Ordered Topological Structures

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Abstract

The paper discusses interactions between order and topology on a given set which do not presuppose any separation conditions for either of the two structures, but which lead to the existing notions established by Nachbin in more special situations. We pursue this discussion at the much more general level of lax algebras, so that our categories do not concern just ordered topological spaces, but also sets with two interacting orders, approach spaces with an additional metric, etc.

Key words: modular topological space, closed-ordered topological space, open-ordered topological space, lax $(\mathbb{T},\mathcal{V})$-algebra, $(\mathbb{T},\mathcal{V})$-category
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Dedicated to Bob Lowen

1 Introduction

Contrary to widespread perception, in his beautiful monograph *Topology and Order* [N2] Nachbin did not formally introduce a notion of topological ordered space, or of ordered topological space. He did introduce normally (pre)ordered and compact ordered spaces, but even the original article [N1] contains no formal definition in the general case, despite the fact that its first paragraph is entitled "On topological ordered spaces". Rather, he simply refers to a topological space equipped with a preorder, which normally is assumed to be closed (as a subset of the product space). About the reasons I can only speculate. But since he often cites the case of the discrete order as the one giving the corresponding ordinary topological notion or result, whereas a topological space with a closed discrete (or any) order must necessarily be Hausdorff, I conclude

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that he tried to avoid formalizing a general definition that in the discretely ordered case would not return the general notion of topological space. This conclusion is consistent with the fact that the monograph carefully avoids inherent separation conditions whenever posssible, working with preorders rather than orders, thus avoiding "antisymmetry", or working with semi-metrics rather than metrics, thus avoiding symmetry and the separation condition that only equal points may have distance 0.

In this article we discuss three possible candidates for a notion of ordered topological space. (We prefer this name over topological ordered space.) These candidates emerge rather naturally when we look at the fundamental adjunction linking order and topology:

$$\mathbf{Ord} \underset{A}{\overset{S}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{Top}$$

Here **Ord** denotes the category of *pre*ordered sets (= sets with a reflexive and transitive relation) and monotone maps. In fact, in order to avoid excessive use of prefixes, we will refer to its objects simply as *ordered sets*, using *separated ordered sets* for what Nachbin calls ordered sets and what most other authors refer to as partially ordered sets. The full embedding A provides an ordered set $(X, \leq)$ with the *Alexandroff topology* $\tau_{\leq}$ generated by the principal downsets $\downarrow x$ $(x \in X)$, while its coreflector S endows a topological space (X, τ) with the (dual of the) *specialization order*:

$$x \leq_\tau y \;\;:\Leftrightarrow\;\; \dot{x} \to y \;\;\Leftrightarrow\;\; \mathcal{N}_x \leq \mathcal{N}_y \;\;\Leftrightarrow\;\; y \in \overline{\{x\}} \quad .$$

(Here $\dot{x}$ is the principal filter over x, and $\mathcal{N}_x$ is the neighbourhood filter of x; $\leq$ for filters is to be read as "finer than".) We take the position that, whatever notion of ordered topological space one wants to adopt, it should include arbitrary topological spaces (X, τ) endowed with the order $\leq_\tau$ and, consequently, arbitrary ordered sets $(X, \leq)$ topologized by $\tau_{\leq}$.

Our first description of the specialization order indicates that it may be useful here to think of topological spaces in terms of convergence. It was Barr [B] who first proved that a topological space may be described as a set X with a relation $\to: \beta X \nrightarrow X$ between the set βX of ultrafilters on X and the set X, satisfying two simple axioms which show best the extent to which arbitrary topological spaces generalize Alexandroff spaces, i.e. ordered sets:

Reflexivity: $\dot{x} \to x$,
Transitivity: $\mathfrak{X} \to \mathfrak{y}, \mathfrak{y} \to z \;\Rightarrow\; \Sigma\mathfrak{X} \to z$,

for all $x, z \in X, \mathfrak{y} \in \beta X, \mathfrak{X} \in \beta\beta X$. Here $\Sigma\mathfrak{X}$ denotes the *Kowalsky sum* of $\mathfrak{X}$, defined by $B \in \Sigma\mathfrak{X} \Leftrightarrow B^\sharp \in \mathfrak{X}$, with $B^\sharp = \{\mathfrak{x} \in \beta X | B \in \mathfrak{x}\}$ the set of ultrafilters on $B \subseteq X$. Furthermore, the convergence relation $\to$ has been extended to $\to: \beta\beta X {\to\!\!\!\!\!+} \beta X$, by

$$\mathfrak{X} \to \mathfrak{y} \quad \Leftrightarrow \quad \forall B \in \mathfrak{y} : \{\mathfrak{x} \in \beta X | \exists y \in B : \mathfrak{x} \to y\} \in \mathfrak{X} \quad .$$

Seal [S] showed that βX may in fact be replaced by the set γX of all filters on X: With the same definitions, a relation $\to: \gamma X {\to\!\!\!\!\!+} X$ satisfying the two axioms will still describe **Top**, where again we take the morphisms to be the maps preserving the convergence relation.

Any order $\leq$ that a topological space $(X, \to)$ may carry can be extended to (ultra)filters via

$$\mathfrak{x} \leq \mathfrak{y} \quad \Leftrightarrow \quad \forall B \in \mathfrak{y} : \downarrow B \in \mathfrak{x}$$

(with $\downarrow B = \bigcup_{x \in B} \downarrow x$); when the order is discrete, this is of course just the "finer than" order. Now, when $\leq$ is the specialization order $\leq_\tau$, we observe that the two relations $\to$ and $\leq$ are linked by the following fundamental property:

Modularity: $\mathfrak{x} \leq \mathfrak{y}, \mathfrak{y} \to y, y \leq z \;\Rightarrow\; \mathfrak{x} \to z$.

In fact, quite trivially, any ordered set with a topology generated by a system of down-closed open sets satisfies the modularity condition. Hence, the Scott topology on (the dual of) an ordered set satisfies it, and so does every discretely ordered topological space. However, the real line with its natural order and Euclidean topology does not. In fact, no Hausdorff space with a non-discrete order satisfies the Modularity condition. Hence, instead of permitting non-Hausdorffness, the condition dictates non-Hausdorffness, except in the case of the discrete order.

While we will underline in this paper the central role of the Modularity condition, as that of a mediator between order and topology, it is clear that it cannot serve as a generally acceptable notion. Two distinct weakenings arise naturally, as follows:

Closedness: $\mathfrak{x} \leq \mathfrak{y}, \mathfrak{y} \to y \;\Rightarrow\; \exists x : \mathfrak{x} \to x, x \leq y$,
Openness: $\mathfrak{x} \to x, x \leq y \;\Rightarrow\; \exists \mathfrak{y} : \mathfrak{x} \leq \mathfrak{y}, \mathfrak{y} \to y$.

