Approximate Composition – Another Approach to Quantitative Concept Analysis?

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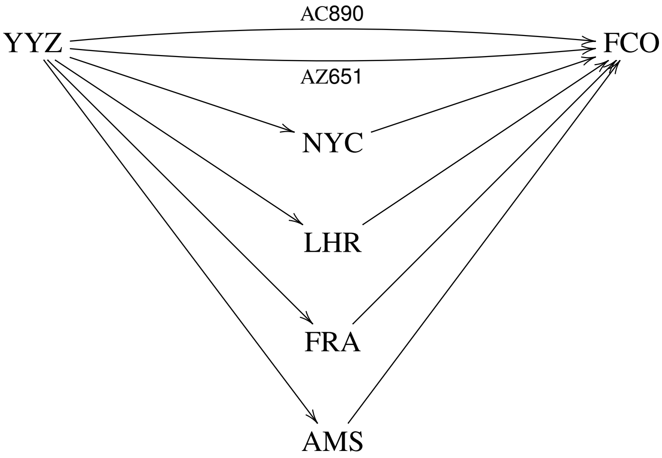
Searching Toronto to Rome: direct flights

.... and many more!

Searching Toronto to Rome: connecting flights



... and many more!

Basic structure: metric(ally enriched) graph $\mathbb{X}$

- $\mathrm{ob}\mathbb{X}$: vertices, objects $x, y, z, ...$ ("*airports*")
- $\mathbb{X}(x, y)$: edges, morphisms $f, f', ... : x \to y$ ("*direct flights $x \to y$*")
- $1_x : x \to x$: zero loop, identity morphism ("*staying grounded at x*")
- $d = d_{x,y}$ symmetric Lawvere metric ("*price difference*") on $\mathbb{X}(x, y)$:
  - $d(f, f) = 0$
  - $d(f, f') = d(f', f)$
  - $d(f, f'') \leq d(f, f') + d(f', f'')$

Note: for $f \neq f'$, $d(f, f') = 0$ or $d(f, f') = \infty$ are permitted.

Major shortcoming:
No comparison between between direct and connecting flights!

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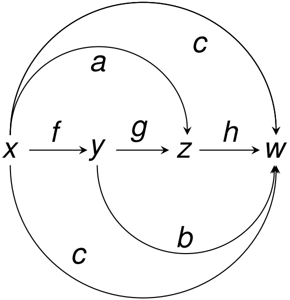
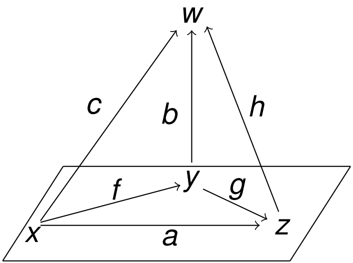
Richer structure: **metric approximate category** $\mathbb{X}$

- $\mathrm{ob}\mathbb{X},\ \mathbb{X}(x,y),\ 1_x \in \mathbb{X}(x,x)$
- "comparator" $\delta = \delta_{x,y,z} : \mathbb{X}(x,y) \times \mathbb{X}(y,z) \times \mathbb{X}(x,z) \to [0,\infty]$
  - $\delta(f, 1_y, f) = 0 = \delta(1_x, f, f)$
  - $|\delta(a,h,c) - \delta(f,b,c)| \leq \delta(f,g,a) + \delta(g,h,b)$





Geometrically: δ represents area, volume, content, ...

Very brief tributes

- Karl Menger 1928: n-metrics, simplex inequality
- Siegfried Gähler 1963: 2-metrics, tetrahedral inequality
- Sammy Eilenberg and Max Kelly 1966: enriched categories
- Bill Lawvere 1973: distances as homs, triangle inequality as composition law, metrically enriched categories
- Abdelkrim Aliouche and Carlos Simpson 2017: approximate categorical structures, “directed tetrahedral” inequalities
- WT and Jiyu (Gates) Wang 2019: (quantalic generalization of) metagories, Yoneda embedding for metagories
- WT 2019 (hopefully): approximate 2-categories

