Monoidal Topology: Advances and Challenges

Walter Tholen

York University, Toronto

Overview

- Categories: internal *vs* enriched

Overview

- Categories: internal *vs* enriched
- Lambek’s multicategories and Burroni’s T-categories and T-orders

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories
- (Monad, Quantale)-enriched categories *vs* T-orders

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories
- (Monad, Quantale)-enriched categories *vs* T-orders
- The fundamental adjunction of Monoidal Topology

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories
- (Monad, Quantale)-enriched categories *vs* T-orders
- The fundamental adjunction of Monoidal Topology
- Some topological properties, defined categorically

Overview

- Categories: internal *vs* enriched
- Lambek's multicategories and Burroni's T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories
- (Monad, Quantale)-enriched categories *vs* T-orders
- The fundamental adjunction of Monoidal Topology
- Some topological properties, defined categorically
- Trading convergence relations for closure operations

Overview

- Categories: internal *vs* enriched
- Lambek’s multicategories and Burroni’s T-categories and T-orders
- Some topological concepts for T-orders
- From enriched multicategories to (Monad, Quantale)-enriched categories
- (Monad, Quantale)-enriched categories *vs* T-orders
- The fundamental adjunction of Monoidal Topology
- Some topological properties, defined categorically
- Trading convergence relations for closure operations
- Challenges, questions, problems

What is a (small) category?

$$\begin{tikzcd}
 & & A \times_X A \arrow[dl, "d'"'] \arrow[dr, "c'"] & & \\
 & A \arrow[dl, "d"'] \arrow[dr, "c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\
X & & X & & X
\end{tikzcd}$$

$$A \xrightarrow{(d,c)} X \times X \qquad \Longleftrightarrow \qquad X \times X \xrightarrow{\hom_A} \mathrm{Set}$$

What is a (small) category?

$$\begin{tikzcd}
& & A \times_X A \arrow[dl, "d'"'] \arrow[dr, "c'"] & & \\
& A \arrow[dl, "d"'] \arrow[dr, "c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\
X & & X & & X
\end{tikzcd}$$

$$A \xrightarrow{(d,c)} X \times X \quad \Longleftrightarrow \quad X \times X \xrightarrow{\hom_A} \mathrm{Set}$$

$$X \xrightarrow{i} A \quad \Longleftrightarrow \quad 1 \xrightarrow{i_x} \hom_A(x, x)$$

What is a (small) category?

$$\begin{tikzcd}
 & & A \times_X A \arrow[dl, "d'"'] \arrow[dr, "c'"] & & \\
 & A \arrow[dl, "d"'] \arrow[dr, "c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\
X & & X & & X
\end{tikzcd}$$

$$A \xrightarrow{(d,c)} X \times X \qquad \Longleftrightarrow \qquad X \times X \xrightarrow{\hom_A} \mathrm{Set}$$

$$X \xrightarrow{i} A \qquad \Longleftrightarrow \qquad 1 \xrightarrow{i_x} \hom_A(x, x)$$

$$A \times_X A \xrightarrow{m} A \qquad \Longleftrightarrow \qquad \hom_A(x, y) \times \hom_A(y, z) \xrightarrow{m_{x,y,z}} \hom_A(x, z)$$

Internal category theory vs Enriched category theory

A (with X, d, c, i, m) category

internal to a category $\mathcal{C}$

with pullbacks,

rather than just $\mathcal{C} = \textbf{Set}$

A (with $\hom_A = A(-,-), i, m$) category

enriched in a category $(\mathcal{V}, \otimes, k)$ that is

(symmetric) monoidal (closed),

rather than just $(\mathcal{V}, \otimes, \mathrm{k}) = (\textbf{Set}, \times, 1)$

Internal category theory vs Enriched category theory

A (with X, d, c, i, m) category
internal to a category $\mathcal{C}$
with pullbacks,
rather than just $\mathcal{C} = \mathbf{Set}$

A (with $\hom_A = A(-,-), i, m$) category
enriched in a category $(\mathcal{V}, \otimes, k)$ that is
(symmetric) monoidal (closed),
rather than just $(\mathcal{V}, \otimes, \mathrm{k}) = (\mathbf{Set}, \times, 1)$

With the internal and enriched notions of functor we obtain the categories

$\mathrm{Cat}(\mathcal{C})$

$\mathcal{V}$-**Cat**

Of course: $\mathrm{Cat}(\mathbf{Set}) = \mathbf{Cat} = \mathbf{Set}\text{-}\mathbf{Cat}$

Internal category theory vs Enriched category theory

A (with X, d, c, i, m) category internal to a category $\mathcal{C}$ with pullbacks, rather than just $\mathcal{C} = \mathbf{Set}$

A (with $\hom_A = A(-,-), i, m$) category enriched in a category $(\mathcal{V}, \otimes, k)$ that is (symmetric) monoidal (closed), rather than just $(\mathcal{V}, \otimes, \mathrm{k}) = (\mathbf{Set}, \times, 1)$

With the internal and enriched notions of functor we obtain the categories

$\mathrm{Cat}(\mathcal{C})$

$\mathcal{V}$-**Cat**

Of course: $\mathrm{Cat}(\mathbf{Set}) = \mathbf{Cat} = \mathbf{Set}\text{-}\mathbf{Cat}$

But already for the (cartesian-closed) category $\mathbf{Ord} = \{\text{pre-ordered sets}\}$:

$$\mathrm{Cat}(\mathbf{Ord}) \neq \mathbf{Ord}\text{-}\mathbf{Cat}$$

Internal category theory vs Enriched category theory

A (with X, d, c, i, m) category
internal to a category $\mathcal{C}$
with pullbacks,
rather than just $\mathcal{C} = \mathbf{Set}$

A (with $\hom_A = A(-,-), i, m$) category
enriched in a category $(\mathcal{V}, \otimes, k)$ that is
(symmetric) monoidal (closed),
rather than just $(\mathcal{V}, \otimes, \mathrm{k}) = (\mathbf{Set}, \times, 1)$

With the internal and enriched notions of functor we obtain the categories

$\mathrm{Cat}(\mathcal{C})$ $\mathcal{V}\text{-}\mathbf{Cat}$

Of course: $\mathrm{Cat}(\mathbf{Set}) = \mathbf{Cat} = \mathbf{Set}\text{-}\mathbf{Cat}$

But already for the (cartesian-closed) category $\mathbf{Ord} = \{$pre-ordered sets$\}$:

$$\mathrm{Cat}(\mathbf{Ord}) \neq \mathbf{Ord}\text{-}\mathbf{Cat}$$

And: $\mathrm{Cat}(\mathbf{Cat}) = \{$(strict) double cats$\} \neq \{$(strict) 2-cats$\} = \mathbf{Cat}\text{-}\mathbf{Cat}$

Playing the same game with multicategories

$$\begin{tikzcd}
 & & & \mathrm{L}A \times_{\mathrm{L}X} A \arrow[dl, "d'"'] \arrow[dr, "c'"] & & \\
 & & \mathrm{L}A \arrow[dl, "\mathrm{L}d"'] \arrow[dr, "\mathrm{L}c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\
 & \mathrm{L}\mathrm{L}X \arrow[dl, "\mu_X = \text{concatenation}"'] & & \mathrm{L}X & & X \\
\mathrm{L}X & & & & &
\end{tikzcd}$$

Playing the same game with multicategories

$$\begin{tikzcd}
 & & & \mathrm{L}A \times_{\mathrm{L}X} A \arrow[dl, "d'"'] \arrow[dr, "c'"] & & \\
 & & \mathrm{L}A \arrow[dl, "\mathrm{L}d"'] \arrow[dr, "\mathrm{L}c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\
 & \mathrm{L}\mathrm{L}X \arrow[dl, "\mu_X = \text{concatenation}"'] & & \mathrm{L}X & & X \\
\mathrm{L}X & & & & &
\end{tikzcd}$$

$$A \xrightarrow{(d,c)} \mathrm{L}X \times X \qquad \Longleftrightarrow \qquad \mathrm{L}X \times X \xrightarrow{\mathrm{hom}_A} \mathsf{Set}$$

$$X \xrightarrow{i} A \qquad \Longleftrightarrow \qquad 1 \xrightarrow{i_x} \mathrm{hom}_A(\eta_X(x), x) = \mathrm{hom}_A((x), x)$$

The composition law needs a closer look!

$$\mathcal{X} = (\overline{x_1}, \ldots, \overline{x_n}) \in \mathrm{LL}X, \ \overline{y} = (y_1, \ldots, y_n) \in \mathrm{L}X, \ z \in X$$

$$\begin{tikzcd}
\overline{x_1} \arrow[r, "f_1"] \arrow[d, dotted, no head] & y_1 \arrow[d, dotted, no head] \\
\overline{x_n} \arrow[r, "f_n"] & y_n
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\overline{f}"] & \overline{y} \arrow[r, "g"] & z
\end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd}
\mu_X(\mathcal{X}) \arrow[r, "g\cdot\overline{f}"] & z
\end{tikzcd}$$

The composition law needs a closer look!

$$\mathcal{X} = (\overline{x_1}, \ldots, \overline{x_n}) \in \mathrm{LL}X,\ \overline{y} = (y_1, \ldots, y_n) \in \mathrm{L}X,\ z \in X$$

$$\begin{tikzcd}
\overline{x_1} \arrow[r, "f_1"] \arrow[d, dotted, no head] & y_1 \arrow[d, dotted, no head] \\
\overline{x_n} \arrow[r, "f_n"] & y_n
\end{tikzcd}$$

$$\mathcal{X} \xrightarrow{\overline{f}} \overline{y} \xrightarrow{g} z \qquad \Rightarrow \qquad \mu_X(\mathcal{X}) \xrightarrow{g \cdot \overline{f}} z$$

$$\mathrm{L}A \times_{\mathrm{L}X} A \xrightarrow{m} A \quad \Longleftrightarrow \quad \hom_A(\mathcal{X}, \overline{y}) \times \hom_A(\overline{y}, z) \xrightarrow{m_{\mathcal{X},\overline{y},z}} \hom_A(\mu_X(\mathcal{X}), z)$$

The composition law needs a closer look!

$\mathcal{X} = (\overline{x_1}, \dots, \overline{x_n}) \in \mathrm{LL}X,\ \overline{y} = (y_1, \dots, y_n) \in \mathrm{L}X,\ z \in X$

$$\begin{tikzcd}
\overline{x_1} \arrow[r, "f_1"] \arrow[d, dotted, no head] & y_1 \arrow[d, dotted, no head] \\
\overline{x_n} \arrow[r, "f_n"] & y_n
\end{tikzcd}$$

$$\mathcal{X} \xrightarrow{\overline{f}} \overline{y} \xrightarrow{g} z \qquad \Rightarrow \qquad \mu_X(\mathcal{X}) \xrightarrow{g \cdot \overline{f}} z$$

$$\mathrm{L}A \times_{\mathrm{L}X} A \xrightarrow{m} A \quad \Longleftrightarrow \quad \mathrm{hom}_A(\mathcal{X}, \overline{y}) \times \mathrm{hom}_A(\overline{y}, z) \xrightarrow{m_{\mathcal{X},\overline{y},z}} \mathrm{hom}_A(\mu_X(\mathcal{X}), z)$$

NOTE: On the RHS, we first had to *define* what $\mathrm{hom}_A(\mathcal{X}, \overline{y})$ stands for.
In particular: $\mathcal{X}$ and $\overline{y}$ had to have the same length to make $\mathrm{hom}_A(\mathcal{X}, \overline{y}) \neq \emptyset$!

Burroni 1971: How to internalize multicategories ...

