Functorial Decomposition of Colimits in Categories and a Generalized Fubini Formula

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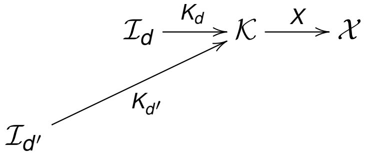
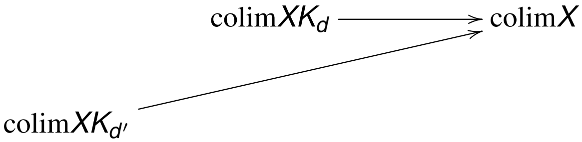
Colimit Decomposition Formula

$D : \mathcal{D} \longrightarrow \mathbf{Cat}$, $d \mapsto \mathcal{I}_d$, $(K_d : \mathcal{I}_d \longrightarrow \mathcal{K})_d$ colimit, $X : \mathcal{K} \longrightarrow \mathcal{X}$

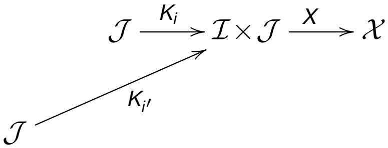
$$\mathrm{colim}^{\mathcal{K}} X \cong \mathrm{colim}^{d \in \mathcal{D}} (\mathrm{colim}^{\mathcal{I}_d} X K_d).$$

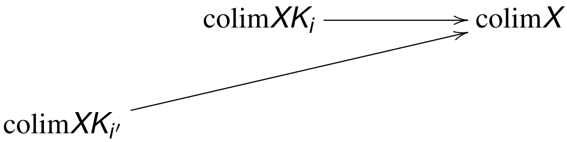




Compare: Fubini Formula for iterated colimits

$$\mathrm{colim}^{\mathcal{I}\times\mathcal{J}} X \cong \mathrm{colim}^{i\in\mathcal{I}}(\mathrm{colim}^{\mathcal{J}} X(i,-)).$$





Questions

- Are these just instances of "(co)limits commute with (co)limits"?
- What about the dualized formulae?
- Is the Fubini Formula an instance of the Decomposition Formula?
- All this should not be new stuff: References?

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"Direct" proof of the Decomposition Formula: Proof 1

(1) $\mathcal{I}_d \xrightarrow{K_d} \mathcal{K}$ colim in **Cat** $\Longrightarrow$

$$\begin{array}{ccc} \mathcal{I}_d & \xrightarrow{\quad K_d \quad} & \mathcal{K} \\ & {\scriptstyle XK_d} \searrow \quad \swarrow {\scriptstyle X} & \\ & \mathcal{X} & \end{array}$$

$\mathcal{K}$ colim in **Cat**$/\mathcal{X}$

(2) **Cat**$/\mathcal{X} \longrightarrow \mathrm{Diag}(\mathcal{X}) =$ **Cat**$//\mathcal{X}$ preserves colimits (does it?)

(3) $\mathcal{X}$ cocomplete $\iff$ $\mathcal{X} \longrightarrow \mathrm{Diag}(\mathcal{X})$ has a left adjoint,

colim, which therefore preserves the colimit of (1)

Conclusion:

Colimit Decomposition Formula holds in every cocomplete category $\mathcal{X}$.

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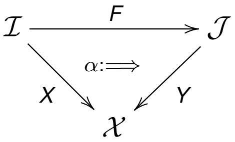
Diag($\mathcal{X}$) : $(F, \alpha) : X \longrightarrow Y$



Trivially: **Cat**$/\mathcal{X} \longrightarrow$ Diag($\mathcal{X}$) preserves coproducts. Coequalizers:

Suffices to show: $\forall X,\ Y : \mathcal{K} \longrightarrow \mathcal{X},\ \nu : XH \longrightarrow YH$:

$$\nu F = \nu G \implies \exists! \rho : X \longrightarrow Y \ (\nu = \rho H)$$

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We better check claim(2)! Finish:

For $k = Hj$, put $\rho_k := \nu_j$; well-defined since $\nu F = \nu G$.

Show naturality in $r = Hf_n \cdot ... \cdot Hf_2 \cdot Hf_1 : k \longrightarrow k'$ in $\mathcal{K}$,

with $f_t : j_t \longrightarrow j'_t \, (t = 1, ..., n)$ in $\mathcal{J}$. Case $n = 2$ should suffice here.