For a Hausdorff space X, Closedness makes the order $\leq$ closed as a subset of $X \times X$, and any such order satisfies the Closedness condition when X is compact. For compact Hausdorff spaces, Closedness is also equivalently expressed by the preservation condition ($B \subseteq X$ closed $\Rightarrow \uparrow B$ closed), contrasting with the following necessary preservation condition for Openness: ($A \subseteq X$ open $\Rightarrow \downarrow A$ open); it is equivalent to Openness in the case of filter convergence. The real line and the Euclidean plane (with its coordinatewise natural order) satisfy both conditions. The emerging categories **COrdTop** and **OOrdTop** of *closed-ordered topological spaces* and *open-ordered topological spaces* both contain the category **ModTop** of *modular topological spaces* as a full bireflective subcategory but fail to possess some of the good properties that **ModTop** enjoys as a topological category over **Ord** (and **Set**).

The setting in which we discuss these categories and their functorial interactions is much more general than these introductory remarks may suggest. We work in the context of so-called $(\mathbb{T}, \mathcal{V})$-*categories* (or *lax* $(\mathbb{T}, \mathcal{V})$-*algebras*) with a quantale $\mathcal{V}$ and a **Set**-monad $\mathbb{T}$ suitably extended to the category $\mathcal{V}$-**Rel** of sets and $\mathcal{V}$-relations. The precise set of axioms is taken from [S], which is based on work presented previously in [CH1], [CT], [CHT]. Ordered topological structures occur in this context when $\mathcal{V}$ is the 2-chain and $\mathbb{T}$ the filter- or ultrafilter monad of **Set**, suitably extended to **Rel**. There are many aspects even in this special case on which we cannot elaborate in this introductory paper, most prominently the question of how continuous lattices fare in our setting. Also, we indicate only briefly how this work provides the basis for a study of sets equipped with a metric and an approach structure [Lo] th! at are adequately linked, i.e. of the case the case when $\mathcal{V} = [0, \infty]$ is Lawvere's extended real half-line [L] and $\mathbb{T}$ the ultrafilter monad.

2 Syntax

Let $\mathcal{V} = (\mathcal{V}, k, \otimes)$ be a commutative unital quantale; hence $\mathcal{V}$ is a complete lattice and $(\mathcal{V}, k, \otimes)$ is a commutative monoid such that the binary operation $\otimes$ distributes over arbitrary suprema : $v \otimes \bigvee w_i = \bigvee v \otimes w_i$. Our two primary examples are the 2-chain $\mathbf{2} = (\{\bot < \top\}, \top, \wedge)$ and the extended half-line $\mathcal{P}_+ = ([0, \infty]^{\mathrm{op}}, 0, +)$, with $[0, \infty]^{\mathrm{op}} = ([0, \infty], \geq)$. A $\mathcal{V}$-*relation* $r : X \nrightarrow Y$ from a set X to a set Y is a function $r : X \times Y \to \mathcal{V}$; its composite with $s : Y \nrightarrow Z$ is defined by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z) \quad .$$

This defines the category $\mathcal{V}$-**Rel**, which is in fact a 2-category: its hom-sets carry the pointwise order

$$r \leq r' \quad \Leftrightarrow \quad \forall x \in X, y \in Y : r(x, y) \leq r'(x, y) \quad .$$

There is also an involution $(r \mapsto r^\circ)$ with $r^\circ(y, x) = r(x, y)$ which is contravariant on 1-cells but covariant on 2-cells.

Every mapping $f : X \to Y$ can be considered as a $\mathcal{V}$-relation $f_\circ : X \nrightarrow Y$ via

$$f_\circ(x, y) = \begin{cases} k & \text{if } f(x) = y \quad , \\ \bot & \text{else} \quad . \end{cases}$$

This defines a functor **Set** $\to$ $\mathcal{V}$-**Rel** which is faithful precisely when $k > \bot$ (i.e., when $\mathcal{V}$ has at least 2 elements). We will assume $k > \bot$ henceforth and write f instead of $f_\circ$. The converse $\mathcal{V}$-relation $f^\circ : Y \nrightarrow X$ is right adjoint to f in $\mathcal{V}$-**Rel**: $f \cdot f^\circ \le 1_Y, 1_X \le f^\circ \cdot f$.

In addition to the quantale $\mathcal{V}$ we consider a monad $\mathbb{T} = (T, e, m)$ of **Set** (i.e. an endofunctor T with natural transformations $e : 1_{\mathbf{Set}} \to T$, $m : TT \to T$ satisfying $m(Te) = m(eT) = 1_T$, $m(Tm) = m(mT)$) and functions

$$\mathcal{V}\text{-}\mathbf{Rel}(X, Y) \to \mathcal{V}\text{-}\mathbf{Rel}(TX, TY) \quad , \quad r \mapsto \hat{T}r \quad ,$$

such that:

(1) $\hat{T}s \cdot \hat{T}r \le \hat{T}(s \cdot r), \quad r \le r' \Rightarrow \hat{T}r \le \hat{T}r'$,
(2) $Tf \le \hat{T}f, \quad (Tf)^\circ \le \hat{T}(f^\circ)$,
(3) $e_Y \cdot r \le \hat{T}r \cdot e_X, \quad m_Y \cdot \hat{T}\hat{T}r \le \hat{T}r \cdot m_X$,

for all $r, r' : X \nrightarrow Y$, $s : Y \nrightarrow Z$, and $f : X \to Y$. Note that from (2) one has in particular $1_{TX} \le \hat{T}1_X$, so that then (1) says that

$$\hat{T} : \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Rel}$$

is a lax functor which coincides with T on objects, whereas (3) stipulates that $e : 1_{\mathcal{V}\text{-}\mathbf{Rel}} \to \hat{T}$, $m : \hat{T}\hat{T} \to \hat{T}$ be op-lax natural transformations. We will refer to $\hat{T}$ as a *lax extension of* $\mathbb{T}$. We note that (3) may be equivalently expressed by:

(3′) $r \cdot e_X^\circ \le e_Y^\circ \cdot \hat{T}r, \quad \hat{T}\hat{T}r \cdot m_X^\circ \le m_Y^\circ \cdot \hat{T}r \quad .$

There are some important identities that one may derive from (1)-(3), as follows:

Proposition 1. *For all r, s, f as above and $g : Z \to Y$ one has:*

1. *(See [S].)* $\hat{T}(s \cdot f) = \hat{T}s \cdot \hat{T}f = \hat{T}s \cdot Tf, \ \hat{T}(g^\circ \cdot r) = \hat{T}(g^\circ) \cdot \hat{T}r = (Tg)^\circ \cdot \hat{T}r \ .$
2. *(See [T2].)* $\hat{T}1_X = \hat{T}(e_X^\circ) \cdot m_X^\circ \ .$

Proof. 1.

$$\begin{aligned}
\hat{T}(s \cdot f) &\le \hat{T}(s \cdot f) \cdot Tf^\circ \cdot Tf && \text{(adjunction)} \\
&\le \hat{T}(s \cdot f) \cdot \hat{T}(f^\circ) \cdot Tf && \text{(by (2))} \\
&\le \hat{T}(s \cdot f \cdot f^\circ) \cdot Tf && \text{(by (1))} \\
&\le \hat{T}s \cdot Tf && \text{(adjunction)} \\
&\le \hat{T}s \cdot \hat{T}f && \text{(by (2))} \\
&\le \hat{T}(s \cdot f) && \text{(by (1))} \quad .
\end{aligned}$$

The second identity follows similarly.