Even richer structure: metric(ally enriched) category $\mathbb{X}$

- category $\mathbb{X}$ with underlying metric graph
- composition $\mathbb{X}(x,y) \times \mathbb{X}(y,z) \to \mathbb{X}(x,z)$ is contractive:

$$d(g \cdot f, g' \cdot f') \leq d(f, f') + d(g, g')$$

Met-Cat→Metag: $\delta(f, g, a) := d(g \cdot f, a)$

Metag→Met-Gph: $d(f, f') := \delta(f, 1_y, f') = \delta(1_x, f, f')$

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Examples

- Every (general) 2-metric space (X, d) gives a chaotic metagory $\mathbb{X}$:

$$\mathrm{ob}\mathbb{X} = X,\ |\mathbb{X}(x, y)| = 1,\ \delta(x \to y, y \to z, x \to z) := d(x, y, z).$$

- In particular: $\mathbb{R}^n$ with its Euclidean 2-metric gives the metagory $\mathbb{R}^n$.
- We already saw: every metric category $\mathbb{X}$ is a metagory.
- In particular: **Met**, **Ban**$_1$, ..., are (large) metagories.

The first example describes a full embedding **2Met**$\to$**Metag**, with reflector

$$(\mathbb{X}, \delta) \mapsto (\mathrm{ob}\mathbb{X}, d),$$

$$d(x, y, z) = \inf\{\delta(f, g, a) \,|\, f : x \to y, g : y \to z, a : x \to z\}.$$

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The quantalic view

$\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathsf{k})$ commutative unital *quantale*, replacing Lawvere's

$$([0,\infty], \geq, +, 0) \cong ([0,1], \leq, \cdot, 1):$$

$(\mathcal{V}, \leq)$ complete lattice, $(\mathcal{V}, \otimes, \mathsf{k})$ commutative monoid,

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i.$$

Examples of principal interest:

- Lawvere quantale
- $2 = (\{0,1\}, \leq, \wedge, 1)$
- $\begin{aligned} \Delta \quad &= \{\text{probability distribribution functions } [0,\infty] \to [0,1]\} \\ &= [0,\infty] \oplus [0,1] \text{ in the cat. of comm. unital quantales} \end{aligned}$

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Tensor, internal hom

- $\mathcal{V}$ symmetric monoidal closed
- $\mathcal{V}$-**Cat** symmetric monoidal closed
- **Met**$_{\mathcal{V}}$ symmetric monoidal closed
- **Met**$_{\mathcal{V}}$-**Gph** symmetric monoidal closed
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$$d_{\mathbb{X}\otimes\mathbb{Y}}((f,g),(f',g')) = d_{\mathbb{X}}(f,f') \otimes d_{\mathbb{Y}}(g,g')$$

$(f, f' : x \to y$ in $\mathbb{X}$ and $g, g' : z \to w$ in $\mathbb{Y})$;

$$d_{[\mathbb{X},\mathbb{Y}]}(\alpha,\alpha') = \bigwedge_{x\in \mathrm{ob}\mathbb{X}} d_{\mathbb{Y}}(\alpha_x, \alpha'_x)$$

$(\alpha, \alpha' : F \to G$ nat. transf. of $\mathcal{V}$-contractive functors $F, G : \mathbb{X} \to \mathbb{Y})$.

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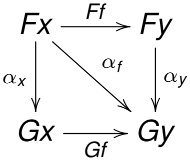
What about $\mathbf{Metag}_{\mathcal{V}}$?

- objects: (small) $\mathcal{V}$-metagories $\mathbb{X}, \mathbb{Y}$, ...
- morphisms: contractors $F : \mathbb{X} \to \mathbb{Y}$
- natural transformations $\alpha : F \to G$:



$$k \leq \delta(Ff, \alpha_y, \alpha_f) \quad \text{and} \quad k \leq \delta(\alpha_x, Gf, \alpha_f).$$

THEOREM: $\mathbf{Metag}_{\mathcal{V}}$ is symmetric monoidal closed!

$$1_F = (Ff)_{f:x \to y}, \quad \delta_{[\mathbb{X},\mathbb{Y}]}(\alpha, \beta, \gamma) = \bigwedge_{x \in \mathrm{ob}\mathbb{X}} \delta_{\mathbb{Y}}(\alpha_x, \beta_x, \gamma_x)$$

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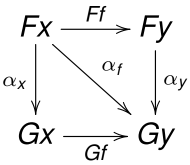