$\mathcal{C}$ a category with pullbacks, $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ any monad on $\mathcal{C}$. Define the category

$$\mathbf{Cat}(\mathrm{T})$$

Objects are (small) T-categories which are monoids in a bicategory of T-spans in $\mathcal{C}$; explicitly, they have an "object of objects" X and an "object of arrows" A, plus

$$\begin{tikzcd}
& X \arrow[dl, "\eta_X"'] \arrow[d, "i"] \arrow[dr, "1_X"] & \\
\mathrm{T}X & A \arrow[l, "d"'] \arrow[r, "c"] & X \\
\mathrm{T}A \arrow[u, "\mu_X \cdot \mathrm{T}d"] & \mathrm{T}A \times_{\mathrm{T}X} A \arrow[l, "d'"'] \arrow[r, "c'"] \arrow[u, "m"'] & A \arrow[u, "c"']
\end{tikzcd}$$

subject to (somewhat cumbersome) unity and associativity laws.

… and multifunctors

Morphisms are T-functors which are *more* than morphisms of such monoids; rather:

$$\begin{tikzcd}
 & \mathrm{T}A \times_{\mathrm{T}X} A \arrow[dl, "\mu_X \cdot \mathrm{T}d \cdot d'"'] \arrow[dr, "m^A"] \arrow[dd, "\mathrm{T}f \times_{\mathrm{T}f_o} f"'] & & X \arrow[dl, "i^A"'] \arrow[dr, "1_X"] \arrow[dd, "f_o"] & \\
\mathrm{T}X \arrow[dd, "\mathrm{T}f_o"'] & & A \arrow[ll, "d"'] \arrow[rr, "c"] \arrow[dd, "f"] & & X \arrow[dd, "f_o"] \\
 & \mathrm{T}B \times_{\mathrm{T}Y} B \arrow[dl, "\mu_Y \cdot \mathrm{T}d \cdot d'"'] \arrow[dr, "m^B"] & & Y \arrow[dl, "i^B"'] \arrow[dr, "1_Y"] & \\
\mathrm{T}Y & & B \arrow[ll, "d"'] \arrow[rr, "c"] & & Y
\end{tikzcd}$$

… and multifunctors

Morphisms are T-functors which are *more* than morphisms of such monoids; rather:

$$\begin{tikzcd}
 & \mathrm{T}A\times_{\mathrm{T}X}A \arrow[dl, swap, "\mu_X\cdot \mathrm{T}d\cdot d'"] \arrow[dr, "m^A"] \arrow[dd, "\mathrm{T}f\times_{\mathrm{T}f_o}f"] & & X \arrow[dl, swap, "i^A"] \arrow[dr, "1_X"] \arrow[dd, "f_o"] & \\
\mathrm{T}X \arrow[dd, swap, "\mathrm{T}f_o"] & & A \arrow[ll, swap, "d"] \arrow[rr, "c"] \arrow[dd, "f"] & & X \arrow[dd, "f_o"] \\
 & \mathrm{T}B\times_{\mathrm{T}Y}B \arrow[dl, "\mu_Y\cdot \mathrm{T}d\cdot d'"] \arrow[dr, "m^B"] & & Y \arrow[dl, swap, "i^B"] \arrow[dr, "1_Y"] & \\
\mathrm{T}Y & & B \arrow[ll, swap, "d"] \arrow[rr, "c"] & & Y
\end{tikzcd}$$

One may, however, set up a double category of T-spans in $\mathcal{C}$ such that T-functors are *precisely* homomorphisms of monoids.

Some properties, not necessarily all easy to prove

- The object-of-objects functor $\mathrm{Cat}(\mathsf{T}) \longrightarrow \mathcal{C},\ (X, A) \longmapsto X,$ is a Grothendieck fibration, with a left adjoint; it has a right adjoint if $\mathcal{C}$ has binary products.

Some properties, not necessarily all easy to prove

- The object-of-objects functor $\mathrm{Cat}(\mathrm{T}) \longrightarrow \mathcal{C},\ (X, A) \longmapsto X,$ is a Grothendieck fibration, with a left adjoint; it has a right adjoint if $\mathcal{C}$ has binary products.
- $\mathrm{Cat}(\mathrm{T})$ has (not easily constructed) pullbacks and is even finitely complete when $\mathcal{C}$ is; that is, when, other than pullbacks, $\mathcal{C}$ has a terminal object.
  If $\mathcal{C}$ is complete, so is $\mathrm{Cat}(\mathrm{T})$.

Some properties, not necessarily all easy to prove

- The object-of-objects functor Cat(T) $\longrightarrow \mathcal{C},\ (X, A) \longmapsto X,$ is a Grothendieck fibration, with a left adjoint; it has a right adjoint if $\mathcal{C}$ has binary products.
- Cat(T) has (not easily constructed) pullbacks and is even finitely complete when $\mathcal{C}$ is; that is, when, other than pullbacks, $\mathcal{C}$ has a terminal object.
  If $\mathcal{C}$ is complete, so is Cat(T).
- The formation of Cat(T) is functorial in T (and $\mathcal{C}$); in particular, the monad morphism $\eta : \mathrm{Id} \longrightarrow$ T induces a functor Cat(T) $\longrightarrow$ Cat($\mathcal{C}$):
  every T-category in $\mathcal{C}$ may be restricted to become an internal category in $\mathcal{C}$.

Some properties, not necessarily all easy to prove

- The object-of-objects functor Cat(T) $\longrightarrow \mathcal{C},\ (X, A) \longmapsto X,$ is a Grothendieck fibration, with a left adjoint; it has a right adjoint if $\mathcal{C}$ has binary products.
- Cat(T) has (not easily constructed) pullbacks and is even finitely complete when $\mathcal{C}$ is; that is, when, other than pullbacks, $\mathcal{C}$ has a terminal object.
  If $\mathcal{C}$ is complete, so is Cat(T).
- The formation of Cat(T) is functorial in T (and $\mathcal{C}$); in particular, the monad morphism $\eta : \mathrm{Id} \longrightarrow$ T induces a functor Cat(T) $\longrightarrow$ Cat($\mathcal{C}$):
  every T-category in $\mathcal{C}$ may be restricted to become an internal category in $\mathcal{C}$.

Theorem (T-Yeganeh 2021)

Cat(T) *has a (Street-Walters) comprehensive factorization system, provided that* $\mathcal{C}$ *has stable reflexive coequalizers that are preserved by* T.

(Surprisingly, no preservation of pullbacks by T is required!)

: Cat($\mathcal{C}$) has such a system, provided that $\mathcal{C}$ has stable reflexive coequalizers (Johnstone 2002), and so does **MultiCat** (Berger-Kaufmann 2017).

Some properties, not necessarily all easy to prove

- The object-of-objects functor $\mathrm{Cat}(\mathrm{T}) \longrightarrow \mathcal{C}, \ (X, A) \longmapsto X$, is a Grothendieck fibration, with a left adjoint; it has a right adjoint if $\mathcal{C}$ has binary products.
- $\mathrm{Cat}(\mathrm{T})$ has (not easily constructed) pullbacks and is even finitely complete when $\mathcal{C}$ is; that is, when, other than pullbacks, $\mathcal{C}$ has a terminal object.
  If $\mathcal{C}$ is complete, so is $\mathrm{Cat}(\mathrm{T})$.
- The formation of $\mathrm{Cat}(\mathrm{T})$ is functorial in T (and $\mathcal{C}$); in particular, the monad morphism $\eta : \mathrm{Id} \longrightarrow \mathrm{T}$ induces a functor $\mathrm{Cat}(\mathrm{T}) \longrightarrow \mathrm{Cat}(\mathcal{C})$:
  every T-category in $\mathcal{C}$ may be restricted to become an internal category in $\mathcal{C}$.

Theorem (T-Yeganeh 2021)

$\mathrm{Cat}(\mathrm{T})$ *has a (Street-Walters) comprehensive factorization system, provided that* $\mathcal{C}$ *has stable reflexive coequalizers that are preserved by* T.

(Surprisingly, no preservation of pullbacks by T is required!)
Corollaries: $\mathrm{Cat}(\mathcal{C})$ has such a system, provided that $\mathcal{C}$ has stable reflexive coequalizers (Johnstone 2002), and so does **MultiCat** (Berger-Kaufmann 2017).

Making things easier: "Spaces" as simplified $\mathbb{T}$-categories

Definition

A $\mathbb{T}$-category (X, A, d, c, i, m) is a $\mathbb{T}$-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

Making things easier: "Spaces" as simplified $\mathbb{T}$-categories

Definition

A $\mathbb{T}$-category (X, A, d, c, i, m) is a $\mathbb{T}$-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

In that case, i and m are uniquely determined by (X, A, d, c), and their existence becomes a property of (X, A, d, c): *reflexivity* and *transitivity*. The unity and associativity laws now come for free! For a $\mathbb{T}$-functor $(f_o, f) : (X, A) \longrightarrow (Y, B)$, the arrow part f is determined by its object part f_o, and its existence becomes a property of f_o: *monotonicity*.

Making things easier: “Spaces” as simplified T-categories

Definition

A T-category (X, A, d, c, i, m) is a T-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

In that case, i and m are uniquely determined by (X, A, d, c), and their existence becomes a property of (X, A, d, c): *reflexivity* and *transitivity*. The unity and associativity laws now come for free! For a T-functor $(f_o, f) : (X, A) \longrightarrow (Y, B)$, the arrow part f is determined by its object part f_o, and its existence becomes a property of f_o: *monotonicity*.

Some properties of the full subcategory Ord(T) of Cat(T):

- $\mathrm{Ord}(\mathrm{T}) \longrightarrow \mathcal{C}$ is *topological* (= fibration + cofibration + fibres are large-complete), provided that $\mathcal{C}$ is complete and wellpowered.

Making things easier: “Spaces” as simplified T-categories

Definition

A T-category (X, A, d, c, i, m) is a T-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

In that case, i and m are uniquely determined by (X, A, d, c), and their existence becomes a property of (X, A, d, c): *reflexivity* and *transitivity*. The unity and associativity laws now come for free! For a T-functor $(f_o, f) : (X, A) \longrightarrow (Y, B)$, the arrow part f is determined by its object part f_o, and its existence becomes a property of f_o: *monotonicity*.

Some properties of the full subcategory Ord(T) of Cat(T):

- $\mathrm{Ord}(\mathrm{T}) \longrightarrow \mathcal{C}$ is *topological* (= fibration + cofibration + fibres are large-complete), provided that $\mathcal{C}$ is complete and wellpowered.
- If $\mathcal{C}$ is also cocomplete, so is Ord(T).

Making things easier: “Spaces” as simplified T-categories

Definition

A T-category (X, A, d, c, i, m) is a T-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

In that case, i and m are uniquely determined by (X, A, d, c), and their existence becomes a property of (X, A, d, c): *reflexivity* and *transitivity*. The unity and associativity laws now come for free! For a T-functor $(f_o, f) : (X, A) \longrightarrow (Y, B)$, the arrow part f is determined by its object part f_o, and its existence becomes a property of f_o: *monotonicity*.

Some properties of the full subcategory Ord(T) of Cat(T):

- $\mathrm{Ord}(\mathrm{T}) \longrightarrow \mathcal{C}$ is *topological* (= fibration + cofibration + fibres are large-complete), provided that $\mathcal{C}$ is complete and wellpowered.
- If $\mathcal{C}$ is also cocomplete, so is Ord(T).
- Every Eilenberg-Moore T-algebra $(X, a : \mathrm{T}X \to X)$ gives the T-order $(X, \mathrm{T}X, 1_{\mathrm{T}X}, a)$; in fact:
  T-*algebras are precisely those* T-*categories with domain map an identity*.