2. From $m_X \cdot Te_X = 1_{TX} = 1_{TX}^\circ$ one obtains:

$$\begin{aligned}
\hat{T}1_X &= \hat{T}1_X \cdot (Te_X)^\circ \cdot m_X^\circ \\
&\leq \hat{T}1_X \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ && \text{(by (2))} \\
&\leq \hat{T}(e_X^\circ) \cdot m_X^\circ && \text{(by (1))} \\
&\leq \hat{T}(e_X^\circ \cdot \hat{T}1_X) \cdot m_X^\circ && \text{(by (1),(2))} \\
&\leq (Te_X)^\circ \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ && \text{(by (1))} \\
&\leq (Te_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X = \hat{T}1_X && \text{(by (3))} \quad .
\end{aligned}$$

□

Example 1. Writing **Rel** instead of **2-Rel** we can translate the natural bijection $\mathbf{Rel}(X,Y) \cong \mathbf{Rel}(Y,X)$, $r \mapsto r^\circ$, to

$$\mathbf{Set}(X, \mathcal{P}Y) \cong \mathbf{Set}(Y, \mathcal{P}X) = \mathbf{Set}^{\mathbf{op}}(\mathcal{P}X, Y) \quad ,$$

showing the self-adjointness of the contravariant powerset functor $\mathcal{P} \vdash \mathcal{P}^{\mathrm{op}}$, with

$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}^{\mathbf{op}} \quad , \quad (f : X \to Y) \mapsto (\mathcal{P}f : B \mapsto f^{-1}(B)) \quad .$$

The induced monad $\mathbb{P}^2 = (\mathcal{P}^{\mathrm{op}}\mathcal{P}, e, m)$ is given by:

$$\begin{array}{lll}
\mathcal{P}^2 f : \mathcal{P}^2 X \longrightarrow \mathcal{P}^2 Y, & e_X : X \longrightarrow \mathcal{P}^2 X, & m_X : \mathcal{P}^2\mathcal{P}^2 X \longrightarrow \mathcal{P}^2 X \\
\mathfrak{x} \mapsto \{B | f^{-1}(B) \in \mathfrak{x}\} & x \mapsto \dot{x} & \mathfrak{X} \mapsto \Sigma\mathfrak{X}
\end{array}$$

with $\dot{x}$ and $\Sigma\mathfrak{X}$ defined as in the Introduction. There are subfunctors

$$1_{\mathbf{Set}} \to \beta \to \gamma \to \delta \to \mathcal{P}^2$$

which give submonads of $\mathbb{P}^2$, with

$$\begin{aligned}
\mathfrak{x} \in \delta X \quad &\Leftrightarrow \quad X \in \mathfrak{x} \text{ and } (A \in \mathfrak{x}, A \subseteq B \Rightarrow B \in \mathfrak{x}) \quad , \\
\mathfrak{x} \in \gamma X \quad &\Leftrightarrow \quad \mathfrak{x} \in \delta X \text{ and } (A \in \mathfrak{x}, B \in \mathfrak{x} \Rightarrow A \cap B \in \mathfrak{x}) \quad , \\
\mathfrak{x} \in \beta X \quad &\Leftrightarrow \quad \mathfrak{x} \in \gamma X \text{ and } (A \cup B \in \mathfrak{x} \Rightarrow A \in \mathfrak{x} \text{ or } B \in \mathfrak{x}) \quad .
\end{aligned}$$

It is not difficult to prove that if T is any of $1, \beta, \gamma, \delta$, then we may define a lax extension of the corresponding **Set** monad to **Rel** by

$$\begin{aligned}
\mathfrak{x}(\hat{T}r)\mathfrak{y} \quad &\Leftrightarrow \quad \forall B \in \mathfrak{y}\ \exists A \in \mathfrak{x} : A \subseteq r^\circ(B) \\
&\Leftrightarrow \quad \forall B \in \mathfrak{y}\ \exists A \in \mathfrak{x}\ \forall x \in A\ \exists y \in B : xry \quad ,
\end{aligned}$$

where $r : X \nrightarrow Y$, writing xry when $r(x,y) = \top$. (We note that T may *not* be taken to be $\mathcal{P}^2$ since in each of (2),(3) the second inequality may no longer be satisfied.)

3 Strict Semantics

Let $\mathcal{V}$ be as in Section 2. A $\mathcal{V}$-*category* $X = (X, a)$ is a set X with a $\mathcal{V}$-relation that is reflexive ($1_X \leq a$) and transitive ($a \cdot a \leq a$). A $\mathcal{V}$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ with $f \cdot a \leq b \cdot f$. This defines the category $\mathcal{V}$-**Cat**. For $\mathcal{V} = \mathbf{2}$, this is the category **Ord**, and with $\mathcal{V} = \mathcal{P}_+$ one obtains the category **Met** whose objects $(X, a : X \times X \to [0, \infty])$ are just required to satisfy $a(x,x) = 0$ and $a(x,z) \leq a(x,y) + a(y,z)$ for all $x, y, z \in X$, and whose morphisms $f : (X, a) \to (Y, b)$ satisfy $b(f(x), f(x')) \leq a(x, x')$ for all $x, x' \in X$; see [L], [CHT].

