quite easy. In order to have this order it is essential that, unlike in the case of more general factorization systems, closure operators provide a factorization for *mono*morphisms only, or a suitable subtype of monomorphism. In the follow-up paper [T] we will present middle-interchange laws for general types of factorization systems (including weak factorization systems as used in Quillen model categories), the proofs of which require the replacement of inequalities by certain coherent morphisms. In this way, the proofs presented in this paper provide essential guidance for the considerably more complicated arrow-based proofs in [T].

In order to emphasize the guiding role of closure operators for more general factorization systems we actually give a new definition of closure operator in this paper which no longer makes any assumptions on the ambient category (in terms of existence of preimages or direct images), and which leads us directly to the general notion of factorization system as presented in [T]. Our notion of closure operator encompasses in particular the notion of normal closure operator as presented in [BG], [CDT].

2 Closure operators

Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathcal{K}$ which contains all isomorphisms, is closed under composition with isomorphisms, and satisfies the left-cancellation condition:

$$n \cdot m \in \mathcal{M},\ n \in \mathcal{M} \implies m \in \mathcal{M} \quad .$$

We refer to morphisms in $\mathcal{M}$ as *subobjects* and write $m \leqslant m'$ whenever $m = hm'$ for some morphism h; that morphism is uniquely determined and must lie in $\mathcal{M}$ when m and m' lie in $\mathcal{M}$.

Recall that the morphisms of $\mathcal{K}$ form the objects of the arrow category $\mathcal{K}^2$ of $\mathcal{K}$; a morphism $(u, v) : m \to n$ in $\mathcal{K}^2$ is given by a pair of morphisms which make the diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \qquad (1)$$

commute. We can now regard $\mathcal{M}$ as a full (and isomorphism-closed) subcategory of $\mathcal{K}^2$; that is, we consider only morphisms in $\mathcal{K}^2$ with m, n in the class $\mathcal{M}$.

Definition 2.1. A *closure operator C of $\mathcal{M}$ in $\mathcal{K}$* is an endofunctor $C : \mathcal{M} \to \mathcal{M}$ with $I \leqslant C$ and $\operatorname{cod} C = \operatorname{cod}$; here I denotes the identity functor of $\mathcal{M}$, and $\operatorname{cod} : \mathcal{M} \to \mathcal{K}$ is the codomain functor $(u, v) \mapsto v$.

Since $m \leqslant Cm$, the closure operator facilitates a factorization

$$m = Cm \cdot \gamma_m$$

of every subobject m, with a uniquely determined morphism γ_m; furthermore, given the morphism (1) in $\mathcal{M}$ one has the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ \downarrow{\scriptstyle \gamma_m} & & \downarrow{\scriptstyle \gamma_n} \\ C_X M & \xrightarrow[\gamma_{u,v}]{} & C_Y N \\ {\scriptstyle Cm}\downarrow & & \downarrow{\scriptstyle Cn} \\ X & \xrightarrow[v]{} & Y \end{array} \tag{2}$$

Here we have put $\gamma_{u,v} := \operatorname{dom} C(u, v)$ (with $\operatorname{dom} : \mathcal{M} \to \mathcal{K}$ the domain functor $(u, v) \mapsto u$). Note that the lower rectangle in (2) commutes since $C(u, v)$ is a morphism in $\mathcal{K}^2$, and that the upper one commutes since the outer rectangle (1) commutes and Cn is monic in $\mathcal{K}$.

Writing $C_X(m)$ instead of Cm (where $X = \operatorname{cod} m$) we see that C is given by a family of maps $C_X : \operatorname{sub} X \to \operatorname{sub} X = \{m \in \mathcal{M} \mid \operatorname{cod} m = X\}$ where

(1) $m \leqslant C_X(m)$ and

(2) $m \leqslant m' \Rightarrow C_X(m) \leqslant C_X(m')$ (consider $v = 1_X$ in (2)).