Aspirational inclusions: Algebra ⊂ Topology ⊂ Category Theory

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{T}} = \mathsf{EM}(\mathsf{T}) \arrow[r, tail] \arrow[d] & \mathsf{Ord}(\mathsf{T}) \arrow[r, tail] \arrow[d] & \mathsf{Cat}(\mathsf{T}) \arrow[d] & \\
\mathcal{C} \arrow[r, tail] & \mathsf{Ord}(\mathcal{C}) \arrow[r, tail] & \mathsf{Cat}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

Aspirational inclusions: Algebra ⊂ Topology ⊂ Category Theory

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{T}} = \mathsf{EM}(\mathsf{T}) \arrow[r, tail] \arrow[d] & \mathsf{Ord}(\mathsf{T}) \arrow[r, tail] \arrow[d] & \mathsf{Cat}(\mathsf{T}) \arrow[d] & \\
\mathcal{C} \arrow[r, tail] & \mathsf{Ord}(\mathcal{C}) \arrow[r, tail] & \mathsf{Cat}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

Role models: $\mathsf{T} = \mathrm{L}$ (list monad) and $\mathsf{T} = \mathrm{U}$ (ultrafilter monad) on **Set**

$$\begin{tikzcd}
\mathbf{Mon} \arrow[r, tail] \arrow[d] & \mathbf{MultiOrd} \arrow[r, tail] \arrow[d] & \mathbf{MultiCat} \arrow[d] & \\
\mathbf{Set} \arrow[r, tail] & \mathbf{Ord} \arrow[r, tail] & \mathbf{Cat} \arrow[r] & \mathbf{Set} \\
\mathbf{CompHaus} \arrow[r, tail] \arrow[u] & \mathbf{Top} \arrow[r, tail] \arrow[u] & \mathbf{UltraCat}_{\mathrm{internal}} \arrow[u] &
\end{tikzcd}$$

Some justifications for the bottom row to be given later!

More properties, and a glimpse at the “topological potential” of $\mathsf{Ord}(\mathrm{T})$

- $\mathsf{Ord}(\mathrm{T}) \rightarrowtail \mathsf{Cat}(\mathrm{T}\,)$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} =$ **Set**, under Choice).

More properties, and a glimpse at the "topological potential" of $\mathsf{Ord}(\mathbb{T})$

- $\mathsf{Ord}(\mathbb{T}) \rightarrowtail \mathsf{Cat}(\mathbb{T})$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} =$ **Set**, under Choice).
- $\mathsf{EM}(\mathbb{T}) \rightarrowtail \mathsf{Ord}(\mathbb{T})$ is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd's GAFT): Stone-Čech if $\mathbb{T} = \mathbb{U}$.

More properties, and a glimpse at the “topological potential” of $\mathsf{Ord}(\mathbb{T})$

- $\mathsf{Ord}(\mathbb{T}) \rightarrowtail \mathsf{Cat}(\mathbb{T})$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} = \mathbf{Set}$, under Choice).
- $\mathsf{EM}(\mathbb{T}) \rightarrowtail \mathsf{Ord}(\mathbb{T})$ is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd’s GAFT): Stone-Čech if $\mathbb{T} = \mathrm{U}$.

In this case, define a $\mathbb{T}$-order $X = (X, A, d, c)$ to be

- *Hausdorff* if the reflection $\beta_X : X \to \beta X$ is monic;

More properties, and a glimpse at the "topological potential" of $\mathsf{Ord}(\mathrm{T})$

- $\mathsf{Ord}(\mathrm{T}) \rightarrowtail \mathsf{Cat}(\mathrm{T})$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} = \mathbf{Set}$, under Choice).
- $\mathsf{EM}(\mathrm{T}) \rightarrowtail \mathsf{Ord}(\mathrm{T})$ is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd's GAFT): Stone-Čech if $\mathrm{T} = \mathrm{U}$.

In this case, define a T-order $X = (X, A, d, c)$ to be

- *Hausdorff* if the reflection $\beta_X : X \to \beta X$ is monic;
- *completely regular* if β_X is cartesian over $\mathcal{C}$;

More properties, and a glimpse at the "topological potential" of $\mathsf{Ord}(\mathrm{T})$

- $\mathsf{Ord}(\mathsf{T}) \rightarrowtail \mathsf{Cat}(\mathsf{T})$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} = \mathbf{Set}$, under Choice).
- $\mathsf{EM}(\mathsf{T}) \rightarrowtail \mathsf{Ord}(\mathsf{T})$ is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd's GAFT): Stone-Čech if $\mathsf{T} = \mathrm{U}$.

In this case, define a T-order $X = (X, A, d, c)$ to be

- *Hausdorff* if the reflection $\beta_X : X \to \beta X$ is monic;
- *completely regular* if β_X is cartesian over $\mathcal{C}$;
- *Tychonoff* if X is Hausdorff and completely regular.

More properties, and a glimpse at the “topological potential” of $\mathsf{Ord}(\mathsf{T})$

- $\mathsf{Ord}(\mathsf{T}) \rightarrowtail \mathsf{Cat}(\mathsf{T})$ is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on T in case $\mathcal{C} = \mathbf{Set}$, under Choice).
- $\mathsf{EM}(\mathsf{T}) \rightarrowtail \mathsf{Ord}(\mathsf{T})$ is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd’s GAFT): Stone-Čech if $\mathsf{T} = \mathrm{U}$.

In this case, define a T-order $X = (X, A, d, c)$ to be

- *Hausdorff* if the reflection $\beta_X : X \to \beta X$ is monic;
- *completely regular* if β_X is cartesian over $\mathcal{C}$;
- *Tychonoff* if X is Hausdorff and completely regular.

$$\begin{tikzcd}
 & & \mathsf{Haus}(\mathsf{T}) \arrow[dr, tail] & & \\
\mathsf{EM}(\mathsf{T}) \arrow[r, tail] & \mathsf{Tych}(\mathsf{T}) \arrow[ur, tail] \arrow[r, tail] & \mathsf{CReg}(\mathsf{T}) \arrow[r, tail] & \mathsf{Ord}(\mathsf{T}) \arrow[r, tail] & \mathsf{Cat}(\mathsf{T})
\end{tikzcd}$$

The categorical meaning of $\mathsf{CReg}(\mathbb{T})$

Theorem (Burroni 1971, slightly modified)

$\mathsf{CReg}(\mathbb{T})$ *is a fibred extension of* $\mathsf{EM}(\mathbb{T})$*, and it is universal as such:*

$$\begin{tikzcd}
\mathsf{EM}(\mathbb{T}) \arrow[r, tail] \arrow[rr, bend left, "F"] \arrow[dr] & \mathsf{CReg}(\mathbb{T}) \arrow[r, dashed, "\overline{F}"] \arrow[d] & \mathcal{E} \arrow[dl, "P \text{ fibration}"] \\
& \mathcal{C} &
\end{tikzcd}$$

$\overline{F}$ *pres. cartesianess of the reflections* β_X

The categorical meaning of $\mathsf{CReg}(\mathrm{T})$

Theorem (Burroni 1971, slightly modified)

$\mathsf{CReg}(\mathrm{T})$ *is a fibred extension of* $\mathsf{EM}(\mathrm{T})$*, and it is universal as such:*

$$\begin{tikzcd}
\mathsf{EM}(\mathrm{T}) \arrow[r, tail] \arrow[rr, bend left, "F"] \arrow[dr] & \mathsf{CReg}(\mathrm{T}) \arrow[r, dashed, "\overline{F}"] \arrow[d] & \mathcal{E} \arrow[dl, "P\ \textit{fibration}"] \\
& \mathcal{C} &
\end{tikzcd}$$

$\overline{F}$ *pres. cartesianess of the reflections* β_X

Similarly for $\mathsf{Tych}(\mathrm{T})$, as a universal *mono*-fibred extension of $\mathsf{EM}(\mathrm{T})$, with $\overline{F}$ preserving cartesian *mono*morphisms.

An example of category theory embracing topology

Let $f : X = (X, A) \longrightarrow Y = (Y, B)$ be in Tych(T). Then:

$$\begin{tikzcd}
\mathrm{T}X \arrow[d, "\mathrm{T}f"'] \arrow[dr, phantom, "\text{pullback in } \mathcal{C}"] & A \arrow[l, "d"'] \arrow[d, "f"] & \Rightarrow & X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\text{pb in Ord}(\mathrm{T})"] & \beta X \arrow[d, "\beta f"] \\
\mathrm{T}Y & B \arrow[l, "d"] & & Y \arrow[r] & \beta Y
\end{tikzcd}$$

An example of category theory embracing topology

Let $f : X = (X, A) \longrightarrow Y = (Y, B)$ be in Tych(T). Then:

$$\begin{tikzcd} \mathsf{T}X \arrow[d, "\mathsf{T}f"'] & & A \arrow[ll, "d"'] \arrow[d, "f"] \\ \mathsf{T}Y & \text{pullback in } \mathcal{C} & B \arrow[ll, "d"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X \arrow[rr] \arrow[d, "f"'] & & \beta X \arrow[d, "\beta f"] \\ Y \arrow[rr] & \text{pb in Ord}(\mathrm{T}) & \beta Y \end{tikzcd}$$

discrete cofibration in Cat(T) ⇒ *perfect* (as in [T 1999])

An example of category theory embracing topology

Let $f : X = (X, A) \longrightarrow Y = (Y, B)$ be in Tych(T). Then:

$$\begin{tikzcd} \mathsf{T}X \arrow[d, "\mathsf{T}f"'] \arrow[dr, phantom, "\text{pullback in } \mathcal{C}"] & A \arrow[l, "d"'] \arrow[d, "f"] \\ \mathsf{T}Y & B \arrow[l, "d"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\text{pb in Ord}(\mathbb{T})"] & \beta X \arrow[d, "\beta f"] \\ Y \arrow[r] & \beta Y \end{tikzcd}$$

discrete cofibration in Cat(T) ⇒ *perfect* (as in [T 1999])

Consequently:

Comprehensive factorization of *f* means (antiperfect, perfect)-factorization of *f*, a.k.a. the fibrewise Stone-Čech compactification of *f*.

How to enrich multicategories?

$$\mathrm{L}X \times X \xrightarrow{\hom_A} \mathsf{Set}$$

$$a : \mathrm{L}X \mathrel{\mapstochar\rightarrow} X$$

How to enrich multicategories?

$$\mathrm{L}X \times X \xrightarrow{\hom_A} \mathrm{Set}$$

$$1 \longrightarrow \hom_A(\eta_X(x), x)$$

$$a : \mathrm{L}X \relbar\joinrel\mapstochar\joinrel\rightarrow X$$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] \arrow[dr, phantom, shift left=4, "\Rightarrow"] & \mathrm{L}X \arrow[d, "a"] \\
& X
\end{tikzcd}$$

How to enrich multicategories?

$$\mathrm{L}X \times X \xrightarrow{\hom_A} \mathsf{Set}$$

$$a : \mathrm{L}X \mapsto X$$

$$1 \longrightarrow \hom_A(\eta_X(x), x)$$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] \arrow[dr, phantom, "\Rightarrow", shift left=4] & \mathrm{L}X \arrow[d, "a"] \\
& X
\end{tikzcd}$$

$$\hom_A(\mathcal{X}, \overline{y}) \times \hom_A(\overline{y}, z) \longrightarrow \hom_A(\mu_X(\mathcal{X}), z)$$

$$\begin{tikzcd}
\mathrm{L}\mathrm{L}X \arrow[r, "\hat{\mathrm{L}}a"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Leftarrow"] & \mathrm{L}X \arrow[d, "a"] \\
\mathrm{L}X \arrow[r, "a"'] & X
\end{tikzcd}$$

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathbb{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

- $\hat{\mathsf{T}}$ is a lax 2-functor with $\hat{\mathsf{T}}(1_X^\circ) = 1_{\mathsf{T}X}^\circ$ [and $\hat{\mathsf{T}}(r^\circ) = (\hat{\mathsf{T}}r)^\circ$ for $r : X {\to\!\!\!\!\!\!\bullet\;\;} Y$];