Let us now consider a **Set**-monad $\mathbb{T} = (T, e, m)$ with a lax extension $\hat{T}$. We can augment $\mathbb{T}$ into a monad of $\mathcal{V}$-**Cat**, by putting

$$T(X, a) = (TX, \hat{T}a) \quad .$$

Indeed, this is again a $\mathcal{V}$-category since

$$1_{TX} \leq \hat{T}1_X \leq \hat{T}a \quad , \quad \hat{T}a \cdot \hat{T}a \leq \hat{T}(a \cdot a) = \hat{T}a \quad .$$

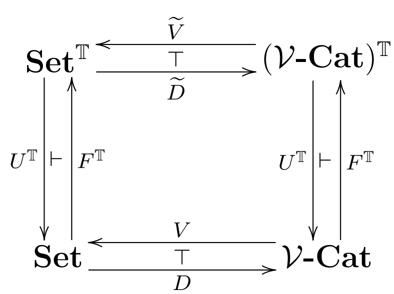
And for a $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$, since

$$Tf \cdot \hat{T}a \leq \hat{T}(f \cdot a) \leq \hat{T}(b \cdot f) = \hat{T}b \cdot Tf \quad ,$$

$Tf : T(X, a) \to T(Y, b)$ is again a $\mathcal{V}$-functor. Finally, condition (3) for $\hat{T}$ makes e_X and m_X $\mathcal{V}$-functors. Hence:

Proposition 2. $\mathbb{T}$ *is a monad of* $\mathcal{V}$-**Cat**.

Considering the respective Eilenberg-Moore categories over **Set** and $\mathcal{V}$-**Cat**, one obtains the commutative diagram



Here V is the forgetful functor which exhibits $\mathcal{V}$**-Cat** as a topological category over **Set** (see [CHT]), and D is its left adjoint $X \mapsto (X, 1_X)$. An object (X, a, c) in $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$ is composed of objects $(X, a) \in \mathcal{V}$**-Cat** and $(X, c) \in \mathbf{Set}^{\mathbb{T}}$ (i.e., $c : TX \to X$ satisfies $c \cdot e_X = 1_X$, $c \cdot Tc = c \cdot m_X$) such that $c : T(X, a) \to (X, a)$ is a $\mathcal{V}$-functor (i.e. $c \cdot \hat{T}a \le a \cdot c$); morphisms of $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$ must live simultaneously in $\mathcal{V}$**-Cat** and in $\mathbf{Set}^{\mathbb{T}}$. The lifted adjunction $\widetilde{D} \dashv \widetilde{V} : (X, a, c) \mapsto (X, c)$ is obtained from:

Proposition 3. *$\widetilde{V}$ is (like V) a topological functor. In particular, $\widetilde{V}$ has both a right adjoint and a left adjoint.*

Proof. In fact, having a family of morphisms $f_i : (X, c) \to (Y_i, d_i)$ in $\mathbf{Set}^{\mathbb{T}}$ with all $(Y_i, b_i, d_i) \in (\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$, we may simply consider the V-initial structure $a = \bigwedge_i f_i^\circ \cdot b_i \cdot f_i$ on X and observe that (X, a, c) lies in $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$:

$$\begin{aligned} c \cdot \hat{T}a &\le \textstyle\bigwedge_i c \cdot \hat{T}(f_i^\circ \cdot b_i \cdot f_i) && \text{(by 1.)} \\ &\le \textstyle\bigwedge_i c \cdot (Tf_i)^\circ \cdot \hat{T}b_i \cdot Tf_i && \text{(Prop. 1)} \\ &\le \textstyle\bigwedge_i f_i^\circ \cdot d_i \cdot \hat{T}b_i \cdot Tf_i && \text{(from } f_i \cdot c = d_i \cdot Tf) \\ &\le \textstyle\bigwedge_i f_i^\circ \cdot b_i \cdot d_i \cdot Tf_i && \text{(from } d_i \cdot \hat{T}b_i \le b_i \cdot d_i) \\ &= \textstyle\bigwedge_i f_i^\circ \cdot b_i \cdot f_i \cdot c = a \cdot c && \text{(from } f_i \cdot c = d_i \cdot Tf) \quad . \end{aligned}$$

The right adjoint of $\widetilde{V}$ is constructed by $\widetilde{V}$-initially lifting the empty family, i.e. by putting a to be constantly the top element $\top$ in $\mathcal{V}$, and to obtain its left adjoint $\widetilde{D}$ one considers the family of all morphisms $(X, c) \to (Y, d)$ with $(Y, b, d) \in (\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$. Hence, $\widetilde{D}(X, c) = (X, a, c)$ with a the least $\mathcal{V}$**-Cat** structure on X that makes (X, a, c) live in $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$. □

We hasten to remark that only when the lax extension $\hat{T}$ of $\mathbb{T}$ is *flat*, that is when $\hat{T}1_X = 1_{TX}$ (and then $\hat{T}f = Tf$, $\hat{T}(f^\circ) = (Tf)^\circ$ in (2) of Section 2, by Prop. 1), are we sure that $\widetilde{D}(X, c)$ may be easily computed as $(X, 1_X, c)$.

Corollary 1. $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$ *is complete and cocomplete, and $\widetilde{V}$ preserves all limits and colimits.*

Example 2. For $T = \beta$, $\mathbf{Set}^{\mathbb{T}}$ is the category **CompHaus** of compact Hausdorff spaces ([M]), and with $\mathcal{V} = \mathbf{2}$ we then obtain the category

$$(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}} = \mathbf{OrdCompHaus}$$

of *ordered compact Hausdorff spaces*, i.e. of compact Hausdorff spaces X which carry a (pre)order making ultrafilter convergence monotone:

$$\mathfrak{x} \to x, \mathfrak{y} \to y, \mathfrak{x} \le \mathfrak{y} \quad \Longrightarrow \quad x \le y \quad .$$

The extension $\hat{T}a$ of the order a of X as considered in Example 1 gives precisely the order on βX as considered in the Introduction. We also note that the extension is flat.

Let us now clarify that Nachbin's compact ordered spaces are precisely those objects of **OrdCompHaus** whose order is separated; we denote the corresponding full subcategory by **SepOrdCompHaus**. This follows easily from:

Proposition 4. *In a topological space X provided with an order $\le$, consider the following conditions:*

(i) *for $\mathfrak{x} \le \mathfrak{y}$ in βX and $\mathfrak{y} \to y$ there is $x \le y$ with $\mathfrak{x} \to x$ (Closedness);*
(ii) *for every closed set $F \subseteq X$, $\uparrow F$ is also closed;*
(iii) *$\{(x,y)|x \le y\}$ is closed in $X \times X$;*
(iv) *$\uparrow x$ is closed for all $x \in X$.*

Then the implications (i) ⇒ (ii) and (iii) ⇒ (iv) hold, while (i) ⇒ (iii) and (ii) ⇒ (iv) hold when X is Hausdorff, and (iv) ⇒ (i) holds when X is compact.

Proof. (i) ⇒ (ii). Let F be closed and consider $\mathfrak{y} \to y$ with $\uparrow F \in \mathfrak{y}$. Since

$$F \cap \downarrow B \neq \emptyset \quad \Leftrightarrow \quad \uparrow F \cap B \neq \emptyset \quad ,$$

F belongs to any ultrafilter $\mathfrak{x}$ containing the filter $\{\downarrow B | B \in \mathfrak{y}\}$. Trivially, $\mathfrak{x} \le \mathfrak{y}$, so that $\mathfrak{x} \to x$ for some $x \le y$, by hypothesis. But $x \in F$ since F is closed, and hence $y \in \uparrow F$.