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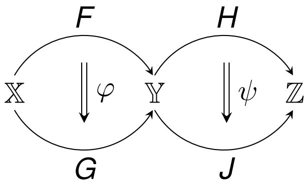
Self-enrichment

- **Cat** is a 2-category, *i.e.*, **Cat** is enriched in **Cat.**
- **Met**$_{\mathcal{V}}$-**Cat** is enriched in **Met**$_{\mathcal{V}}$-**Cat**:

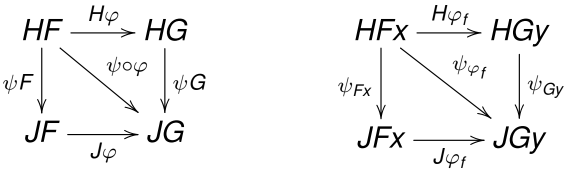
$$c_{\mathbb{X},\mathbb{Y},\mathbb{Z}} : [\mathbb{X}, \mathbb{Y}] \otimes [\mathbb{Y}, \mathbb{Z}] \to [\mathbb{X}, \mathbb{Z}],$$

$$(\varphi : F \to G,\ \psi : H \to J)\ \mapsto\ (\psi \circ \varphi : HF \to JG)$$



is $\mathcal{V}$-contractive!

Again: what about $\mathbf{Metag}_{\mathcal{V}}$?



If F, G, H, J are just contractors of $\mathcal{V}$-metagories, put

$$(\psi \circ \varphi)_f := \psi_{\varphi_f} : HFx \to JGy,$$

for all $f : x \to y$ in $\mathbb{X}$.

THEOREM: $\mathbf{Metag}_{\mathcal{V}}$ is enriched in $\mathbf{Metag}_{\mathcal{V}}$!

$\mathcal{V}$ as an internal monoid in $\mathbf{Met}_{\mathcal{V}}$-$\mathbf{Cat}$

Internal hom:

$$z \leq u \multimap v \iff z \otimes u \leq v$$

$\mathcal{V}$-metric:

$$d_{\mathcal{V}}(u, v) = (u \multimap v) \wedge (v \multimap u)$$

$\otimes : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ is $\mathcal{V}$-contractive:

$$d_{\mathcal{V}}(u, v) \otimes d_{\mathcal{V}}(w, z) \leq d_{\mathcal{V}}(u \otimes w, v \otimes z)$$

Strategy:

Expand these facts to $\mathcal{V}$-distributors (= $\mathcal{V}$-(bi)modules, $\mathcal{V}$-profunctors)!

Goal:
Embed a $\mathcal{V}$-metagory into a $\mathcal{V}$-metric category!

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$\mathcal{V}$-distributors of $\mathcal{V}$-metric spaces

$X, Y \quad \mathcal{V}$-metric spaces

$\varphi = (\varphi(x,y))_{x \in X, y \in Y}$ $\mathcal{V}$-*distributor* $\iff$ $\varphi(x,y) \in \mathcal{V}$ with

$$\varphi(x,y) \otimes d_Y(y,y') \leq \varphi(x,y'), \quad d_X(x',x) \otimes \varphi(x,y) \leq \varphi(x',y)$$

Composition of $\varphi : X \nrightarrow Y$ followed by $\psi : Y \nrightarrow Z$:

$$(\psi \circ \varphi)(x,z) = \bigvee_{y \in Y} \varphi(x,y) \otimes \psi(y,z)$$

The one-object category $\mathcal{V}$ embeds fully into $\mathbf{Dist}_{\mathcal{V}}$:

$$\mathcal{V} \to \mathbf{Dist}_{\mathcal{V}}, v \mapsto (v : \mathsf{I} \nrightarrow \mathsf{I})$$

What about its $\mathcal{V}$-metric structure?