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

- $\hat{\mathsf{T}}$ is a lax 2-functor with $\hat{\mathsf{T}}(1_X^\circ) = 1_{\mathsf{T}X}^\circ$ [and $\hat{\mathsf{T}}(r^\circ) = (\hat{\mathsf{T}}r)^\circ$ for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$];
- $\hat{\mathsf{T}}(s \cdot f_\circ) \cong \hat{\mathsf{T}}s \cdot (\mathsf{T}f)_\circ$ for $f : X \to Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$;

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

- $\hat{\mathsf{T}}$ is a lax 2-functor with $\hat{\mathsf{T}}(1_X^\circ) = 1_{\mathsf{T}X}^\circ$ [and $\hat{\mathsf{T}}(r^\circ) = (\hat{\mathsf{T}}r)^\circ$ for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$];
- $\hat{\mathsf{T}}(s \cdot f_\circ) \cong \hat{\mathsf{T}}s \cdot (\mathsf{T}f)_\circ$ for $f : X \to Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$;
- $\eta_\circ : \mathrm{Id}_{\mathsf{Mat}(\mathcal{V})} \to \hat{\mathsf{T}}$, $\mu_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ are oplax natural transformations,

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

- $\hat{\mathsf{T}}$ is a lax 2-functor with $\hat{\mathsf{T}}(1_X^\circ) = 1_{\mathsf{T}X}^\circ$ [and $\hat{\mathsf{T}}(r^\circ) = (\hat{\mathsf{T}}r)^\circ$ for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$];
- $\hat{\mathsf{T}}(s \cdot f_\circ) \cong \hat{\mathsf{T}}s \cdot (\mathsf{T}f)_\circ$ for $f : X \to Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$;
- $\eta_\circ : \mathrm{Id}_{\mathsf{Mat}(\mathcal{V})} \to \hat{\mathsf{T}}, \quad \mu_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ are oplax natural transformations,

subject to (many) coherence and compatibility conditions; here $\mathsf{Mat}(\mathcal{V})$ has

Clementino-T 2003, after Rosebrugh-Wood 2002, Betti *et al.* 1983

- $(\mathcal{V}, \otimes, \mathrm{k})$ cocomplete symmetric monoidal-closed category,
- $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ monad on **Set** equipped with a *flat lax extension* $\hat{\mathsf{T}}$ of T as in

$$\begin{tikzcd}
\mathsf{Mat}(\mathcal{V}) \arrow[r, "\hat{\mathsf{T}}"] & \mathsf{Mat}(\mathcal{V}) \\
\mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "\mathsf{T}"] & \mathbf{Set} \arrow[u, "(-)_\circ"']
\end{tikzcd}$$

that preserves $[(-)^\circ : \mathsf{Mat}(\mathcal{V})^{\mathrm{op}} \to \mathsf{Mat}(\mathcal{V})$ and$]$ whiskering by maps; explicitly:

- $\hat{\mathsf{T}}$ is a lax 2-functor with $\hat{\mathsf{T}}(1_X^\circ) = 1_{\mathsf{T}X}^\circ$ [and $\hat{\mathsf{T}}(r^\circ) = (\hat{\mathsf{T}}r)^\circ$ for $r : X \mapsto Y$];
- $\hat{\mathsf{T}}(s \cdot f_\circ) \cong \hat{\mathsf{T}}s \cdot (\mathsf{T}f)_\circ$ for $f : X \to Y$ and $s : Y \mapsto Z$;
- $\eta_\circ : \mathrm{Id}_{\mathsf{Mat}(\mathcal{V})} \to \hat{\mathsf{T}}, \quad \mu_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ are oplax natural transformations,

subject to (many) coherence and compatibility conditions; here $\mathsf{Mat}(\mathcal{V})$ has objects = sets; functors $X \times Y \to \mathcal{V}$; natural transfs.; horizontal composition:

$$(s \cdot r)(x, z) = \coprod_{y \in Y} r(x, y) \otimes s(y, z)\ .$$

$(\mathsf{T}, \mathcal{V})$-categories as lax $\hat{\mathsf{T}}$-algebras

$(\mathsf{T}, \mathcal{V})$-Cat	$(X, a : \mathsf{T}X \mapsto X)$	$\mathrm{k} \to a(\eta_X(x), x)$	$(x, z \in X)$
		$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \to a(\mu_X(\mathcal{X}), z)$	$(\mathcal{X} \in \mathsf{T}\mathsf{T}X, \overline{y} \in \mathsf{T}X)$
	$(X, a) \xrightarrow{f} (Y, b)$	$a(\overline{x}, y) \to b(\mathsf{T}f(\overline{x}), f(y))$	$(\overline{x} \in \mathsf{T}X, y \in X)$

$(\mathsf{T}, \mathcal{V})$-categories as lax $\hat{\mathsf{T}}$-algebras

$(\mathsf{T}, \mathcal{V})$-Cat $(X, a : \mathsf{T}X \mathrel{\mapstochar\rightarrow} X)$ $\mathrm{k} \to a(\eta_X(x), x)$ $(x, z \in X)$

$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \to a(\mu_X(\mathcal{X}), z)$ $(\mathcal{X} \in \mathsf{TT}X, \overline{y} \in \mathsf{T}X)$

$(X, a) \xrightarrow{f} (Y, b)$ $a(\overline{x}, y) \to b(\mathsf{T}f(\overline{x}), f(y))$ $(\overline{x} \in \mathsf{T}X, y \in X)$

$$\begin{tikzcd}
\mathsf{TT}X \arrow[r, "\hat{\mathsf{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{T}X \arrow[d, "a"'] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathsf{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}X \arrow[r, "(\mathsf{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

subject to an extensive array of coherence and compatibility conditions.

$(\mathsf{T},\mathcal{V})$-categories as lax $\hat{\mathsf{T}}$-algebras

$(\mathsf{T},\mathcal{V})$-Cat	$(X, a\colon \mathsf{T}X \relbar\joinrel\mapstochar\joinrel\rightarrow X)$	$\mathrm{k} \to a(\eta_X(x), x)$	$(x, z \in X)$
		$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \to a(\mu_X(\mathcal{X}), z)$	$(\mathcal{X} \in \mathsf{TT}X, \overline{y} \in \mathsf{T}X)$
	$(X, a) \xrightarrow{f} (Y, b)$	$a(\overline{x}, y) \to b(\mathsf{T}f(\overline{x}), f(y))$	$(\overline{x} \in \mathsf{T}X, y \in X)$

$$\begin{tikzcd}
\mathsf{TT}X \arrow[r, "\hat{\mathsf{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{T}X \arrow[d, "a"'] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathsf{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}X \arrow[r, "(\mathsf{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

subject to an extensive array of coherence and compatibility conditions.

$(\mathrm{Id}, \mathcal{V})$-**Cat** $=$ $\mathcal{V}$-**Cat**

$(\mathsf{T}, \mathcal{V})$-categories as lax $\hat{\mathsf{T}}$-algebras

$(\mathsf{T}, \mathcal{V})$-**Cat** $\quad (X, a : \mathsf{T}X \nrightarrow X) \quad \mathrm{k} \to a(\eta_X(x), x) \qquad (x, z \in X)$

$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \to a(\mu_X(\mathcal{X}), z) \qquad (\mathcal{X} \in \mathsf{TT}X, \overline{y} \in \mathsf{T}X)$

$(X, a) \xrightarrow{f} (Y, b) \quad a(\overline{x}, y) \to b(\mathsf{T}f(\overline{x}), f(y)) \qquad (\overline{x} \in \mathsf{T}X, y \in X)$

$$\begin{tikzcd}
\mathsf{TT}X \arrow[r, "\hat{\mathsf{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{T}X \arrow[d, "a"] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathsf{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}X \arrow[r, "(\mathsf{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

subject to an extensive array of coherence and compatibility conditions.

$(\mathrm{Id}, \mathcal{V})$-**Cat** $= \mathcal{V}$-**Cat**

$(\mathrm{L}, \mathcal{V})$-**Cat** $= \mathcal{V}$-**MultiCat**, $\quad$ where $(\hat{\mathrm{L}}r)(\overline{x}, \overline{y}) = r(x_1, y_1) \otimes \ldots \otimes r(x_n, y_n)$
if $\mathrm{length}(\overline{x}) = n = \mathrm{length}(\overline{y})$; $\quad =$ initial obj. 0 else.

$(\mathsf{T},\mathcal{V})$-categories as lax $\hat{\mathsf{T}}$-algebras

$(\mathsf{T},\mathcal{V})$-Cat $\quad (X, a : \mathsf{T}X \relbar\mapstochar\rightarrow X) \quad \mathrm{k} \to a(\eta_X(x), x) \qquad (x, z \in X)$

$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \to a(\mu_X(\mathcal{X}), z) \qquad (\mathcal{X} \in \mathsf{TT}X, \overline{y} \in \mathsf{T}X)$

$(X, a) \xrightarrow{f} (Y, b) \quad a(\overline{x}, y) \to b(\mathsf{T}f(\overline{x}), f(y)) \qquad (\overline{x} \in \mathsf{T}X, y \in X)$

$$\begin{tikzcd}
\mathsf{TT}X \arrow[r, "\hat{\mathsf{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{T}X \arrow[d, "a"'] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathsf{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}X \arrow[r, "(\mathsf{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

subject to an extensive array of coherence and compatibility conditions.

$(\mathrm{Id},\mathcal{V})$-**Cat** $= \mathcal{V}$-**Cat**

$(\mathrm{L},\mathcal{V})$-**Cat** $= \mathcal{V}$-**MultiCat**, $\qquad$ where $(\hat{\mathrm{L}}r)(\overline{x}, \overline{y}) = r(x_1, y_1) \otimes \ldots \otimes r(x_n, y_n)$
if $\mathrm{length}(\overline{x}) = n = \mathrm{length}(\overline{y})$; $\quad$ = initial obj. 0 else.

What about $\mathsf{T} = \mathrm{U}$, even for $\mathcal{V} =$ **Set** ?

Making things easy again: let $\mathcal{V}$ be "thin"!

V unital and (for convenience) commutative *quantale*

= a complete lattice with a commutative monoid structure, $\mathsf{V} = (\mathsf{V}, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, cocomplete symmetric monoidal-closed category

= a commutative monoid in the symmetric monoidal-closed category **Sup**

Some examples:

Making things easy again: let $\mathcal{V}$ be “thin”!

V unital and (for convenience) commutative *quantale*

= a complete lattice with a commutative monoid structure, $V = (V, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, cocomplete symmetric monoidal-closed category

= a commutative monoid in the symmetric monoidal-closed category **Sup**

Some examples:

- $V = 2$ with $u \otimes v = u \,\&\, v$, $\quad \mathrm{k} = \mathrm{true}$ (Boolean 2-chain)

Making things easy again: let $\mathcal{V}$ be "thin"!

V unital and (for convenience) commutative *quantale*

= a complete lattice with a commutative monoid structure, $\mathsf{V} = (\mathsf{V}, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, cocomplete symmetric monoidal-closed category

= a commutative monoid in the symmetric monoidal-closed category **Sup**

Some examples:

- ► $\mathsf{V} = 2$ with $u \otimes v = u \,\&\, v, \quad \mathrm{k} = \mathrm{true}$ (Boolean 2-chain)
- ► $\mathsf{V} = [0, \infty]$ with $u \otimes v = u + v, \quad \mathrm{k} = 0$ (Lawvere quantale)

Making things easy again: let $\mathcal{V}$ be “thin”!