(iii) ⇒ (iv). See Prop. 1, p. 26 of [N2].

(i) ⇒ (iii). Consider $\mathfrak{z} \in \beta(X \times X)$ with $G = \{(x,y)|x \le y\} \in \mathfrak{z}$ and $\mathfrak{z} \to (x,y)$, so that with $\mathfrak{x} = Tp_1(\mathfrak{z})$, $\mathfrak{y} = Tp_2(\mathfrak{z})$ one has $\mathfrak{x} \to x$, $\mathfrak{y} \to y$ (where $p_1, p_2 : X \times X \to X$ are the projections). We claim $\mathfrak{x} \le \mathfrak{y}$; indeed, if $B \in \mathfrak{y}$, then $X \times B \in \mathfrak{z}$ and then $\downarrow B = p_1((X \times B) \cap G) \in \mathfrak{x}$. Consequently $\mathfrak{x} \to x'$ for some $x' \le y$. When X is Hausdorff, with $\mathfrak{x} \to x$ one obtains $x' = x$ and therefore $(x,y) \in G$.

(ii) ⇒ (iv) is trivial when X is a T_1 space.

(iv) ⇒ (i). Assume $\mathfrak{x} \le \mathfrak{y}$ and $\mathfrak{y} \to y$. When X is compact one has $\mathfrak{x} \to x$, and $\uparrow x$ is closed, by hypothesis. Since for all $B \in \mathfrak{y}$ one has $\downarrow B \in \mathfrak{x}$, so that $\downarrow B \cap \uparrow x \neq \emptyset$ and then even $B \cap \uparrow x \neq \emptyset$, $\mathfrak{y}$ restricts to $\uparrow x$, so that its limit y must lie in $\uparrow x$. □

Compactness without separation conditions does not yield equivalence of the conditions in Prop. 4. For example, the Sierpinski space $X = \{0 \leq 1\}$ endowed with $\{1\}$ open satisfies conditions (i),(ii) but neither of (iii), (iv). If, instead, we make $\{0\}$ the only non-trivial open set while keeping the order, also (iv) holds true while (iii) still fails.

The reflector of **Ord** onto the full subcategory **SepOrd** (obtained by $X \mapsto X / \sim$ with $(x \sim y \Leftrightarrow x \leq y$ and $y \leq x))$ may be lifted to a reflector of **OrdCompHaus** onto **SepOrdCompHaus**: it follows from condition (iii) of Prop. 4 (which equivalently says that whenever $x \not\leq y$ there are neighbourhoods V of x and W of y with $V = \uparrow V$, $W = \downarrow W$, and $V \cap W = \emptyset$) that the quotient space $X / \sim$ is Hausdorff, and the quotient is also easily seen to satisfy condition (ii) of Prop. 4. Hence:

Corollary 2. *Nachbin's compact ordered spaces form a quotient-reflective subcategory of* **OrdCompHaus**. *Hence, the subcategory is complete and cocomplete.*

4 Lax Versus Strict Semantics

Let $\mathcal{V}$ and $\mathbb{T}$ be as in Section 2, with a lax extension $\hat{T}$. The category

$$(\mathbb{T}, \mathcal{V})\text{-}\mathbf{Cat}$$

of $(\mathbb{T}, \mathcal{V})$-*categories* (or *lax* $(\mathbb{T}, \mathcal{V})$-*algebras*) has as objects pairs (X, c) with a set X and a $\mathcal{V}$-relation $c : TX \nrightarrow X$ with $1_X \leq c \cdot e_X$ and $c \cdot \hat{T}c \leq c \cdot m_X$; a morphism $f : (X, c) \to (Y, d)$ must satisfy $f \cdot c \leq d \cdot Tf$ or, equivalently, $f \cdot c \leq d \cdot \hat{T}f$. In fact, one has $d \cdot Tf = d \cdot \hat{T}f$ since

$$\begin{aligned} d \cdot Tf &\leq d \cdot \hat{T}(1_Y \cdot f) && \text{(by (2))} \\ &= d \cdot \hat{T}1_Y \cdot Tf && \text{(by Prop. 1)} \\ &= d \cdot \hat{T}(e_Y^\circ) \cdot m_Y^\circ \cdot Tf && \text{(by Prop. 1)} \\ &\leq d \cdot \hat{T}d \cdot m_Y^\circ \cdot Tf && \text{(since } e_Y^\circ \leq d) \\ &\leq d \cdot m_Y \cdot m_Y^\circ \cdot Tf && \text{(since } d \cdot \hat{T}d \leq d \cdot m_Y) \\ &\leq d \cdot Tf && \text{(adjunction)} \quad . \end{aligned}$$

For $\mathbb{T}$ the identity monad (and $\hat{T} = 1$), $(\mathbb{T}, \mathcal{V})$-**Cat** = $\mathcal{V}$-**Cat**. For $\mathcal{V} = \mathbf{2}$ and $\mathbb{T}$ either the ultrafilter monad or the filter monad (extended as in Example 1), $(\mathbb{T}, \mathcal{V})$-**Cat** is the category **Top** of topological spaces, as shown by Barr [B] and Seal [S]. In general, $(\mathbb{T}, \mathcal{V})$-**Cat** is a topological category over **Set** (see

[CHT]) which may be linked to $\mathcal{V}$**-Cat** by the following adjunction which, under more restrictive conditions on $\mathbb{T}$ and $\mathcal{V}$, was established in [CH2]:

$$\mathcal{V}\text{-}\mathbf{Cat} \underset{A}{\overset{S}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} (\mathbb{T},\mathcal{V})\text{-}\mathbf{Cat}$$

with $S(X,c) = (X, c \cdot e_X)$, $A(X,a) = (X, e_X^\circ \cdot \hat{T}a)$. Verification that $A(X,a)$ is a $(\mathbb{T},\mathcal{V})$-category requires the full range of conditions (1)-(3):

$$1_X \leq e_X^\circ \cdot e_X \leq e_X^\circ \cdot \hat{T}1_X \cdot e_X \leq (e_X^\circ \cdot \hat{T}a) \cdot e_X \quad ,$$

$$\begin{aligned} (e_X^\circ \cdot \hat{T}a) \cdot \hat{T}(e_X^\circ \cdot \hat{T}a) &= e_X^\circ \cdot \hat{T}a \cdot \hat{T}(e_X^\circ) \cdot \hat{T}\hat{T}a && \text{(by Prop. 1)} \\ &\leq e_X^\circ \cdot \hat{T}(a \cdot e_X^\circ) \cdot \hat{T}\hat{T}a \\ &\leq e_X^\circ \cdot \hat{T}(e_X^\circ \cdot \hat{T}a) \cdot \hat{T}\hat{T}a \\ &= e_X^\circ \cdot (Te_X)^\circ \cdot \hat{T}\hat{T}a \cdot \hat{T}\hat{T}a && \text{(by Prop. 1)} \\ &\leq e_X^\circ \cdot m_X \cdot \hat{T}\hat{T}(a \cdot a) && \text{(since } (Te_X)^\circ \leq m_X) \\ &\leq (e_X^\circ \cdot \hat{T}a) \cdot m_X \quad . && \square \end{aligned}$$