When $\mathcal{K}$ has pullbacks of subobjects, so that for all $f : X \to Y$ in $\mathcal{K}$ and $n \in \operatorname{sub} Y$ there is a pullback diagram

$$\begin{array}{ccc} f^{-1}N & \longrightarrow & N \\ {\scriptstyle f^{-1}(n)}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3}$$

with $f^{-1}(n)$ in $\operatorname{sub} X$, we may apply C to (3) in lieu of (1) and obtain

(3) $C_X(f^{-1}(n)) \leqslant f^{-1}(C_Y(n))$.

When $\mathcal{K}$ has right $\mathcal{M}$-factorizations (see [DT]), that is, when $\mathcal{M}$ is reflective in $\mathcal{K}^2$, then for all $f : X \to Y$ in $\mathcal{K}$ and $m \in \operatorname{sub} X$ there is a factorization

$$\begin{array}{ccc} M & \longrightarrow & f(M) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f(m)} \\ X & \xrightarrow{f} & Y \end{array} \tag{4}$$

we may apply C to (4) in lieu of (1) to obtain

(3') $f(C_X(m)) \leqslant C_Y(f(m))$.

Proposition 2.2. *A closure operator C of $\mathcal{M}$ in $\mathcal{K}$ may equivalently be defined by a family of maps $(C_X : \operatorname{sub} X \to \operatorname{sub} X)_{X \in \operatorname{ob} \mathcal{K}}$ satisfying conditions* (1),(2),(3) *or* (1),(2),(3'), *for all $f : X \to Y$ in $\mathcal{K}$ and $m, m' \in \operatorname{sub} X$, $n \in \operatorname{sub} Y$, provided that the required pullbacks or factorizations exist in $\mathcal{K}$, respectively.*

Proof. The necessity of the conditions follows from the above considerations, and for sufficiency see [DT], Lemma 2.4. □

The advantage of the Definition 2.1 lies in the fact that it minimizes the conditions on the category: neither the existence of inverse images nor that of direct images in the category is required. Hence, it encompasses not only the notion of closure operator as coined by Dikranjan and Giuli [DG], but also that of a *normal closure operator* [BG], [CDT].

Remark 2.3. In 2.1 we have (implicitly) introduced a closure operator C as a pointed endofunctor $\Gamma : I \to C$ with $\operatorname{cod} \Gamma = 1_{\operatorname{cod}}$, where

$$\Gamma m = (\gamma_m, 1) \quad : \quad \begin{array}{ccc} \cdot & \xrightarrow{\gamma_m} & \cdot \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Cm} \\ \cdot & \xrightarrow[1]{} & \cdot \end{array}$$

lies componentwise in the class $\mathcal{M}$, for all subobjects m in $\mathcal{M}$. But we could have equivalently introduced a closure operator as a copointed endofunctor $\Delta : \widetilde{C} \to I$ with $\operatorname{dom} \Delta = 1_{\operatorname{dom}}$, where

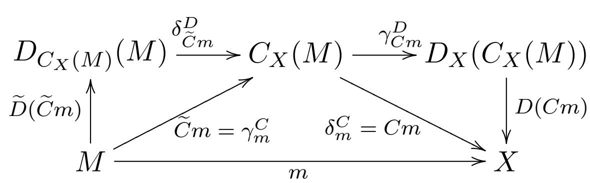
$$\Delta m = (1, \delta_m) \quad : \quad \begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle \gamma_m = }\downarrow{\scriptstyle \widetilde{C}m} & & \downarrow{\scriptstyle m} \\ \cdot & \xrightarrow[\delta_m = Cm]{} & \cdot \end{array}$$

lies componentwise in the class $\mathcal{M}$, for all subobjects m. We call $\widetilde{C}$ (with Δ) the *companion* of C and note that we may define a closure operator by just defining its companion.