V unital and (for convenience) commutative *quantale*

= a complete lattice with a commutative monoid structure, $\mathsf{V} = (\mathsf{V}, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, cocomplete symmetric monoidal-closed category

= a commutative monoid in the symmetric monoidal-closed category **Sup**

Some examples:

- ► $\mathsf{V} = 2$ with $u \otimes v = u \,\&\, v, \quad \mathrm{k} = \mathrm{true}$ (Boolean 2-chain)
- ► $\mathsf{V} = [0, \infty]$ with $u \otimes v = u + v, \quad \mathrm{k} = 0$ (Lawvere quantale)
- ► V any frame with $u \otimes v = u \wedge v, \quad \mathrm{k} = \top$ (a cartesian quantale)

Making things easy again: let $\mathcal{V}$ be “thin”!

V unital and (for convenience) commutative *quantale*

= a complete lattice with a commutative monoid structure, $\mathsf{V} = (\mathsf{V}, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, cocomplete symmetric monoidal-closed category

= a commutative monoid in the symmetric monoidal-closed category **Sup**

Some examples:

- $\mathsf{V} = 2$ with $u \otimes v = u \,\&\, v, \quad \mathrm{k} = \mathrm{true}$ (Boolean 2-chain)
- $\mathsf{V} = [0, \infty]$ with $u \otimes v = u + v, \quad \mathrm{k} = 0$ (Lawvere quantale)
- V any frame with $u \otimes v = u \wedge v, \quad \mathrm{k} = \top$ (a cartesian quantale)
- $\mathsf{V} = 2^M$, for any commutative monoid M (the free quantale over M), with $A \otimes B = \{\alpha \cdot \beta \mid \alpha \in A, \beta \in B\}, \quad \mathrm{k} = \{\varepsilon\}, \ \varepsilon$ neutral in M

How do lax monad extensions fare now?

Writing V-**Rel** for $\mathsf{Mat}(\mathsf{V})$, our **Set**-monad $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ comes with a lax 2-functor $\hat{\mathsf{T}} : \mathsf{V}\text{-}\mathbf{Rel} \to \mathsf{V}\text{-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \mathsf{V}\text{-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathsf{T} = (T, \eta, \mu)$ comes with a lax 2-functor $\hat{T} : \mathsf{V}\text{-}\mathbf{Rel} \to \mathsf{V}\text{-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \mathsf{V}\text{-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathsf{T} = (T, \eta, \mu)$ comes with a lax 2-functor $\hat{T} : \text{V-}\mathbf{Rel} \to \text{V-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \text{V-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

Possible relaxation of the above conditions, by *foregoing flatness* of the lax extension (Seal 2004, adopted in *Monoidal Topology* 2014):

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ comes with a lax 2-functor $\hat{\mathrm{T}} : \mathrm{V}\text{-}\mathbf{Rel} \to \mathrm{V}\text{-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \mathrm{V}\text{-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

Possible relaxation of the above conditions, by *foregoing flatness* of the lax extension (Seal 2004, adopted in *Monoidal Topology* 2014):

- $\hat{\mathrm{T}}$ is a lax 2-functor, coinciding with T on objects;

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ comes with a lax 2-functor $\hat{\mathsf{T}} : \text{V-}\mathbf{Rel} \to \text{V-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \text{V-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

Possible relaxation of the above conditions, by *foregoing flatness* of the lax extension (Seal 2004, adopted in *Monoidal Topology* 2014):

- $\hat{\mathsf{T}}$ is a lax 2-functor, coinciding with T on objects;
- $(\mathsf{T}f)_\circ \leq \hat{\mathsf{T}}(f_\circ)$ and $(\mathsf{T}f)^\circ \leq \hat{\mathsf{T}}(f^\circ)$, for every map f of sets;

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ comes with a lax 2-functor $\hat{\mathsf{T}} : \mathsf{V}\text{-}\mathbf{Rel} \to \mathsf{V}\text{-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \mathsf{V}\text{-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

Possible relaxation of the above conditions, by *foregoing flatness* of the lax extension (Seal 2004, adopted in *Monoidal Topology* 2014):

- $\hat{\mathsf{T}}$ is a lax 2-functor, coinciding with T on objects;
- $(\mathsf{T}f)_\circ \leq \hat{\mathsf{T}}(f_\circ)$ and $(\mathsf{T}f)^\circ \leq \hat{\mathsf{T}}(f^\circ)$, for every map f of sets;
- $\eta_\circ : \mathrm{Id}_{\mathsf{V}\text{-}\mathbf{Rel}} \to \hat{\mathsf{T}}$ and $\mu_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ are oplax natural transformations.

How do lax monad extensions fare now?

Writing V-**Rel** for Mat(V), our **Set**-monad $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ comes with a lax 2-functor $\hat{\mathsf{T}} : \text{V-}\mathbf{Rel} \to \text{V-}\mathbf{Rel}$, which extends T along $(-)_\circ : \mathbf{Set} \to \text{V-}\mathbf{Rel}$ [and commutes with the involution $(-)^\circ$ of V-**Rel**]; it preserves whiskering with **Set**-maps and makes $\eta_\circ$ and $\mu_\circ$ oplax.

Main simplifying effect: As 2-cells are given by order, all coherence and compatibility conditions will be satisfied for free!

Possible relaxation of the above conditions, by *foregoing flatness* of the lax extension (Seal 2004, adopted in *Monoidal Topology* 2014):

- $\hat{\mathsf{T}}$ is a lax 2-functor, coinciding with T on objects;
- $(\mathsf{T}f)_\circ \leq \hat{\mathsf{T}}(f_\circ)$ and $(\mathsf{T}f)^\circ \leq \hat{\mathsf{T}}(f^\circ)$, for every map f of sets;
- $\eta_\circ : \mathrm{Id}_{\text{V-}\mathbf{Rel}} \to \hat{\mathsf{T}}$ and $\mu_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ are oplax natural transformations.

These conditions *imply* the whiskering conditions

$$\hat{\mathsf{T}}(s \cdot f_\circ) = (\hat{\mathsf{T}}s) \cdot (\mathsf{T}f)_\circ \text{ and } \hat{\mathsf{T}}(g^\circ \cdot r) = (\mathsf{T}g)^\circ \cdot (\hat{\mathsf{T}}r) .$$

(T, V)-categories

(T, V)-Cat	$(X, a : \mathsf{T}X \mathrel{\mapstochar\rightarrow} X)$	$\mathrm{k} \leq a(\eta_X(x), x)$	$(x, z \in X)$
		$\hat{\mathsf{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \leq a(\mu_X(\mathcal{X}), z)$	$(\mathcal{X} \in \mathsf{T}\mathsf{T}X, \overline{y} \in \mathsf{T}X)$
	$(X, a) \xrightarrow{f} (Y, b)$	$a(\overline{x}, y) \leq b(\mathsf{T}f(\overline{x}), f(y))$	$(\overline{x} \in \mathsf{T}X, y \in Y)$

(T, V)-categories

(T, V)-Cat	$(X, a : \mathrm{T}X \relbar\joinrel\mapstochar\joinrel\rightarrow X)$	$\mathrm{k} \leq a(\eta_X(x), x)$	$(x, z \in X)$
		$\hat{\mathrm{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \leq a(\mu_X(\mathcal{X}), z)$	$(\mathcal{X} \in \mathrm{TT}X, \overline{y} \in \mathrm{T}X)$
	$(X, a) \xrightarrow{f} (Y, b)$	$a(\overline{x}, y) \leq b(\mathrm{T}f(\overline{x}), f(y))$	$(\overline{x} \in \mathrm{T}X, y \in Y)$

$$\begin{tikzcd}
\mathrm{TT}X \arrow[r, "\hat{\mathrm{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\geq"] & \mathrm{T}X \arrow[d, "a"'] \arrow[dr, phantom, "\geq"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathrm{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T}X \arrow[r, "(\mathrm{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\leq"] & \mathrm{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

(T, V)-categories

(T, V)-Cat

$$
\begin{array}{lll}
(X, a : \mathrm{T}X \relbar\joinrel\mapstochar\joinrel\rightarrow X) & \mathrm{k} \leq a(\eta_X(x), x) & (x, z \in X) \\
 & \hat{\mathrm{T}}a(\mathcal{X}, \overline{y}) \otimes a(\overline{y}, z) \leq a(\mu_X(\mathcal{X}), z) & (\mathcal{X} \in \mathrm{TT}X, \overline{y} \in \mathrm{T}X) \\
(X, a) \xrightarrow{f} (Y, b) & a(\overline{x}, y) \leq b(\mathrm{T}f(\overline{x}), f(y)) & (\overline{x} \in \mathrm{T}X, y \in Y)
\end{array}
$$

$$\begin{tikzcd}
\mathrm{TT}X \arrow[r, "\hat{\mathrm{T}}a"] \arrow[d, "(\mu_X)_\circ"'] \arrow[dr, phantom, "\geq"] & \mathrm{T}X \arrow[d, "a"'] \arrow[dr, phantom, "\geq"] & X \arrow[l, "(\eta_X)_\circ"'] \arrow[dl, "1_X^\circ"] \\
\mathrm{T}X \arrow[r, "a"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T}X \arrow[r, "(\mathrm{T}f)_\circ"] \arrow[d, "a"'] \arrow[dr, phantom, "\leq"] & \mathrm{T}Y \arrow[d, "b"] \\
X \arrow[r, "f_\circ"'] & Y
\end{tikzcd}$$

Equivalently: $\eta_X^\circ \leq a$

$a \circ a \leq a$ (Kleisli convolution)

$a \leq f^\circ \cdot b \cdot (\mathrm{T}f)_\circ$

Kleisli convolution for $r : \mathrm{T}X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s : \mathrm{T}Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$: $s \circ r := s \cdot \hat{\mathrm{T}}r \cdot \mathrm{m}_X^\circ : \mathrm{T}X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$

Some properties and results, comparison with $\mathsf{Ord}(\mathsf{T})$

- (T, V)-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;

Some properties and results, comparison with $\mathsf{Ord}(\mathsf{T})$

- (T, V)-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;
- (T, V)-**Cat** is complete and cocomplete, with (co)limits of diagrams formed by (co)cartesian lifting of the (co)limits of the underlying **Set**-diagrams;

Some properties and results, comparison with $\mathsf{Ord}(\mathsf{T})$

- (T, V)-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;
- (T, V)-**Cat** is complete and cocomplete, with (co)limits of diagrams formed by (co)cartesian lifting of the (co)limits of the underlying **Set**-diagrams;
- the formation of (T, V)-**Cat** is functorial in T (contra-) and V (co-variantly);

Some properties and results, comparison with Ord(T)

- (T, V)-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;
- (T, V)-**Cat** is complete and cocomplete, with (co)limits of diagrams formed by (co)cartesian lifting of the (co)limits of the underlying **Set**-diagrams;
- the formation of (T, V)-**Cat** is functorial in T (contra-) and V (co-variantly);
- (Lowen-Vroegrijk 2008, Hofmann 2014) For every T, laxly extended to V-**Rel**, one can find a monad Π (encoding both, T and V) , laxly extended to **Rel** $=$ 2-**Rel**, such that (T, V)-**Cat** $\cong$ (Π, 2)-**Cat**.

Some properties and results, comparison with $\mathsf{Ord}(\mathbb{T})$

- $(\mathbb{T}, \mathsf{V})$-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;
- $(\mathbb{T}, \mathsf{V})$-**Cat** is complete and cocomplete, with (co)limits of diagrams formed by (co)cartesian lifting of the (co)limits of the underlying **Set**-diagrams;
- the formation of $(\mathbb{T}, \mathsf{V})$-**Cat** is functorial in $\mathbb{T}$ (contra-) and V (co-variantly);
- (Lowen-Vroegrijk 2008, Hofmann 2014) For every $\mathbb{T}$, laxly extended to V-**Rel**, one can find a monad Π (encoding both, $\mathbb{T}$ and V) , laxly extended to **Rel** $=$ 2-**Rel**, such that $(\mathbb{T}, \mathsf{V})$-**Cat** $\cong$ $(\Pi, 2)$-**Cat**.