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$\mathbf{Dist}_{\mathcal{V}}$ as a $\mathbf{Met}_{\mathcal{V}}$-enriched category

$\varphi, \varphi' : X \mapsto Y$:

$$d(\varphi, \varphi') := \bigwedge_{x \in X, y \in Y} d_{\mathcal{V}}(\varphi(x,y), \varphi'(x,y))$$

makes every $\mathbf{Dist}_{\mathcal{V}}(X, Y)$ a *separated* $\mathcal{V}$-metric space, that is

$$d(\varphi, \varphi') \geq \mathsf{k} \text{ only if } \varphi = \varphi'.$$

THEOREM:

For every (small) $\mathcal{V}$-metagory $\mathbb{X}$, the $\mathcal{V}$-metagory $[\mathbb{X}^{\mathrm{op}}, \mathbf{Dist}_{\mathcal{V}}]$ is (induced by) a separated $\mathcal{V}$-metric category.

Now we got to embed $\mathbb{X}$ into this $\mathcal{V}$-metric category! But:
Not every metagory may be isometrically embedded into a metric category! (Aliouche and Simpson)

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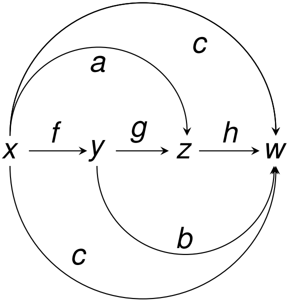
Transitive $\mathcal{V}$-metagories



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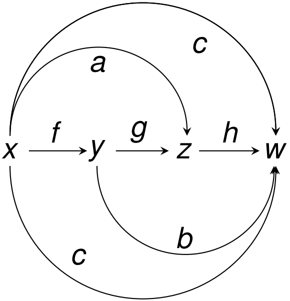

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Yoneda

THEOREM

$\mathbb{X}$ transitive $\mathcal{V}$-metagory. Then there is an isometric $\mathcal{V}$-contractor

$$\mathsf{y} : \mathbb{X} \to [\mathbb{X}^{\mathrm{op}}, \mathbf{Dist}_{\mathcal{V}}], \quad w \mapsto \quad (\mathsf{y}_w : \mathbb{X}^{\mathrm{op}} \to \mathbf{Dist}_{\mathcal{V}}, x \mapsto \mathbb{X}(x, w))$$

mapping $\mathbb{X}$ into a separated $\mathcal{V}$-metric category, as follows:
For every object w and $f : x \to y$ in $\mathbb{X}$, one has the $\mathcal{V}$-distributor

$$\mathsf{y}_w(f) : \mathbb{X}(y, w) \leftrightarrow \mathbb{X}(x, w), \quad \mathsf{y}_w(f)(b, c) = \delta(f, b, c),$$

for all $b : y \to w, c : x \to w$; this gives the contractor y_w;
for $m : w \to v$ in $\mathbb{X}$ one has the natural transf. $\mathsf{y}_m : \mathsf{y}_w \to \mathsf{y}_v$ with

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y maps objects injectively; same for morphisms if $\mathbb{X}$ is separated.

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Comparison with the standard $\mathcal{V}$-Yoneda

If $\mathbb{X}$ is a $\mathcal{V}$-metric category:

$$/ < -`- > `- > / < 500, 300 > [[\mathbb{X}^{\mathrm{op}}, \mathbf{Met}_{\mathcal{V}}]`\mathbb{X}`[\mathbb{X}^{\mathrm{op}}, \mathbf{Dist}_{\mathcal{V}}]; \tilde{\mathrm{y}}``\mathrm{y}]$$

Note: generally, y is not full!

The isometry $\mathbb{X} \to \mathrm{Path}\mathbb{X}$

COROLLARY

The unit $\mathbb{X} \to \mathrm{Path}\mathbb{X}$ of the right-adjoint functor

$$\mathbf{Met}_{\mathcal{V}}\text{-}\mathbf{Cat} \to \mathbf{Metag}_{\mathcal{V}}$$

at the transitive $\mathcal{V}$-metagory $\mathbb{X}$ is an isometry.

$$/ < -`- > `- > / < 500, 300 > [\mathrm{Path}\mathbb{X}`\mathbb{X}`[\mathbb{X}^{\mathrm{op}}, \mathbf{Dist}_{\mathcal{V}}].; ``y]$$

The metric category Path $\mathbb{R}^n$ (Aliouche-Simpson)

Warning:

A 2-metric space, seen as a metagory, is generally **not** transitive!

In particular: $\mathbb{R}^n$ is not transitive!

Nevertheless:

The metagory Path $\mathbb{R}^n$ is metric!