Of course, for $\mathcal{V} = \mathbf{2}$ and $T = \beta$ or γ we get back the adjunction considered in the Introduction. We now establish the $(\mathbb{T},\mathcal{V})$-generalization of the category **ModTop**: the category

$$(\mathbb{T},\mathcal{V})\text{-}\mathbf{ModCat}$$

has objects (X,a,c) with $(X,a) \in \mathcal{V}$-**Cat**, $(X,c) \in (\mathbb{T},\mathcal{V})$-**Cat**,and

$$a \cdot c \leq c \quad \text{and} \quad c \cdot \hat{T}a \leq c \quad ;$$

hence, in the language of [L], $c : T(X,a) \to (X,a)$ is a $\mathcal{V}$-module. A morphism in $(\mathbb{T},\mathcal{V})$-**ModCat** must simultaneously live in $\mathcal{V}$-**Cat** and in $(\mathbb{T},\mathcal{V})$-**Cat**.

Lemma 1. *For a $(\mathbb{T},\mathcal{V})$-category (X,c) and any $\mathcal{V}$-relation $a : X \nrightarrow X$, the following are equivalent:*

(i) $a \leq c \cdot e_X$;
(ii) $a \cdot c \leq c$ *and* $c \cdot \hat{T}a \leq c$;
(iii) $a \cdot c \leq c$.

Proof. (i) $\Rightarrow$ (ii). From (i) we obtain

$$a \cdot c \leq c \cdot e_X \cdot c \leq c \cdot \hat{T}c \cdot e_{TX} \leq c \cdot m_X \cdot e_{TX} = c \quad ,$$

$$c \cdot \hat{T}a \leq c \cdot \hat{T}(c \cdot e_X) = c \cdot \hat{T}c \cdot Te_X \leq c \cdot m_X \cdot Te_X = c \quad .$$

(iii) $\Rightarrow$ (i). From (iii) one derives $a \leq a \cdot c \cdot e_X \leq c \cdot e_X$. $\square$

The Lemma says in particular that (X, a, c) is in **ModTop** whenever the topological space (X, c) is provided with an order a that is contained in the (dual of the) specialization order given by (X, c), i.e. when $x \leq y$ w.r.t. a implies $\dot{x} \to y$ for all $x, y \in X$.

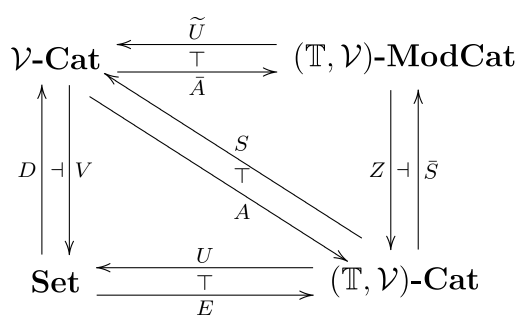
Theorem 1. $(\mathbb{T}, \mathcal{V})$**-ModCat** *is a topological category over* $\mathcal{V}$**-Cat***, and the adjunction* $A \dashv S$ *factors through* $(\mathbb{T}, \mathcal{V})$**-ModCat***.*

Proof. Initial structures with respect to the forgetful functor $\widetilde{U} : (\mathbb{T}, \mathcal{V})$**-ModCat** $\to$ $\mathcal{V}$**-Cat** are obtained by lifting those structures that are initial with respect to the topological functor $U : (\mathbb{T}, \mathcal{V})$**-Cat** $\to$ **Set**. Hence, having morphisms $f_i : (X, a) \to (Y_i, b_i)$ in $\mathcal{V}$**-Cat** with all $(Y_i, b_i, d_i) \in (\mathbb{T}, \mathcal{V})$**-ModCat**, one considers the U-initial $(\mathbb{T}, \mathcal{V})$-category structure

$$c = \bigwedge_i f_i^\circ \cdot d_i \cdot T f_i$$

on X which is easily seen to satisfy $a \cdot c \leq c$. Hence, by Lemma 1, (X, a, c) is an object of $(\mathbb{T}, \mathcal{V})$**-Cat** and provides the desired $\widetilde{U}$-initial lifting.

We now establish a commutative diagram of adjunctions



Here $\bar{S}$ and $\bar{A}$ are liftings of S and A:

$$\bar{S}(X, c) = (X, c \cdot e_X, c) \quad , \quad \bar{A}(X, a) = (X, a, e_X^\circ \cdot \hat{T} a) \quad .$$

The modularity condition for $\bar{S}(X, c)$ follows from

$$(c \cdot e_X) \cdot c \leq c \cdot \hat{T} c \cdot e_{TX} \leq c \cdot m_X \cdot e_{TX} = c \quad .$$

In the diagram, of the forgetful functors $U, \widetilde{U}, V, Z$, all but Z are topological. Trivially,

$$Z\bar{A} = A, \;\; \widetilde{U}\bar{S} = S, \;\; VS = U, \;\text{ and }\; AD = E : X \mapsto (X, e_X^\circ \cdot \hat{T} 1_X);$$

all other verifications are left to the reader. □

Since $\widetilde{U}\bar{A} = 1$ and $Z\bar{S} = 1$ we note:

Corollary 3. *$\bar{A}$ is a full coreflective embedding and $\bar{S}$ a full reflective embedding.*

We now revisit the adjunction

$$\mathcal{V}\textbf{-Cat} \underset{F^{\mathbb{T}}}{\overset{U^{\mathbb{T}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} (\mathcal{V}\textbf{-Cat})^{\mathbb{T}}$$

of Section 3 and show that it factors through $(\mathbb{T}, \mathcal{V})$**-ModCat** as well. To this end we consider the "composition functor"

$$C : (\mathcal{V}\textbf{-Cat})^{\mathbb{T}} \to (\mathbb{T}, \mathcal{V})\textbf{-ModCat} \quad , \quad (X, a, c) \mapsto (x, a, a \cdot c) \quad .$$

