Monoidal Topology

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Tychonoff's Theorem

$$\prod_{i \in I} X_i \text{ compact if all } X_i \text{ compact}$$

Proof:

Geometric Argument

Most Books: Engelking, ...

Involved

Convergence Argument:

Few Books: Willard, ...

Trivial

Why?

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Initial toplogy

$f_i : X \longrightarrow Y_i,\ Y_i \in \mathbf{Top}\ (i \in I)$

Geometric Description:

- Collect all $f_i^{-1}(V)$, $V \subseteq Y_i$ open
- **Generate** a topology from these!

Convergence Description:

- $\mathfrak{x} \to y :\Leftrightarrow \forall i \in I : f_i[\mathfrak{x}] \longrightarrow f_i(y)$

This **is** (the conv. of) a topology!

By contrast:

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$f_i : X_i \longrightarrow Y, X_i \in \mathbf{Top}\ (i \in I)$

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- $V \subseteq Y$ open:$\Leftrightarrow$ $\forall i \in I : f_i^{-1}(V) \subseteq X_i$ open

This **is** a topology!

Convergence Description:

- Collect all $f_i[\mathfrak{x}] \to f_i(y)$ for $\mathfrak{x} \to y$ in X_i

- **Generate** (the conv. of) a topology!

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Convergence Description:

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- **Generate** (the conv. of) a topology!

First conclusions by a categorical topologist

- Appreciate the importance of topological functors, such as **Top** $\longrightarrow$ **Set**, **Unif** $\longrightarrow$ **Set**, **TopGrp** $\longrightarrow$ **Grp**, ...
- While it is beautiful to have self-duality of topological functors: all "initials" (infs) exist $\Leftrightarrow$ all "finals" (sups) exist, ...
- ... it may not always be convenient to express infs in terms of sups, or conversely.
- Treat opens/closeds/neighbourhoods and convergence side by side!

This talk is about a categorical formalization of convergence that has many predecessors :
$\geq$ 1968: Manes, Wyler, Gähler, Möbus, Höhle, Flagg, Kopperman, ...
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The "Two-Axiom Miracle" in Algebra

Example: M-sets (M a monoid) $\quad M \times X \xrightarrow{a} X, \quad a(\alpha, x) = \alpha \cdot x$

$e_M \cdot x = x \qquad (\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x) \qquad f(\alpha \cdot x) = \alpha \cdot f(x)$

- $M \times X$ (with the obvious action) is the free M-set over a set X.
- Eilenberg-Moore: One may replace $M \times X$ by the free group, free ring, free Lie algebra, or any free algebra in a variety, to see that...
- ... the Two-Axiom Miracle continues throughout Algebra.

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$$\begin{array}{ccc} X & \xrightarrow{(e_M, 1_X)} & M \times X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} \\ & & X \end{array} \qquad \begin{array}{ccc} M \times M \times X & \xrightarrow{1_M \times a} & M \times X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ M \times X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} M \times X & \xrightarrow{1_M \times f} & M \times Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

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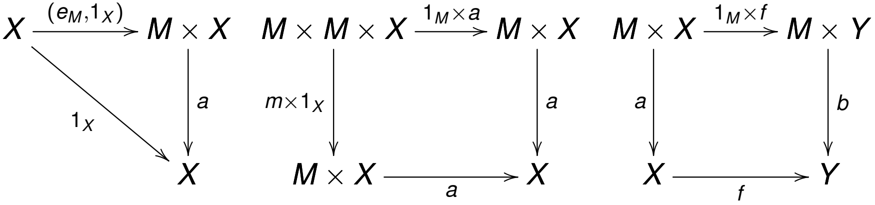

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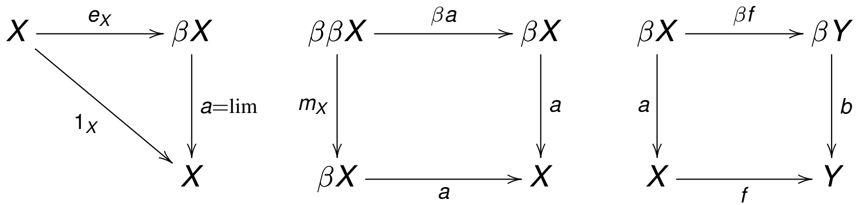
Manes 1968: compact Hausdorff spaces