Sufficient conditions for transitivity

For a $\mathcal{V}$-metagory $\mathbb{X}$, consider:

(i) $\mathbb{X}$ is (induced by) a $\mathcal{V}$-metric category;

(ii) for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\mathsf{k} \leq \delta(f, g, a)$;

(iii) for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, $\mathsf{k} \leq \bigvee_{a:x\to z} \delta(f, g, a) \otimes \delta(f, g, a)$;

(iv) $\mathbb{X}$ is transitive.

Then $(\mathrm{i}) \Longrightarrow (\mathrm{ii}) \Longrightarrow (\mathrm{iii}) \Longrightarrow (\mathrm{iv})$, and $(\mathrm{i}) \Longleftrightarrow (\mathrm{ii})$ if $\mathbb{X}$ is separated.

Furthermore, if $\mathcal{V}$ satisfies $\bigvee\{\varepsilon \mid \varepsilon \ll \mathsf{k}\} = \mathsf{k}$, then (iii) is equivalent to

(iii′) for all $\varepsilon \ll \mathsf{k}$ in $\mathcal{V}$, $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a) \otimes \delta(f, g, a)$.

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Then $\mathrm{(i)} \Longrightarrow \mathrm{(ii)} \Longrightarrow \mathrm{(iii)} \Longrightarrow \mathrm{(iv)}$, and $\mathrm{(i)} \iff \mathrm{(ii)}$ if $\mathbb{X}$ is separated.
Furthermore, if $\mathcal{V}$ satisfies $\bigvee\{\varepsilon \,|\, \varepsilon \ll \mathsf{k}\} = \mathsf{k}$, then $\mathrm{(iii)}$ is equivalent to

(iii′) for all $\varepsilon \ll \mathsf{k}$ in $\mathcal{V}$, $f : x \to y, g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a) \otimes \delta(f, g, a)$.

Sufficient conditions for transitivity, continued

Let $\varepsilon \in \mathcal{V}$.

A $\mathcal{V}$-metagory $\mathbb{X}$ is ε-*categorical* if for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \le \delta(f, g, a)$.

THEOREM:
Assume that the quantale $\mathcal{V}$ satisfies $\bigvee\{\varepsilon \otimes \varepsilon \,|\, \varepsilon \ll \mathsf{k}\} = \mathsf{k}$. Then

$$\mathsf{k}\text{-}\mathbf{Metag}_{\mathcal{V}} \subseteq \bigcap_{\varepsilon \ll \mathsf{k}} \varepsilon\text{-}\mathbf{Metag}_{\mathcal{V}} \subseteq \mathbf{TransMetag}_{\mathcal{V}}.$$

For $\mathcal{V}$-metagories $\mathbb{X}$ and $\mathbb{Y}$, if $\mathbb{Y}$ is k-categorical, so is $[\mathbb{X}, \mathbb{Y}]$. If both $\mathbb{X}$ and $\mathbb{Y}$ are k-categorical or transitive, $\mathbb{X} \otimes \mathbb{Y}$ has the respective property; likewise for the property of being ε-categorical for all $\varepsilon \ll \mathsf{k}$.

Sufficient conditions for transitivity, continued

Let $\varepsilon \in \mathcal{V}$.

A $\mathcal{V}$-metagory $\mathbb{X}$ is ε-*categorical* if for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a)$.

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Sufficient conditions for transitivity, continued

Let $\varepsilon \in \mathcal{V}$.

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For $\mathcal{V}$-metagories $\mathbb{X}$ and $\mathbb{Y}$, if $\mathbb{Y}$ is k-categorical, so is $[\mathbb{X}, \mathbb{Y}]$. If both $\mathbb{X}$ and $\mathbb{Y}$ are k-categorical or transitive, $\mathbb{X} \otimes \mathbb{Y}$ has the respective property; likewise for the property of being ε-categorical for all $\varepsilon \ll \mathsf{k}$.

Back to the frequent flyer

Let $\mathbb{X}$ be a metagory given by airports, "sufficiently many" flights between them, and a comparator function for direct versus connecting flights.

Then Path$\mathbb{X}$ carries a *metric* structure into which $\mathbb{X}$ is isometrically embedded as a submetagory.

Thanks!