Theorem (Burroni 1971)

Let $\mathcal{C} =$ **Set** *and* $\mathbb{T}$ *be laxly extended to* **Rel** *á la Barr. Then:* $(\mathbb{T}, 2)$-**Cat** $\cong \mathsf{Ord}(\mathbb{T})$.

Barr 1971: Given $r = (\ X \xleftarrow{d} R \xrightarrow{c} Y)$, put $\hat{\mathsf{T}}r = (\ \mathsf{T}X \xleftarrow{\mathsf{T}d} \mathsf{T}R \xrightarrow{\mathsf{T}c} \mathsf{T}Y\ )$.

Some properties and results, comparison with $\mathsf{Ord}(\mathbb{T})$

- $(\mathbb{T}, \mathsf{V})$-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints;
- $(\mathbb{T}, \mathsf{V})$-**Cat** is complete and cocomplete, with (co)limits of diagrams formed by (co)cartesian lifting of the (co)limits of the underlying **Set**-diagrams;
- the formation of $(\mathbb{T}, \mathsf{V})$-**Cat** is functorial in $\mathbb{T}$ (contra-) and V (co-variantly);
- (Lowen-Vroegrijk 2008, Hofmann 2014) For every $\mathbb{T}$, laxly extended to V-**Rel**, one can find a monad Π (encoding both, $\mathbb{T}$ and V) , laxly extended to **Rel** $=$ 2-**Rel**, such that $(\mathbb{T}, \mathsf{V})$-**Cat** $\cong$ $(\Pi, 2)$-**Cat**.

Theorem (Burroni 1971)

Let $\mathcal{C} =$ **Set** *and* $\mathbb{T}$ *be laxly extended to* **Rel** *á la Barr. Then:* $(\mathbb{T}, 2)$-**Cat** $\cong \mathsf{Ord}(\mathbb{T})$.

Barr 1971: Given $r = (X \xleftarrow{d} R \xrightarrow{c} Y)$, put $\hat{\mathsf{T}}r = (\mathsf{T}X \xleftarrow{\mathsf{T}d} \mathsf{T}R \xrightarrow{\mathsf{T}c} \mathsf{T}Y)$.

But the Theorem may *not* be applied to Π above – Π is not extended á la Barr!

Guiding examples for $\mathsf{V} = 2$, as vaguely envisioned by Hausdorff 1914!

T	Reflexivity	Transitivity	$(\mathsf{T}, 2)$-**Cat**
Id	$x \leq x$	$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$	**Ord**

Guiding examples for $\mathsf{V} = 2$, as vaguely envisioned by Hausdorff 1914!

T	Reflexivity	Transitivity	$(\mathsf{T}, 2)$-**Cat**
Id	$x \leq x$	$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$	**Ord**
L	$(x) \leq x$	$\mathcal{X} \leq \overline{y} \;\&\; \overline{y} \leq z \Rightarrow \mu_X(\mathcal{X}) \leq z$	**MultiOrd**

Guiding examples for $\mathsf{V} = 2$, as vaguely envisioned by Hausdorff 1914!

T	Reflexivity	Transitivity	$(\mathsf{T}, 2)$-**Cat**
Id	$x \leq x$	$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$	**Ord**
L	$(x) \leq x$	$\mathcal{X} \leq \overline{y} \;\&\; \overline{y} \leq z \Rightarrow \mu_X(\mathcal{X}) \leq z$	**MultiOrd**
U	$\dot{x} \rightsquigarrow x$	$\mathcal{X} \rightsquigarrow \overline{y} \;\&\; \overline{y} \rightsquigarrow z \Rightarrow \mu_X(\mathcal{X}) \rightsquigarrow z$	**Top** (Manes 1967 → Barr 1970)

Guiding examples for $\mathsf{V} = 2$, as vaguely envisioned by Hausdorff 1914!

T	Reflexivity	Transitivity	$(\mathsf{T}, 2)$-**Cat**
Id	$x \leq x$	$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$	**Ord**
L	$(x) \leq x$	$\mathcal{X} \leq \overline{y} \;\&\; \overline{y} \leq z \Rightarrow \mu_X(\mathcal{X}) \leq z$	**MultiOrd**
U	$\dot{x} \rightsquigarrow x$	$\mathcal{X} \rightsquigarrow \overline{y} \;\&\; \overline{y} \rightsquigarrow z \Rightarrow \mu_X(\mathcal{X}) \rightsquigarrow z$	**Top** (Manes 1967 → Barr 1970)

$$\begin{array}{llllll}
\overline{x_1} = (x_{1,1}, \dots) & \rightsquigarrow y_1 & & & & \\
\quad\vdots & \vdots & & & & \\
\overline{x_n} = (x_{n,1}, \dots) & \rightsquigarrow y_n & & & & \\
\quad\vdots & \vdots & & & & \\
\mathcal{X} = (\overline{x_1}, \dots\dots) & \rightsquigarrow \overline{y} & \rightsquigarrow z & \Rightarrow & \mu_X(\mathcal{X}) = (x_{1,1}, .., x_{n,n}, ..) & \rightsquigarrow z
\end{array}$$

Trading 2 for $[0, \infty]$– also envisioned by Hausdorff 1914

T	Reflexivity	Transitivity $=$ ∇-Inequality	$(\mathsf{T}, [0, \infty])$-**Cat**
Id	$0 \geq d(x, x)$	$d(x, y) + d(y, z) \geq d(x, z)$	**Met** (Lawvere 1973)

Trading 2 for $[0,\infty]$– also envisioned by Hausdorff 1914

T	Reflexivity	Transitivity $=$ ∇-Inequality	$(\mathrm{T}, [0,\infty])$-**Cat**
Id	$0 \geq d(x,x)$	$d(x,y) + d(y,z) \geq d(x,z)$	**Met** (Lawvere 1973)
L	$0 \geq d((x),x)$	$d(\mathcal{X},\overline{y}) + d(\overline{y},z) \geq d(\mu_X(\mathcal{X}),z)$	**MultiMet**
		$d(\mathcal{X},\overline{y}) := \hat{\mathrm{L}}d(\mathcal{X},\overline{y}) = \sum_i d(\overline{x_i}, y_i)$	

Trading 2 for [0, ∞]– also envisioned by Hausdorff 1914

T	Reflexivity	Transitivity $=$ ∇-Inequality	(T, [0, ∞])-**Cat**
Id	$0 \geq d(x, x)$	$d(x, y) + d(y, z) \geq d(x, z)$	**Met** (Lawvere 1973)
L	$0 \geq d((x), x)$	$d(\mathcal{X}, \overline{y}) + d(\overline{y}, z) \geq d(\mu_X(\mathcal{X}), z)$	**MultiMet**
		$d(\mathcal{X}, \overline{y}) := \hat{\mathrm{L}}d(\mathcal{X}, \overline{y}) = \sum_i d(\overline{x_i}, y_i)$	
U	$0 \geq d(\dot{x}, x)$	$d(\mathcal{X}, \overline{y}) + d(\overline{y}, z) \geq d(\mu_X(\mathcal{X}), z)$	**App** (Lowen 1989 →

$$d(\mathcal{X}, \overline{y}) := \hat{\mathrm{U}}d(\mathcal{X}, \overline{y}) = \sup_{\mathcal{A} \in \mathcal{X}, B \in \overline{y}} \inf_{\overline{x} \in \mathcal{A}, y \in B} d(\overline{x}, y)$$

→ Clementino-Hofmann 2003)

Extending U to quantales other than 2 or $[0, \infty]$, and beyond

Theorem (Clementino-T 2003)

The ultrafilter monad may be laxly extended (á la Barr) to V-**Rel** *when the underlying lattice of the quantale* V *is constructively completely distributive.*

Extending U to quantales other than 2 or $[0,\infty]$, and beyond

Theorem (Clementino-T 2003)

The ultrafilter monad may be laxly extended (á la Barr) to V-**Rel** *when the underlying lattice of the quantale* V *is constructively completely distributive.*

For $r : X \relbar\mapsto Y$ define $\hat{\mathrm{U}}r : \mathrm{U}X \relbar\mapsto \mathrm{T}Y$ by

$$\hat{\mathrm{U}}r(\overline{x},\overline{y}) = \bigwedge_{A\in\overline{x},B\in\overline{y}} \ \bigvee_{x\in A,y\in B} r(x,y).$$

Extending U to quantales other than 2 or $[0, \infty]$, and beyond

Theorem (Clementino-T 2003)

The ultrafilter monad may be laxly extended (á la Barr) to V-**Rel** *when the underlying lattice of the quantale* V *is constructively completely distributive.*

For $r : X \nrightarrow Y$ define $\hat{\mathrm{U}}r : \mathrm{U}X \nrightarrow \mathrm{T}Y$ by

$$\hat{\mathrm{U}}r(\overline{x}, \overline{y}) = \bigwedge_{A \in \overline{x}, B \in \overline{y}} \ \bigvee_{x \in A, y \in B} r(x, y).$$

This theorem may be generalized from such quantales to those complete and cocomplete symmetric monoidal-closed categories $\mathcal{V}$ in which every object is a coproduct of connected objects, in particular to $\mathcal{V} = \textbf{Set}$. This leads to the category

$$(\mathrm{U}, \textbf{Set})\text{-}\textbf{Cat} = \textbf{UltraCat}_{\mathrm{enriched}}$$

of Clementino-T 2003, which I conjecture to coincide with $\textbf{UltraCat}_{\mathrm{internal}}$.

A fundamental adjunction

The **Set**-monad T with its lax extension $\hat{\mathsf{T}}$ to V-**Rel**
may be considered as a (KZ-)monad on V-**Cat** (T 2009): $\mathsf{T}(X, a_0) = (\mathsf{T}X, \hat{\mathsf{T}}a_0)$

A fundamental adjunction

The **Set**-monad T with its lax extension $\hat{T}$ to V-**Rel**
may be considered as a (KZ-)monad on V-**Cat** (T 2009): $T(X, a_0) = (TX, \hat{T}a_0)$

If the Kleisli convolution is associative, then (Clementino-Hofmann 2009):

$$(X, a_0 : X \relbar\joinrel\mapstochar\joinrel\rightarrow X, \xi : TX \to X) \longmapsto (X, a_0 \cdot \xi_\circ : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$$

$$\begin{tikzcd}
(V\text{-}\mathbf{Cat})^{T} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\top"] & & (T, V)\text{-}\mathbf{Cat} \arrow[ll, bend left, "M"]
\end{tikzcd}$$

$$(TX, \underbrace{\hat{T}a \cdot \mu_X^\circ}_{=:\, \hat{a} \,:\, TX \relbar\joinrel\mapstochar\joinrel\rightarrow TX}, \mu_X : TTX \to TX) \longleftarrow\!\!\shortmid (X, a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$$

A fundamental adjunction

The **Set**-monad T with its lax extension $\hat{T}$ to V-**Rel**
may be considered as a (KZ-)monad on V-**Cat** (T 2009): $T(X, a_0) = (TX, \hat{T}a_0)$

If the Kleisli convolution is associative, then (Clementino-Hofmann 2009):

$$(X, a_0 : X \relbar\joinrel\mapstochar\joinrel\rightarrow X, \xi : TX \to X) \longmapsto (X, a_0 \cdot \xi_\circ : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$$

$$\begin{tikzcd}
(\text{V-}\mathbf{Cat})^{T} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\top"] & & (T, \text{V})\text{-}\mathbf{Cat} \arrow[ll, bend left, "M"]
\end{tikzcd}$$

$$(TX, \underbrace{\hat{T}a \cdot \mu_X^\circ}_{=: \hat{a} : TX \relbar\joinrel\mapstochar\joinrel\rightarrow TX}, \mu_X : TTX \to TX) \longmapsfrom (X, a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$$

In particular (Hofmann 2007): If the V-category (V, hom) has a *good* T-structure ξ, then K makes V a (T, V)-category, enables dualization, Yoneda embedding, ...