Replace $M \times X$ by βX = set of ultrafilters on X:



$e_X(x) = \dot{x}$ $\quad$ $m_X(\mathfrak{X}) = \Sigma\mathfrak{X}$ ("Kowalsky sum"):

$A \in \Sigma\mathfrak{X} \Leftrightarrow \{\mathfrak{x} \in \beta X \mid A \in \mathfrak{x}\} \in \mathfrak{X}$

$A \in \beta a(\mathfrak{X}) = a[\mathfrak{X}]$ ("image" of $\mathfrak{X}$)

$\Leftrightarrow \{\mathfrak{x} \in \beta X \mid a(\mathfrak{x}) \in A\} \in \mathfrak{X}$

$\beta f(\mathfrak{x}) = f[\mathfrak{x}]$ ("image")

$B \in f[\mathfrak{x}] \Leftrightarrow f^{-1}(B) \in \mathfrak{x}$

$\lim \dot{x} = x$ $\quad$ $\lim(\lim \mathfrak{X}) = \lim \Sigma\mathfrak{X}$ $\quad$ $f(\lim \mathfrak{x}) = \lim(f[\mathfrak{x}])$

$\dot{x} \to x$ $\quad$ $\mathfrak{X} \to \mathfrak{y}$ and $\mathfrak{y} \to z \Rightarrow \Sigma\mathfrak{X} \to z$ $\quad$ $\mathfrak{x} \to y \Rightarrow f[\mathfrak{x}] \to f(y)$

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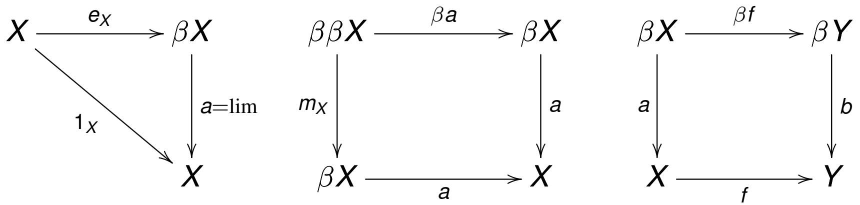

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Barr 1970: arbitrary topological spaces?

Replace the map $\beta X \xrightarrow{a} X$ by a relation $\beta X \overset{a}{\relbar\!\!\!+\!\!\!\rightarrow} X$.

Recall that a relation *a* is a map precisely when

- defined everywhere: existence of convergence points: compactness;
- defined uniquely: uniqueness of convergence points: Hausdorffness.

What are the axioms on *a* characterizing it as a topological convergence relation?

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a conv. rel. of a top. sp. $X \Leftrightarrow$

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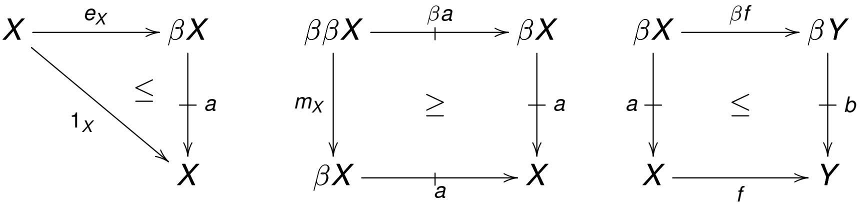

What does βa mean when a is just a relation?

More generally:
For a relation $r : X \rightarrow\!\!\!\!\!\!+\;\; Y$, what does $\beta r : \beta X \rightarrow\!\!\!\!\!\!+\;\; Y$ mean?