Definition 2.4. For closure operators C, D the *composite* $D \cdot C$ is obtained by composing the functor C with D while the *cocomposite* $D * C$ is obtained by composing the (copointed) functor $\widetilde{C}$ with $\widetilde{D}$, so that

$$D \cdot C := DC \quad , \quad \widetilde{D * C} := \widetilde{D}\widetilde{C} \quad .$$

Explicitly, for a subobject m one has the diagram



and

$$(D \cdot C)(m) = D(Cm), \quad \gamma^{D \cdot C}_m = \gamma^D_{Cm} \cdot \gamma^C_m.$$

$$(\widetilde{D * C})(m) = \widetilde{D}(\widetilde{C}m) \ , \quad \delta_m^{D*C} = \delta_m^C \cdot \delta_{\widetilde{C}m}^D \ .$$

Of course the symmetry of composition and cocomposition is lost if one insists in presenting $D * C$ 'directly', as

$$(D * C)(m) = Cm \cdot D(\gamma_m^C) \ , \quad \gamma_m^{D*C} = \gamma_{\gamma_m^C}^D \ .$$

Remark 2.5. A closure operator C is in fact a *wellpointed* ([K]) endofunctor of $\mathcal{M}$, that is, $\Gamma C = C\Gamma$. Likewise, its companion $\widetilde{C}$ is *wellcopointed*, that is, $\Delta\widetilde{C} = \widetilde{C}\Delta$.

Definition 2.6. A closure operator C is *idempotent* if $\Gamma C : C \to CC$ is an isomorphism, and C is *weakly hereditary* if $\Delta\widetilde{C} : \widetilde{C}\widetilde{C} \to \widetilde{C}$ is an isomorphism. Hence C is idempotent if $\gamma_{Cm} : C_X(M) \to C_X(C_X(M))$ is an isomorphism, and weakly hereditary if $\delta_{\widetilde{C}m} = C\gamma_m : C_{C_X(M)}(M) \to C_X(M)$ is an isomorphism, for all subobjects $m : M \to X$. In terms of the pointwise-defined preorder for closure operators, C *is idempotent if* $C \cdot C \leqslant C$, *and weakly hereditary if* $C \leqslant C * C$.

The preordered conglomerate **CLOP** of all closure operators of $\mathcal{M}$ in $\mathcal{K}$ has a least element I and a largest element T, the trivial closure operator with $T(m : M \to X) = 1_X$. From [DT] we recall the following easily established facts:

Proposition 2.7.

(1) Composition and cocomposition are associative binary operations on **CLOP** *that are monotone in each variable.*

(2) I is neutral w.r.t. composition and absorbing w.r.t. cocomposition (so that $I * C = C * I = I$), while T is neutral w.r.t. cocomposition and absorbing w.r.t. composition (so that $T \cdot C = C \cdot T = T$).

Note that, as usual, we have written $C = D$ (instead of $C \simeq D$) when $C \leqslant D$ and $D \leqslant C$. We will follow this practice also in what follows.

3 The lax middle-interchange law

Throughout this section we consider closure operators C, D, E, F of $\mathcal{M}$ in $\mathcal{K}$ and first prove the lax middle-interchange law (mil):

Proposition 3.1. $(C * D) \cdot (E * F) \leqslant (C \cdot E) * (D \cdot F)$.

Proof. For subobjects $m : M \to X$ and $n : N \to X$ we write $M \leqslant N$ instead of $m \leqslant n$ and first note that one has $E_N(M) \leqslant E_K(M)$ whenever $N \leqslant K$, all to be considered subobjects of X. In particular,

$$L := E_{F_X(M)}(M) \leqslant E_{D_X(F_X(M))}(M) \quad \textit{and} \quad L \leqslant F_X(M) \quad .$$

Consequently,

$$C_{D_X(L)}(L) \leqslant C_{D_X(F_X(M))}(L) \leqslant C_{D_X(F_X(M))}(E_{D_X(F_X(M))}(M)) \quad .$$

But the left-hand side is $((C * D) \cdot (E * F))_X(M)$, and the right-hand side is $((C \cdot E) * (D \cdot F))_X(M)$. ☐

Remark 3.2.

(1) A 'morphism-based' proof of 3.1 will be presented in [T] in greater generality.