$M \dashv K$ is a factor of the Eilenberg-Moore adjunction

$$\begin{tikzcd}
 & (X, a) \arrow[r, mapsto] & (X, a \cdot (\eta_X)_\circ) \\
(\mathsf{V}\text{-}\mathbf{Cat})^{\mathrm{T}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & (\mathsf{T}, \mathsf{V})\text{-}\mathbf{Cat} \arrow[l, bend left, "M"] \arrow[r, bend left, "A_\circ"] \arrow[r, phantom, "\top"] & \mathsf{V}\text{-}\mathbf{Cat} \arrow[l, bend left, "A^\circ"] \\
 & (X, \eta_X^\circ \cdot \hat{\mathsf{T}} a_0) & (X, a_0) \arrow[l, mapsto]
\end{tikzcd}$$

$M \dashv K$ is a factor of the Eilenberg-Moore adjunction

$$\begin{tikzcd}
 & (X, a) \arrow[r, mapsto] & (X, a \cdot (\eta_X)_\circ) \\
(\mathsf{V}\text{-}\mathbf{Cat})^{\mathrm{T}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & (\mathsf{T}, \mathsf{V})\text{-}\mathbf{Cat} \arrow[l, bend left, "M"] \arrow[r, bend left, "A_\circ"] \arrow[r, phantom, "\top"] & \mathsf{V}\text{-}\mathbf{Cat} \arrow[l, bend left, "A^\circ"] \\
 & (X, \eta_X^\circ \cdot \hat{\mathsf{T}} a_0) & (X, a_0) \arrow[l, mapsto]
\end{tikzcd}$$

$\mathsf{T} = \mathrm{U},\ \mathsf{V} = 2$:

$$\begin{tikzcd}
\mathbf{OrdCompHaus} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Top} \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left]
\end{tikzcd}$$

K topologizes $(X, \leq, \xi)$ by $(\overline{x} \rightsquigarrow y \iff \xi(\overline{x}) \leq y)$; $A_\circ =$(dual) specialization order

M orders $\mathrm{U}X$ by $(\overline{x} \leq \overline{y} \iff \forall A \in \overline{x}$ closed in $X : A \in \overline{y})$; $A^\circ =$ Alexandroff topol.

The greater picture (when T is flat and V integral)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(\mathsf{V}\text{-}\mathbf{Cat})^{\mathsf{T}} \arrow[r, bend left=15, "K"] \arrow[r, phantom, "\top"] \arrow[d, bend right=40] \arrow[d, bend left=25, "\pi_0"] & (\mathsf{T},\mathsf{V})\text{-}\mathbf{Cat} \arrow[l, bend left=15, "M"] \arrow[r, bend left=15, "A_\circ"] \arrow[r, phantom, "\top"] & \mathsf{V}\text{-}\mathbf{Cat} \arrow[l, bend left=15, "A^\circ"] \arrow[d, bend right=40] \arrow[d, bend left=25, "\pi_0"] \\
\mathbf{Set}^{\mathsf{T}} \arrow[u, bend left=60, "c"] \arrow[u, phantom, shift left=16, "\vdash"] \arrow[u, phantom, shift left=11, "\vdash"] \arrow[u, bend left=15, "d"] \arrow[u, phantom, "\vdash"] \arrow[rr, bend left=8] \arrow[rr, phantom, "\top"] & & \mathbf{Set} \arrow[u, bend left=60, "c"] \arrow[u, phantom, shift left=16, "\vdash"] \arrow[u, phantom, shift left=11, "\vdash"] \arrow[u, bend left=15, "d"] \arrow[u, phantom, "\vdash"] \arrow[ll, bend left=8]
\end{tikzcd}$$

The greater picture (when $\mathrm{T} = \mathrm{U}$ and $\mathsf{V} = \mathsf{2}$)

$$\begin{tikzcd}
\textbf{OrdCompHaus} \arrow[r, bend left] \arrow[r, phantom, "\top"] \arrow[d, <-, shift right=6, "c"'] \arrow[d, phantom, shift right=4.5, "\vdash"] \arrow[d, shift right=3] \arrow[d, phantom, shift right=1.5, "\vdash"] \arrow[d, <-, "d"'] \arrow[d, phantom, shift left=1.5, "\vdash"] \arrow[d, shift left=3, "\pi_0"] & \textbf{Top} \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\top"] \arrow[dl, bend left, "\beta"] & \textbf{Ord} \arrow[l, bend left] \arrow[d, <-, shift right=6, "c"'] \arrow[d, phantom, shift right=4.5, "\vdash"] \arrow[d, shift right=3] \arrow[d, phantom, shift right=1.5, "\vdash"] \arrow[d, <-, "d"'] \arrow[d, phantom, shift left=1.5, "\vdash"] \arrow[d, shift left=3, "\pi_0"] \\
\textbf{CompHaus} \arrow[ur, bend left, "\mathrm{incl}"] \arrow[ur, phantom, "\top"] \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \textbf{Set} \arrow[ll, bend left]
\end{tikzcd}$$

Note:
So far, we are able to justify the name "π_0" only when $X \in$ **Top** is normal; that is: when X is normal, βX is homeomorphic to the space of connected components wrt the order that is imposed on $\mathrm{U}X$ by the functor M.

Replacing inequalities by equalities: T_1-separation, core compactness

$(X, a : \mathsf{T}X \nrightarrow X)$

(R) $1_X \leq a \cdot (\eta_X)_\circ$ **T_1** : $1_X \geq a \cdot (\eta_X)_\circ$

$\mathrm{T} = \mathrm{U}, \mathsf{V} = 2 : \quad (\dot{x} \rightsquigarrow y \Rightarrow x = y)$

Replacing inequalities by equalities: T_1-separation, core compactness

$(X, a : \mathrm{T}X \nrightarrow X)$

(R) $1_X \leq a \cdot (\eta_X)_\circ$ **T₁** : $1_X \geq a \cdot (\eta_X)_\circ$

$\mathrm{T} = \mathrm{U}, \mathsf{V} = 2$: $(\dot{x} \rightsquigarrow y \Rightarrow x = y)$

(T) $a \cdot \hat{\mathrm{T}}a \leq a \cdot (\mathrm{m}_X)_\circ$ **core compact:** $a \cdot \hat{\mathrm{T}}a \geq a \cdot (\mu_X)_\circ$

Pisani 1999: $\mathrm{T} = \mathrm{U}, \mathsf{V} = 2$: $\mu_X(\mathcal{X}) \rightsquigarrow z \Rightarrow \exists\, \overline{y}\, (\mathcal{X} \rightsquigarrow \overline{y} \rightsquigarrow z)$

$\iff \forall x \in B \subseteq X$ open

$\exists A \subseteq X$ open $(x \in A \ll B)$

$\iff X$ exponentiable in **Top**

Replacing inequalities by equalities: T_1-separation, core compactness

$(X, a : \mathsf{T}X \nrightarrow X)$

(R) $1_X \leq a \cdot (\eta_X)_\circ$ **T_1** : $1_X \geq a \cdot (\eta_X)_\circ$

$\mathrm{T} = \mathrm{U}, \mathsf{V} = 2$: $(\dot{x} \rightsquigarrow y \Rightarrow x = y)$

(T) $a \cdot \hat{\mathsf{T}}a \leq a \cdot (\mathrm{m}_X)_\circ$ **core compact:** $a \cdot \hat{\mathsf{T}}a \geq a \cdot (\mu_X)_\circ$

Pisani 1999: $\mathrm{T} = \mathrm{U}, \mathsf{V} = 2$: $\mu_X(\mathcal{X}) \rightsquigarrow z \Rightarrow \exists\, \overline{y}\, (\mathcal{X} \rightsquigarrow \overline{y} \rightsquigarrow z)$

$\iff \forall x \in B \subseteq X$ open

$\exists A \subseteq X$ open $(x \in A \ll B)$

$\iff X$ exponentiable in **Top**

Note: If we express
(R) and (T) equivalently as $\eta_X^\circ \leq a$ and $a \circ a \leq a$ resp., and “strictify” these inequalities, *different* properties will emerge: discrete and no condition at all!

Replacing inequalities by equalities: proper maps, open maps

$f : (X, a) \to (Y, b)$

$f_\circ \cdot a \leq b \cdot (\mathrm{T}f)_\circ$ **proper:** $f_\circ \cdot a \geq b \cdot (\mathrm{T}f)_\circ$ $\bigvee_{x \in f^{-1} y} a(\overline{x}, x) \geq b(\mathrm{T}f(\overline{x}), y))$

Manes 1974: $\mathrm{T} = \mathrm{U}, \mathsf{V} = 2:$

$$\begin{tikzcd}
\overline{x} \arrow[r, dotted] \arrow[d, dash] & x \arrow[d, dashed, dash] \\
f[\overline{x}] \arrow[r] & y
\end{tikzcd}$$

Replacing inequalities by equalities: proper maps, open maps

$f : (X, a) \to (Y, b)$

$f_\circ \cdot a \leq b \cdot (\mathrm{T}f)_\circ$ **proper:** $f_\circ \cdot a \geq b \cdot (\mathrm{T}f)_\circ$ $\displaystyle\bigvee_{x\in f^{-1}y} a(\overline{x}, x) \geq b(\mathrm{T}f(\overline{x}), y))$

Manes 1974: $\mathrm{T} = \mathrm{U}, \mathrm{V} = 2 :$

$$\begin{tikzcd}
\overline{x} \arrow[r, dotted] \arrow[d, dash] & x \arrow[d, dash, dashed] \\
f[\overline{x}] \arrow[r] & y
\end{tikzcd}$$

$a \cdot (\mathrm{T}f)^\circ \leq f^\circ \cdot b$ **open:** $a \cdot (\mathrm{T}f)^\circ \geq f^\circ \cdot b$ $\displaystyle\bigvee_{\overline{x}\in(\mathrm{T}f)^{-1}\overline{y}} a(\overline{x}, x) \geq b(\overline{y}, f(x))$

Möbus 1981: $\mathrm{T} = \mathrm{U}, \mathrm{V} = 2 :$

$$\begin{tikzcd}
\overline{x} \arrow[r, dotted] \arrow[d, dash, dashed] & x \arrow[d, dash] \\
\overline{y} \arrow[r] & f(x)
\end{tikzcd}$$

Some stability properties for proper and open maps

- Isomorphisms are proper/open
- Proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\implies f$ proper/open
- $g \cdot f$ proper/open, f surjective $\implies g$ proper/open
- f proper/open $\implies$ every pullback of f is proper/open

Some stability properties for proper and open maps

- Isomorphisms are proper/open
- Proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\implies$ f proper/open
- $g \cdot f$ proper/open, f surjective $\implies$ g proper/open
- f proper/open $\implies$ every pullback of f is proper/open

Theorem (Tychonoff-Frolík-Bourbaki Theorem)

Let V be completely distributive. Then:

$$f_i : X_i \to Y_i \quad \textit{proper} \ \ (i \in I) \implies \prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \quad \textit{proper}$$

Some stability properties for proper and open maps

- Isomorphisms are proper/open
- Proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\Longrightarrow$ f proper/open
- $g \cdot f$ proper/open, f surjective $\Longrightarrow$ g proper/open
- f proper/open $\Longrightarrow$ every pullback of f is proper/open

Theorem (Tychonoff-Frolík-Bourbaki Theorem)