Present r as a *span* $r = \left(\begin{array}{ccc} & R & \\ {}^{r_1}\swarrow & & \searrow^{r_2} \\ X & & Y \end{array} \right)$

The *Barr extension* of β to a relation r is given by:

$$\begin{array}{ccc} & R & \\ {}^{r_1^\circ}\nearrow & & \searrow^{r_2} \\ X & \xrightarrow[r]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} & \beta R & \\ {}^{(\beta r_1)^\circ}\nearrow & & \searrow^{\beta r_2} \\ \beta X & \xrightarrow[\beta r]{} & \beta Y \end{array}$$

$$r = r_2 \cdot r_1^\circ \qquad \longmapsto \qquad \beta r := (\beta r_2) \cdot (\beta r_1)^\circ$$

$$x\, r\, y \Leftrightarrow \exists p \in R : \qquad\qquad \mathfrak{x}(\beta r)\mathfrak{y} :\Leftrightarrow \exists \mathfrak{p} \in \beta R :$$

$$r_1(p) = x \qquad\qquad r_1[\mathfrak{p}] = \mathfrak{x}$$

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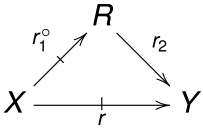

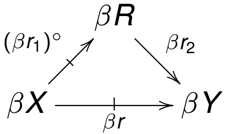

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Filters instead of ultrafilters?

YES One may replace βX by $\gamma X =$ set of filters on X and describe topological spaces with the *same* two axioms, but:

NO It is *not* sufficient to just mimic Barr's extension to relations!

More significantly:
One loses the ability to do meaningfully topology in this environment

See: Seal 2005, "*Monoidal Topology*" 2014

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From β to any **Set**-monad T

$e_X : X \longrightarrow TX$ nat. $\quad m_X : TTX \longrightarrow TX$ nat. $\quad Tf : TX \longrightarrow TY$ functorial

$+$ Two axioms making (T, m, e) look like a monoid: *"monad"*

$+$ Provision for "extending" T from maps to relations

T-relational spaces (X, a) and *continuous* maps $f : (X, a) \longrightarrow (Y, b)$:

$e_X(x)\, a\, x \quad (\mathfrak{X}\,(Ta)\,\mathfrak{y}$ and $\mathfrak{y}\, a\, z \Rightarrow m_X(\mathfrak{X})\, a\, z) \quad (\mathfrak{x}\, a\, y \Rightarrow Tf(\mathfrak{x})\, b\, f(y))$

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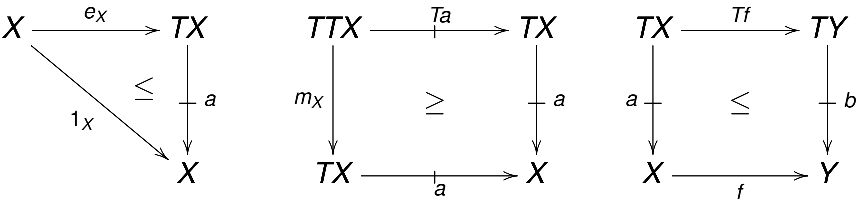

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$(T,2)$-**Cat**

$T = \mathrm{Id}$: **Ord** = (pre)ordered sets

$x \le x, \quad (x \le y \text{ and } y \le z \Rightarrow x \le z)$

$T = M \times (-)$: M-**Ord** = "M-ordered sets"

$x \le_{e_M} x, \quad (x \le_\alpha y \text{ and } y \le_\beta z \Rightarrow x \le_{\beta\alpha} z)$

$T = \beta$: **Top** = topological spaces

From Boolean relations to quantale-valued relations

$r : X \times Y \longrightarrow 2$ to become $r : X \times Y \longrightarrow V$ for
V unital (commutative) *quantale*
= complete lattice with monoid structure $V = (V, \otimes, k)$ s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i, \quad (\bigvee_{i \in I} v_i) \otimes u = \bigvee_{i \in I} v_i \otimes u$$

V-relational composition of $r : X \nrightarrow Y$ followed by $s : Y \nrightarrow Z$:

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} s(y, z) \otimes r(x, y)$$

- $V = 2 = \{0 < 1\}$ with $u \otimes v = u \wedge v, k = 1$
- $V = ([0, \infty], \geq)$ with $u \otimes v = u + v, k = 0$ (Lawvere 1973)
- $V = (2^M, \subseteq)$ with $A \otimes B = \{\alpha\beta \mid \alpha \in A, \beta \in B\}, k = \{e_M\}$

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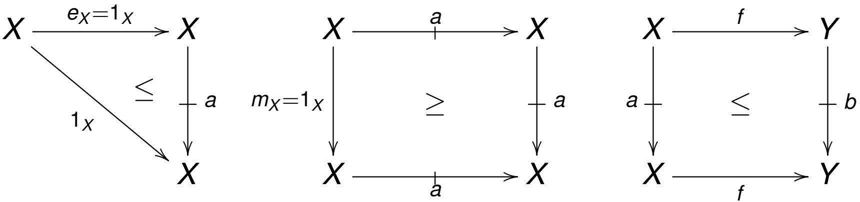
V-**Cat** = (Id, V)-**Cat** $\quad (T = \mathrm{Id})$



$1_X \leq a$	$a \cdot a \leq a$	$f \cdot a \leq b \cdot f$
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- $V = 2^M$: $\quad$ V-**Cat** = $(M \times (-), 2)$-**Cat** = M-ordered sets
- $V = [0, \infty]^{\mathrm{op}}$: $\quad$ V-**Cat** = **Met** = (generalized) metric spaces

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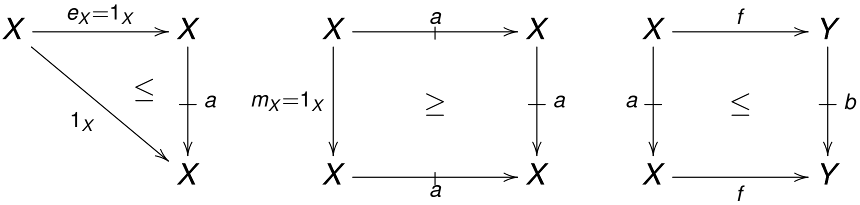

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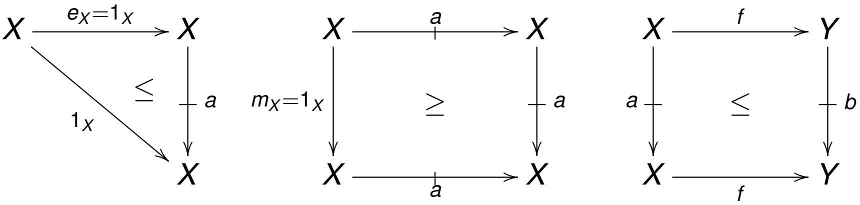

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Why "V-**Cat**"? Eilenberg and Kelly 1966

General case: $(V, \otimes, k)$ (symmetric) monoidal-closed category

A V-category (X, a) has a set X of objects with "hom-objects" $a(x, y) = \hom_X(x, y) \in V$ and V-arrows

$$k \longrightarrow a(x, x) \qquad\qquad a(y, z) \otimes a(x, y) \longrightarrow a(x, z)$$

subject to natural "monoidal" conditions

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Why "V-**Cat**"? Eilenberg and Kelly 1966

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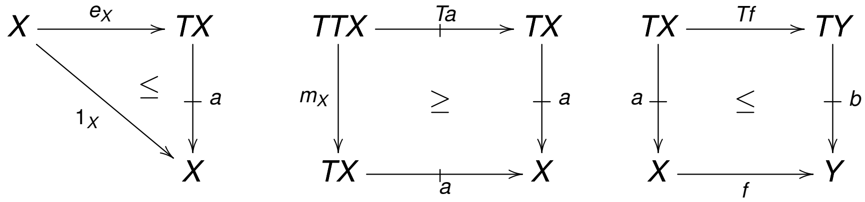