Since $\widetilde{U}C = U^{\mathbb{T}}$, it suffices to show that C has a left adjoint, and for that, according to the generalized version of Wyler's "Taut Lift Theorem" ([W1],[T1]), we just need to show that every source of morphisms in $(\mathcal{V}\textbf{-Cat})^{\mathbb{T}}$ factors into an epimorphism followed by a $U^{\mathbb{T}}$-initial family $f_i : (X, a, c) \to (Y, b_i, d_i)$ which is mapped to a $\widetilde{U}$-initial family by C. Indeed, standard factorization techniques show that the family (f_i) may be chosen to be monic (hence surely $U^{\mathbb{T}}$-initial) and $\widetilde{V}$-initial, so that $a = \bigwedge_i f_i^\circ \cdot b_i \cdot f_i$ (see Prop. 3). The C-image of (f_i) is $\widetilde{U}$-initial since $f_i : (X, a \cdot c) \to (Y_i, b_i \cdot d_i)$ is U-initial (see Thm. 1); indeed,

$$\bigwedge_i f_i^\circ \cdot (b_i \cdot d_i) \cdot T f_i = \bigwedge_i f_i^\circ \cdot b_i \cdot f_i \cdot c = a \cdot c \quad .$$

This shows that C has a left adjoint W, and it completes the proof of:

Theorem 2. *The adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ factors as*

$$\mathcal{V}\textbf{-Cat} \underset{\bar{A}}{\overset{\widetilde{U}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} (\mathbb{T}, \mathcal{V})\textbf{-ModCat} \underset{W}{\overset{C}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} (\mathcal{V}\textbf{-Cat})^{\mathbb{T}} \quad .$$

In summary, $(\mathcal{V}\textbf{-Cat})^{\mathbb{T}}$ behaves very nicely as a category over $\mathcal{V}$**-Cat**: the forgetful functor is monadic and factors through the topological category $(\mathbb{T}, \mathcal{V})$**-ModCat**. However, the situation is less satisfactory when we want to consider $(\mathcal{V}\textbf{-Cat})^{\mathbb{T}}$ as a category over $(\mathbb{T}, \mathcal{V})$**-Cat**, by composing C with the functor $Z : (\mathbb{T}, \mathcal{V})\textbf{-ModCat} \to (\mathbb{T}, \mathcal{V})\textbf{-Cat}$ of Theorem 1. In fact, since we are composing a right adjoint functor with a left adjoint, no good preservation properties are to be expected of the composite functor which, however, is still a concrete functor over $\mathcal{V}$**-Cat** since the diagram

$$
\begin{array}{ccc}
(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}} & \xrightarrow{\quad ZC \quad} & (\mathbb{T},\mathcal{V})\text{-}\mathbf{Cat} \\
& {\scriptstyle U^{\mathbb{T}}} \searrow \quad \swarrow {\scriptstyle S} & \\
& \mathcal{V}\text{-}\mathbf{Cat} &
\end{array}
$$

commutes. Indeed, $ZC(X,a,c) = (X, a \cdot c)$, hence $SZC(X,a,c) = (X, a \cdot c \cdot e_X) = (X,a)$. The reader should note that ZC does not simply forget structure. In fact, for $(X,a,c) \in (\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}}$, (X,c) will generally fail to be in $(\mathbb{T},\mathcal{V})$-**Cat**, unless the extension $\hat{T}$ is flat (as defined in 3). In that case, however, $\hat{T}c = Tc$ when c is a map, so that one has a functor

$$(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \to (\mathbb{T},\mathcal{V})\text{-}\mathbf{Cat} \quad , \quad (X,a,c) \mapsto (X,c) \quad .$$

The extension provided by Example 1 is flat when $T = \beta$ (and yields the forgetful functor **OrdCompHaus** $\to$ **Top**), but not when $T = \gamma$ or δ. A thorough discussion of the case $\mathcal{V} = \mathbf{2}$ and $T = \gamma$, where $\mathbf{Set}^{\mathbb{T}}$ is the category of continuous lattices (see [D], [W2], [GHKLMS]), must appear elsewhere.

5 More Lax Semantics

A *closed* $\mathcal{V}$*-structured* $(\mathbb{T},\mathcal{V})$*-category* (X,a,c) must satisfy $(X,a) \in \mathcal{V}$-**Cat**, $(X,c) \in (\mathbb{T},\mathcal{V})$-**Cat**, and

$$c \cdot \hat{T}a \le a \cdot c \quad \text{and} \quad \hat{T}(a \cdot c) \le \hat{T}a \cdot \hat{T}c \quad .$$

These are the objects of the category

$$(\mathbb{T},\mathcal{V})\text{-}\mathbf{CCat}$$

whose morphisms are maps that are morphisms in both $\mathcal{V}$-**Cat** and $(\mathbb{T},\mathcal{V})$-**Cat**. Before considering the standard example, let us first point out:

Proposition 5. $(\mathbb{T},\mathcal{V})$-**ModCat** *is a full bireflective subcategory of* $(\mathbb{T},\mathcal{V})$-**CCat**.

Proof. For $(X,a,c) \in (\mathbb{T},\mathcal{V})$-**ModCat** we have quite trivially

$$c \cdot \hat{T}a \le c = 1_X \cdot c \le a \cdot c \quad \text{and} \quad \hat{T}(a \cdot c) \le \hat{T}c = \hat{T}1_X \cdot \hat{T}c \le \hat{T}a \cdot \hat{T}c \quad .$$

The reflector K is (again) given by composition of structures: for $(X,a,c) \in (\mathbb{T},\mathcal{V})$-**CCat**, put $K(X,a,c) = (X,a,a\cdot c)$. This object lives in $(\mathbb{T},\mathcal{V})$-**ModCat**, and $f : (X,a,c) \to (Y,b,d)$ with $(Y,b,d) \in (\mathbb{T},\mathcal{V})$-**ModCat** is a morphism precisely when $f : (X,a,a \cdot c) \to (Y,b,d)$ is a morphism:

$$f \cdot c \le f \cdot a \cdot c \le b \cdot f \cdot c \le b \cdot d \cdot Tf \le d \cdot Tf \quad .$$

□

Example 3. For $\mathcal{V} = \mathbf{2}$ and $T = \beta$ extended as in Example 1, $(\mathbb{T}, \mathcal{V})$-**CCat** is the category **COrdTop** mentioned in the Introduction, characterized by the Closedness condition, i.e. property (i) of Prop. 4. (We note that for $T = \beta, \gamma$, or δ, the condition $\hat{T}(a \cdot c) \leq \hat{T}a \cdot \hat{T}c$ is redundant; for $T = \beta$ it is satisfied for all relations a, c since $\hat{T} : \mathbf{Rel} \to \mathbf{Rel}$ is actually a functor, but also for $T = \gamma$ or δ it still holds when a is reflexive and transitive.) The category **COrdTop** has products that are formed as in **Ord** and in **Top**, but the corresponding statement for equalizers fails. However, its full subcategory **COrdHaus** of those objects whose topology is Hausdorff is complete, with both products and equalizers formed as in **Ord** and in **Top**.