Let V be completely distributive. Then:

$$f_i : X_i \to Y_i \quad \textit{proper} \;\; (i \in I) \Longrightarrow \prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i \quad \textit{proper}$$

Note that, by contrast (*not* by categorical dualization!), one has:

$$f_i : X_i \to Y_i \quad \text{open} \;\; (i \in I) \Longrightarrow \coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i \quad \text{open}$$

Hausdorff separation and compactness as adjoints

Under light assumptions on V (excluding 2^M, but none of the other examples):

(X, a) **Hausdorff:** $a \cdot a^\circ \leq 1_X$ $\quad \bot < a(\overline{z}, x) \otimes a(\overline{z}, y) \;\Rightarrow\; x = y$

(X, a) **compact:** $1_{TX} \leq a^\circ \cdot a$ $\quad \forall\, \overline{z} \in TX : \mathrm{k} \leq \bigvee_{x \in X} a(\overline{z}, x)$

Hausdorff separation and compactness as adjoints

Under light assumptions on V (excluding 2^M, but none of the other examples):

(X, a) **Hausdorff:** $a \cdot a^\circ \leq 1_X$ $\quad \bot < a(\overline{z}, x) \otimes a(\overline{z}, y) \;\Rightarrow\; x = y$

(X, a) **compact:** $1_{TX} \leq a^\circ \cdot a$ $\quad \forall\, \overline{z} \in TX : \mathrm{k} \leq \bigvee_{x \in X} a(\overline{z}, x)$

Theorem (Manes, Lawvere, Clementino-Hofmann, T)

$$\mathbf{Set}^{\mathrm{T}} = (\mathrm{T}, \mathrm{V})\text{-}\mathbf{Cat}_{\mathrm{Comp}} \cap (\mathrm{T}, \mathrm{V})\text{-}\mathbf{Cat}_{\mathrm{Haus}}$$

Hausdorff separation and compactness as adjoints

Under light assumptions on V (excluding 2^M, but none of the other examples):

(X, a) **Hausdorff:** $a \cdot a^\circ \leq 1_X \qquad \bot < a(\mathfrak{z}, x) \otimes a(\mathfrak{z}, y) \;\Rightarrow\; x = y$

(X, a) **compact:** $1_{TX} \leq a^\circ \cdot a \qquad \forall\, \mathfrak{z} \in TX : \mathrm{k} \leq \bigvee_{x \in X} a(\mathfrak{z}, x)$

Theorem (Manes, Lawvere, Clementino-Hofmann, T)

$$\mathbf{Set}^{\mathbb{T}} = (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat}_{\mathrm{Comp}} \cap (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat}_{\mathrm{Haus}}$$

Proof:

$$(a \cdot a^\circ \leq 1_X \text{ and } 1_{TX} \leq a^\circ \cdot a) \iff a \dashv a^\circ \iff a \text{ is a map}$$

Normality and extremal disconnectedness

Reminder:

$X \in \mathbf{Top}$ normal $\iff$ disjoint closed sets have disjoint nbhds in X

X extremally disconnected $\iff$ closures of open sets are open in X

How do these properties fare in our setting?

Normality and extremal disconnectedness

Reminder:

$X \in \mathbf{Top}$ normal $\iff$ disjoint closed sets have disjoint nbhds in X

X extremally disconnected $\iff$ closures of open sets are open in X

How do these properties fare in our setting? Recall:

$$(\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \xrightarrow{M} \mathsf{V}\text{-}\mathbf{Cat}^{\mathbb{T}} \to \mathsf{V}\text{-}\mathbf{Cat}, \qquad (X, a) \mapsto (TX, \hat{a}, \mu_X) \mapsto (TX, \hat{a}),$$

$$\text{with } \hat{a} = (TX \xrightarrow{\mu_X^\circ} TTX \xrightarrow{\hat{T}a} TX)$$

Normality and extremal disconnectedness

Reminder:

$X \in \mathbf{Top}$ normal $\iff$ disjoint closed sets have disjoint nbhds in X

X extremally disconnected $\iff$ closures of open sets are open in X

How do these properties fare in our setting? Recall:

$$(\mathrm{T},\mathsf{V})\text{-}\mathbf{Cat} \xrightarrow{M} \mathsf{V}\text{-}\mathbf{Cat}^{\mathrm{T}} \to \mathsf{V}\text{-}\mathbf{Cat}, \quad (X,a) \mapsto (\mathrm{T}X, \hat{a}, \mu_X) \mapsto (\mathrm{T}X, \hat{a}),$$

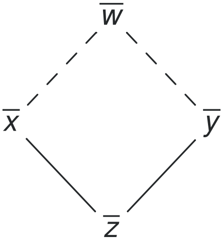
$$\text{with } \hat{a} = (\mathrm{T}X \xrightarrow{\mu_X^\circ} \mathrm{T}\mathrm{T}X \xrightarrow{\hat{T}a} \mathrm{T}X)$$

For $\mathrm{T} = \mathrm{U}$, $\mathsf{V} = 2$ and $X \in \mathbf{Top}$, the functor provides $\mathrm{U}X$ with the order

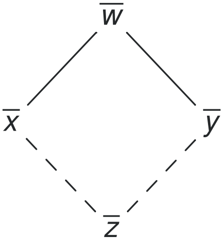
$$\overline{x} \leq \overline{y} \iff \forall A \subseteq X \text{ closed} : (A \in \overline{x} \Rightarrow A \in \overline{y})$$

Normality and extremal disconnectedness are dual to each other!

$X \in$ **Top** normal

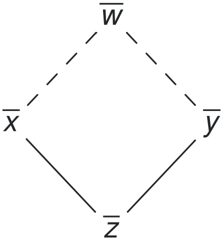


X extremally disconnected

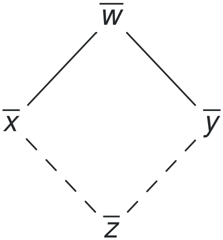


Normality and extremal disconnectedness are dual to each other!

$X \in \mathbf{Top}$ normal



X extremally disconnected



$(X, a) \in (\mathsf{T}, \mathsf{V})$-**Cat** **normal**

$$\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}$$

$$\Longleftrightarrow$$

$(\mathsf{T}X, \hat{a})$ normal in V-**Cat**

(X, a) **extremally disconnected**

$$\hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ$$

$$\Longleftrightarrow$$

$(\mathsf{T}X, \hat{a}^\circ)$ normal in V-**Cat**

Monoidal topology without convergence relations?

Cls $c : \mathsf{P}X \to 2^X$ (R) $A \subseteq cA$

(T) $B \subseteq cA \Rightarrow cB \subseteq cA$

Top c finitely additive: (A) $c(A \cup B) = cA \cup cB$

$c\varnothing = \varnothing$

(C) $f(c_X A) \subseteq c_Y(fA)$

Monoidal topology without convergence relations?

Cls $c : \mathsf{P}X \to 2^X$ (R) $A \subseteq cA$

(T) $B \subseteq cA \Rightarrow cB \subseteq cA$

Top c finitely additive: (A) $c(A \cup B) = cA \cup cB$

$c\varnothing = \varnothing$

(C) $f(c_X A) \subseteq c_Y(fA)$

[Seal 2009]

V-**Cls** $c : \mathsf{P}X \to \mathsf{V}^X$ (R) $\forall x \in A : \mathrm{k} \leq (cA)(x)$

$= (\mathrm{P}, \mathsf{V})$-**Cat** (T) $\displaystyle \Big(\bigwedge_{y \in B} (cA)(y)\Big) \otimes (cB)(x) \leq (cA)(x)$

[Lai-T 2016]

V-**Top** c finitely additive: (A) $c(A \cup B)(x) = (cA)(x) \vee (cB)(x)$

$(c\varnothing)(x) = \bot$

(C) $(c_X A)(x) \leq c_Y(fA)(fx)$

The case $\mathsf{V} = [0, \infty]$ gives approach spaces

$[0, \infty]$-**Cls**

$\delta : X \times \mathsf{P}X \to [0, \infty]$ (R) $\forall x \in A : 0 \geq \delta(x, A)$

$\delta(x, A) = (cA)(x)$ (T) $(\sup_{y \in B} \delta(y, A)) + \delta(x, B) \geq \delta(x, A)$

$[0, \infty]$-**Top** $=:$ **App** [Lowen 1989] (but his condition (T) is different!):

δ finitely additive (A) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$

$\delta(x, \varnothing) = \infty$

$f : X \to Y$ (C) $\delta_X(x, A) \geq \delta_Y(fx, fA)$

How to reconcile closure and ultrafilter convergence?

For V completely distributive:

P and U interact via the V-relation $\varepsilon_X : \mathsf{P}X \nrightarrow \mathsf{U}X$ via $\varepsilon_X(A, \overline{x}) = \begin{cases} \mathrm{k} & \text{if } A \in \overline{x} \\ \bot & \text{else} \end{cases}$

With suitable lax extensions obtain

$$A_\varepsilon : (\mathrm{U}, \mathsf{V})\text{-}\mathbf{Cat} \longrightarrow (\mathrm{P}, \mathsf{V})\text{-}\mathbf{Cat}, \quad (X, a) \longmapsto (X, c_a = a \cdot \varepsilon_X)$$

$$(c_a A)(y) = \bigvee_{\overline{x} \ni A} a(\overline{x}, y)$$

A_ε has a right adjoint $(X, c) \longmapsto (X, a_c)$, with

$$a_c(\overline{x}, y) = \bigwedge_{A \in \overline{x}} (cA)(y)$$

$(\mathbb{U}, \mathsf{V})$-**Cat** $\cong$ V-**Top**

Theorem (Lai-T 2016)

Let V *be completely distributive. Then:*

$$A_\varepsilon : (\mathbb{U}, \mathsf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{P}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cls}$$

is a full coreflective embedding; its image is V-**Top** $\cong (\mathbb{U}, \mathsf{V})$-**Cat**.

(U, V)-**Cat** $\cong$ V-**Top**

Theorem (Lai-T 2016)

Let V *be completely distributive. Then:*

$$A_\varepsilon : (\mathrm{U}, \mathsf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathrm{P}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cls}$$

is a full coreflective embedding; its image is V-**Top** $\cong (\mathrm{U}, \mathsf{V})$-**Cat**.

Corollary (Clementino-Hofmann 2003)

$$\mathbf{App} \cong (\mathrm{U}, [0, \infty])\text{-}\mathbf{Cat}$$

To-Do list

- Pursue monoidal topology (enriched) in Burroni’s (internal) context ...
- ... and conversely!
- Explore the “algebra-topology” gap in Burroni’s setting; partial algebras!
- Apply (T, V)-category theory to “probabilistic” quantales or monads.
- To which extent are (T, V)-categories covered by Burroni?
- Apply the emerging theory in particular in topological algebra.
- Dualization, Yoneda, (monoidal) closedness, 2-categorical structure, ...

Some references

- E. G. Manes, Ph.D. thesis, 1967; SLNM 80, 1969.
- M. Barr, SLNM 137, 1970.
- A. Burroni, Cahiers 12, 1971.
- F. W. Lawvere, Milano 1973; TAC Reprints 1, 2002.
- A. Möbus, Ph.D. thesis, 1981; Arch. Math. (Basel) 40, 1983.
- R. Lowen, Math. Nachr. 141, 1989; Oxford UP 1997.
- M. M. Clementino, D. Hofmann, ACS 11, 2003.
- M. M. Clementino, W. Tholen, JPAA 179, 2003.
- M. M. Clementino, D. Hofmann, W. Tholen, ACS 11, 2003.
- G. J. Seal, TAC 14, 2005.
- D. Hofmann, Adv. Math. 215, 2007.
- D. Hofmann, G. J. Seal, W. Tholen (eds), Cambridge UP, 2014.
- W. Tholen, L. Yeganeh, TAC 36 , 2021.