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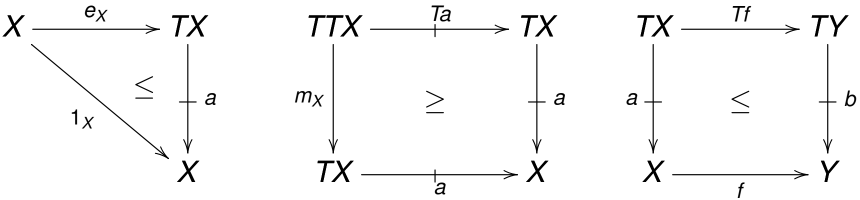

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Basic Theorem

•(T, V)-**Cat** is topological over **Set**, hence complete, cocomplete, *etc.*

• The forgetful functor has both a left- and a right adjoint (discrete and indiscrete structures);

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Initial structure a on X with respect to $f_i : X \longrightarrow (Y_i, b_i)$:

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Principal Examples

T/V	2	$[0,\infty]^{\mathrm{op}}$
Id	**Ord**	**Met**
β	**Top**	**App** = approach spaces: Lowen 1997
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Compact + Hausdorff is algebraic

T	V	(T,V)-**Cat**$_{\mathrm{Comp}}$	(T,V)-**Cat**$_{\mathrm{Haus}}$
Id	2	**Ord**	discrete ordered sets
Id	$[0,\infty]^{\mathrm{op}}$	**Met**	discrete (generalized) metric spaces
β	2	**Comp**	**Haus**
β	$[0,\infty]^{\mathrm{op}}$	**App**$_{0\text{-Comp}}$	approach spaces whose induced pseudotopology is Hausdorff

Manes' Theorem generalized:

$$(T,V)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} = (T,V)\text{-}\mathbf{Cat}_{\mathrm{Comp}} \cap (T,V)\text{-}\mathbf{Cat}_{\mathrm{Haus}} = \mathbf{Set}^{T}$$
$$= \text{ Eilenberg-Moore algebras w.r.t. } T$$

Proof (Lawvere, Clementino-Hofmann)
$(a \cdot a^{\circ} \leq 1_X \text{ and } 1_{TX} \leq a^{\circ} \cdot a) \Leftrightarrow a \dashv a^{\circ} \Leftrightarrow a$ is (induced by) a map.

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V completely distributive
$(\forall i \in I : X_i = (X_i, a_i) \text{ compact}) \Rightarrow (X, a) = \prod_{i \in I} X_i$ compact

Proof (Schubert 2005) For all $\mathfrak{z} \in TX$:

$$\bigvee_{x \in X} a(\mathfrak{z}, x) = \bigvee_{x \in X} \bigwedge_{i \in I} a_i(Tp_i(\mathfrak{z}), p_i(x)) = \bigwedge_{i \in I} \bigvee_{x_i \in X_i} a_i(Tp_i(\mathfrak{z}), x_i) \geq k$$

Equationally-def'd properties cont'd: T1, core-compact

$(X, a : TX \to\!\!\!\!\!+\, X)$

- $1_X \leq a \cdot e_X$ **T1:** $1_X \geq a \cdot e_X$
  $(T = \beta, V = 2 :)$ $(\dot{x} \to y \Rightarrow x = y)$

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Normal, extremally disconnected

Preparation: Induced "order"

on βX :
Top $\longrightarrow$ **Ord**
$X \mapsto (\beta X, \leq)$

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(T, V)-**Cat** $\longrightarrow$ V-**Cat**
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"Adjoint significance" of $\leq$:
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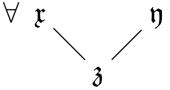

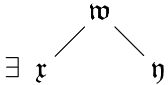

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(X, a) *extremally disconnected*

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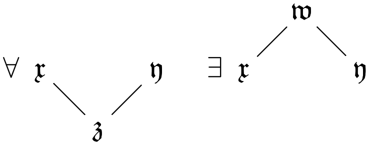

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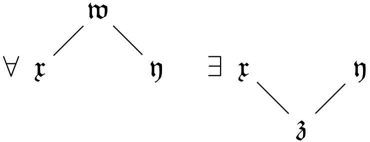

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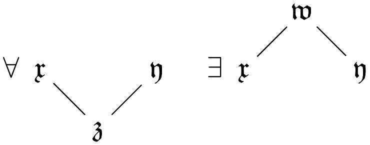

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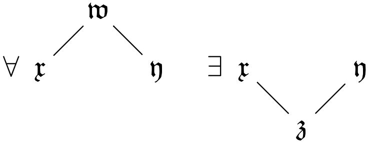

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The categorical imperative: What about morphisms?