(2) The inequality in 3.1 may be as strict as it possibly could get: For $C = F = I$ and $D = E = T$ one has

$$(C * D) \cdot (E * F) = I < T = (C \cdot E) * (D \cdot F) \quad .$$

Here is an easy application of the lax middle-interchange law 3.1:

Corollary 3.3. *$D \cdot C$ is weakly hereditary if D and C are weakly hereditary, and $D * C$ is idempotent if D and C are idempotent.*

Proof. From $D \leqslant D * D$ and $C \leqslant C * C$ one obtains with 3.1

$$D \cdot C \leqslant (D * D) \cdot (C * C) \leqslant (D \cdot C) * (D \cdot C) \quad .$$

Likewise, $C \cdot C \leqslant C$ and $D \cdot D \leqslant D$ imply

$$(D * C) \cdot (D * C) \leqslant (D \cdot D) * (C \cdot C) \leqslant D * C \quad .$$

□

In what follows we exhibit four situations in which the lax middle-interchange law holds strictly:

$$(C * D) \cdot (E * F) = (C \cdot E) * (D \cdot F) \quad . \qquad \text{(MIL)}$$

In each of the four situations described, the application of either side reduces to the application of one of the four participating closure operators.

Theorem 3.4.

(1) C is idempotent and weakly hereditary if, and only if, (MIL) *holds for all D, E, F with $E \leqslant C \leqslant D$; in this case $(C \cdot E) * (D \cdot F) = C$.*

(2) D is idempotent and weakly hereditary if, and only if, (MIL) *holds for all C, E, F with $F \leqslant D \leqslant C$; in this case $(C \cdot E) * (D \cdot F) = D$.*

(3) E is idempotent and weakly hereditary if, and only if, (MIL) *holds for all C, D, F with $C \leqslant E \leqslant F$; in this case $(C \cdot E) * (D \cdot F) = E$.*

(4) F is idempotent and weakly hereditary if, and only if, (MIL) *holds for all C, D, E with $D \leqslant F \leqslant E$; in this case $(C \cdot E) * (D \cdot F) = F$.*

Proof. (1) When C is idempotent and weakly hereditary one has

$$C \cdot E \leqslant C \cdot C \leqslant C \leqslant C * C \leqslant C * D \quad .$$

Consequently,

$$\begin{array}{rcl} (C\cdot E)*(D\cdot F) & \leqslant & C*(D\cdot F) \\ & \leqslant & C*T \\ & = & C \\ & = & C\cdot I \\ & \leqslant & C\cdot(E*F) \\ & \leqslant & (C*D)\cdot(E*F) \quad . \end{array}$$

Conversely, assuming (MIL) one obtains $C*C=C$ when putting $D=C$, $E=I$, $F=T$, and $C=C\cdot C$ when putting $E=C$, $D=T$, $F=I$.

(2) For D idempotent and weakly hereditary one obtains similarly to (1)

$$\begin{array}{rcl} (C\cdot E)*(D\cdot F) & \leqslant & (C\cdot E)*D \\ & \leqslant & D \\ & \leqslant & D\cdot(E*F) \\ & \leqslant & (C*D)\cdot(E*F) \quad . \end{array}$$

Conversely one exploits the choices $C=T$, $E=I$, $F=D$ to obtain $D\cdot D=D$, and $C=D$, $E=T$, $F=I$ to obtain $D=D*D$.

(3), (4) are shown similarly to (1), (2). □

Corollary 3.5. *The following assertions are equivalent for a closure operator C:*

(i) C is idempotent and weakly hereditary;

(ii) for all D, $\quad C\cdot(C*D)=C*(C\cdot D)$;

(iii) for all D, $\quad C\cdot(D*C)=(C\cdot D)*C$;

(iv) for all D, $\quad (C*D)\cdot C=C*(D\cdot C)$;

(v) for all D, $\quad (D*C)\cdot C=(D\cdot C)*C$.

If the equivalent conditions hold, the value of each of the composite closure operators appearing in (ii)-(v) *is C.*

Proof. One obtains (ii)-(iv) as necessary consequences of (i) by choosing the comparable closure operators in (1)-(4) of Theorem 3.4 to be equal; hence, in (1) of 3.4 one chooses $E=C=D$, and so on. One sees that each of (ii)-(v) is sufficient for (i) by choosing $D=I$ or $D=T$. □

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