An *open* $\mathcal{V}$*-structured* $(\mathbb{T}, \mathcal{V})$*-category* (X, a, c) must satisfy $(X, a) \in \mathcal{V}$-**Cat**, $(X, c) \in (\mathbb{T}, \mathcal{V})$-**Cat**, and

$$a \cdot c \leq c \cdot \hat{T}a \quad .$$

A morphism of such objects is again a map that lives in both $\mathcal{V}$-**Cat** and $(\mathbb{T}, \mathcal{V})$-**Cat**. The resulting category is denoted by

$$(\mathbb{T}, \mathcal{V})\text{-}\mathbf{OCat} \quad .$$

Quite similarly to Proposition 5 one can prove:

Proposition 6. $(\mathbb{T}, \mathcal{V})$-**ModCat** *is a full bireflective subcategory of* $(\mathbb{T}, \mathcal{V})$-**OCat**.

Proof. The reflector L is given by $L(X, a, c) = (X, a, c \cdot \hat{T}a)$. Indeed, for a morphism $f : (X, a, c) \to (Y, b, d)$ in $(\mathbb{T}, \mathcal{V})$-**OCat** with $(Y, b, d) \in (\mathbb{T}, \mathcal{V})$-**ModCat** one has (with Prop. 1):

$$f \cdot (c \cdot \hat{T}a) \leq d \cdot Tf \cdot \hat{T}a \leq d \cdot \hat{T}(f \cdot a) \leq d \cdot \hat{T}(b \cdot f) = d \cdot \hat{T}b \cdot Tf \leq d \cdot Tf \quad .$$

□

Example 4. For $\mathcal{V} = \mathbf{2}$ and $T = \beta$ extended as in Example 1, $(\mathbb{T}, \mathcal{V})$-**OCat** is the category **OOrdTop** whose objects must satisfy the Openness condition for ultrafilters of the Introduction. These objects satisfy the preservation condition

$$U \text{ open} \quad \Rightarrow \quad \downarrow U \text{ open} \quad .$$

If we take $T = \gamma$, then $(\mathbb{T}, \mathcal{V})$-**OCat** contains precisely those topological spaces with an order such that the preservation condition is satisfied. Indeed, assuming the preservation condition, if $\mathfrak{x} \to x$ and $x \leq y$ for a filter $\mathfrak{x}$, the neighbourhood filter $\mathfrak{y} := \mathcal{N}_y$ of y trivially satisfies $\mathfrak{y} \to y$, and also $\mathfrak{x} \leq \mathfrak{y}$ since for every open neighbourhood U of y, the down-set $\downarrow U$ is an open neighbourhood of x and, hence, lies in $\mathfrak{x}$.

We also note that for $T = \gamma$, the category $(\mathbb{T}, \mathcal{V})$**-OCat** is easily seen to have products (formed as in **Ord** and in **Top**) but fails to have all equalizers.

6 Outlook

The generality of our approach reaches far beyond ordered topological structures. For $\mathcal{V} = \mathbf{2}$ and $T = 1$ (and $\hat{T} = 1$), our categories describe sets with two interacting orders, similarly to the interaction of convergence and order. One enters truly new territory when $\mathcal{V}$ is taken to be $\mathcal{P}_+$ (see Section 2). With $T = 1$, the category $(\mathbb{T}, \mathcal{V})$**-ModCat** becomes

$$\mathbf{ModMet}$$

whose objects (X, a, c) are such that $(X, a), (X, c) \in \mathbf{Met}$ and $c(x, y) \leq a(x, y)$ for all $x, y \in X$. (If a and c are metrics in the ordinary sense, by Lemma 1 this inequality means equivalently that $c : X \times X \to [0, \infty]$ as a function of the metric space (X, a) is non-expanding.) The functors $\bar{A}$ and $\bar{S}$ of Theorem 1 both describe the embedding

$$\mathbf{Met} \to \mathbf{ModMet} \quad , \quad (X, a) \mapsto (X, a, a) \quad ,$$

(and so does C of Theorem 2), which therefore is both reflective and coreflective: $(X, a, c) \mapsto (X, c)$ is the reflector, $(X, a, c) \mapsto (X, a)$ is the coreflector. (All of these statements remain valid for general $\mathcal{V}$.)

Via $(X, a, c) \mapsto (X, c^\circ, a^\circ)$ the categories $(\mathbb{T}, \mathcal{V})$**-CCat** and $(\mathbb{T}, \mathcal{V})$**-OCat** are easily seen to be isomorphic and described as

$$\mathbf{MetMet}$$

whose objects (X, a, c) must satisfy

$$\inf_u (c(x, u) + a(u, z)) \leq a(x, y) + c(y, z)$$

for all $x, y, z \in X$, in addition to $(X, a), (X, c) \in \mathbf{Met}$.

The next step now is to consider $\mathcal{V} = \mathcal{P}_+$, $T = \beta$, with the extension $\hat{T}r$ for $r : X \nrightarrow Y$ defined by

$$\hat{T}r(\mathfrak{x}, \mathfrak{y}) = \sup_{A \in \mathfrak{x}, B \in \mathfrak{y}} \inf_{x \in A, y \in B} r(x, y) \quad .$$

As first shown in [CH1], $(\mathbb{T},\mathcal{V})$**-Cat** is precisely Lowen's category **App** of approach spaces, which may therefore be thought of as sets X provided with a function $c : \beta X \times X \to [0,\infty]$ which measures "degrees of convergence" ($c(\mathfrak{x},x) = 0$ means "$\mathfrak{x}$ converges to x" while $c(\mathfrak{x},x) = \infty$ says "$\mathfrak{x}$ does not converge to x", but there is a continuum of intermediate degrees of convergence). Less esoterically, an approach space comes with a function $\delta : X \times \mathcal{P}X \to [0,\infty]$ which must satisfy a set of conditions that one naturally would expect point-set distances to satisfy (see [Lo]). Now, an object (X,a,c) of $(\mathbb{T},\mathcal{V})$**-ModCat** =

ModApp

is an approach space (X,c) provided with a metric a (i.e. $(X,a) \in$ **Met**) such that $c(\dot{x},y) \le a(x,y)$ for all $x,y \in X$.

A thorough investigation of this category and of its supercategories $(\mathbb{T},\mathcal{V})$**-CCat** and $(\mathbb{T},\mathcal{V})$**-OCat** must appear elsewhere.

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