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$f : X \longrightarrow Y$ $\qquad$ proper $\qquad$ open

Ord $=$ 2-**Cat**

$$\begin{array}{ccc} x & \leq & z \\ | & & \vdots \\ f(x) & \leq & y \end{array} \qquad \begin{array}{ccc} z & \leq & x \\ \vdots & & | \\ y & \leq & f(x) \end{array}$$

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Basic Stability Properties for proper/open maps

- Isomorphisms are proper/open
- proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\Leftarrow$ f proper/open
- $g \cdot f$ proper open, f surjective $\Rightarrow$ g proper/open

In addition: Proper/open is *stable under pullback:*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p_2} & Z \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

f proper/open $\Rightarrow$ p_2 proper/open

Kuratowski-Mrówka Theorem

Under mild hypotheses on T, V:

$(X, a) \longrightarrow 1$ proper $\quad\Leftrightarrow\quad (X, a)$ compact

Theorem (Clementino-T 2007)

$f : (X, a) \longrightarrow (Y, b)$ proper $\Leftrightarrow$
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(in **Top, App, ...**) $\Leftrightarrow$
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- X compact $\Leftrightarrow \forall Z : X \times Z \longrightarrow Z$ closed (equ'ly: proper)
- $(X \xrightarrow{f} Y)$ proper $\Leftrightarrow \forall (Z \longrightarrow Y) : (X \times_Y Z \longrightarrow Z)$ closed (proper)

Tychonoff-Frolík-Bourbaki Theorem

Conclusion: proper = fibred version of compact
Consequently: categorically proven statements for compact objects transfer to proper morphisms, and conversely.

Theorem: V completely distributive. Then:

$$f_i : X_i \longrightarrow Y_i \quad \text{proper} \quad (i \in I) \Rightarrow \prod_{i \in I} f_i : \prod_{i \in I} X_i \longrightarrow \prod_{i \in I} Y_i \quad \text{proper}$$

Note, by contrast (*not* by categorical dualization!):

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Some Remarks

Starting with an axiomatically given class of “closed morphisms” one establishes a categorical theory of compactness and Hausdorff separation:

Pénon 1972, T 1999,
Clementino-Giuli-T 2004, Clementino-Colebunders-T 2014, ...

Recently this approach has been exploited for the category **TopGrp** of topological groups by He-T, extending the Dikranjan-Uspenskij product theorem for categorically compact groups to *categorically proper* homomorphisms of topological groups.

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Q1: Do we really need the V of (T, V)-**Cat**?

YES:
All topological notions presented depend on T, V,
not just on (T, V)-**Cat**.
This is so already for **Top** when presented via filter convergence instead of ultrafilter convergence! But

NO:
It *is* possible to always replace V by 2 (*i.e.*, have no "fuzziness"!) if

- you are only interested in the category itself and
- you accept a more complicated T:

Theorem (Hofmann-Lowen 2014)
Given T, V, there is a monad $\Pi = \Pi(T, V)$ such that

$$(T, V)\text{-}\mathbf{Cat} \cong (\Pi, 2)\text{-}\mathbf{Cat}$$

Special case: $T = \mathrm{Id} \Rightarrow \Pi = P_V = V-$presheaf monad:

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Q2: Should one consider a quantaloid Q instead of V?

YES
First indication:
Take $Q = DV$ (Stubbe, Zhang, ...) and obtain:

$$(T, Q)\text{-}\mathbf{Cat} = \{\text{partial } (T, V)\text{-spaces}\}$$

In particular:

$$D[0, \infty]^{\mathrm{op}} - \mathbf{Cat} = \{\text{partial metric spaces}\}$$

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Your questions?

THANK YOU !