Since $k = Hj_1$, $Hj'_1 = Hj_2$, $Hj'_2 = k'$:

$$\begin{array}{ccccc} Xk = XHj_1 & \xrightarrow{XHf_1} & XHj'_1 = XHj_2 & \xrightarrow{XHf_2} & XHj'_2 = Xk' \\ \rho_k = \downarrow \nu_{j_1} & & \nu_{j'_1} = \downarrow \nu_{j_2} & & \nu_{j'_2} \downarrow = \rho_{k'} \\ Yk = YHj_1 & \xrightarrow[YHf_1]{} & YHj'_1 = YHj_2 & \xrightarrow[YHf_2]{} & YHj'_2 = Yk' \end{array}$$

commutes – done!

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Connection with Fubini?

Given

$$D : \mathcal{D} \longrightarrow \mathbf{Cat},\ (u : d \to e) \mapsto (u_! : \mathcal{I}_d \to \mathcal{I}_e),$$

form the Grothendieck category

$$\mathcal{G} = \int_{d \in \mathcal{D}}^{\circ} \mathcal{I}_d$$

$(u, \varphi) : (d, x) \longrightarrow (e, y)$ in $\mathcal{G}$ means $u : d \longrightarrow e$ in $\mathcal{D}$, $\varphi : u_!(x) \longrightarrow y$ in $\mathcal{I}_e$

Example:

$D : \mathcal{I} \longrightarrow \mathbf{Cat},\ i \mapsto \mathcal{J}$. Then

$$\int_{i \in \mathcal{I}}^{\circ} \mathcal{J} = \mathcal{I} \times \mathcal{J}$$

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Generalized Fubini Formula

For $D : \mathcal{D} \longrightarrow \mathbf{Cat}$, the fibres of the opfibration $\mathcal{G} \longrightarrow \mathcal{D}$, $(d, x) \longrightarrow d$, are (the images of)

$$J_d : \mathcal{I}_d \longrightarrow \mathcal{G},\ (\varphi : x \longrightarrow x') \mapsto ((1_d, \varphi) : (d, x) \longrightarrow (d, x')),$$

THEOREM

For $Y : \int^{\circ} D \longrightarrow \mathcal{X}$, let the colimits $\tilde{Y}d \cong \mathrm{colim}^{\mathcal{I}_d} YJ_d$ exist in $\mathcal{X}$, for all objects $d \in \mathcal{D}$. Then the colimit of Y exists in $\mathcal{X}$ if, and only if, the colimit of $\tilde{Y} : \mathcal{D} \longrightarrow \mathcal{X}$ exists, in which case the two colimits coincide:

$$\mathrm{colim}^{\int^{\circ} D} Y \;\cong\; \mathrm{colim}^{d \in \mathcal{D}}(\mathrm{colim}^{\mathcal{I}_d} YJ_d).$$

Fubini implies the Decomposition Formula: Proof 2

Given $D : \mathcal{D} \longrightarrow \textbf{Cat}$, consider

$$Q : \int^{\circ} D \longrightarrow \mathrm{colim} D$$

with $QJ_d = K_d$ $(d \in \mathcal{D})$:

$Q : \mathcal{G} \longrightarrow \mathcal{K},\ ((u,\varphi) : (d,x) \longrightarrow (e,y)) \mapsto (K_e\varphi : K_d x = K_e u_! x \longrightarrow K_e y).$

PROPOSITION

Q is final, *i.e.*, the categories $k \downarrow Q$ $(k \in \mathcal{K})$ are connected.

COROLLARY

For every $X : \mathcal{K} \longrightarrow \mathcal{X}$ such that $\mathrm{colim}^{\mathcal{G}} XQ$ exists in $\mathcal{X}$, also $\mathrm{colim}^{\mathcal{K}} X$ exists, and the canonical $\mathrm{colim} XQ \longrightarrow \mathrm{colim} X$ is an isomorphism.

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Observation: Diag($\mathcal{X}$) as a Grothendieck category!

$$\mathcal{X}^{(-)} : \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{CAT},\ (F : \mathcal{I} \longrightarrow \mathcal{J}) \mapsto (F^{\star} : \mathcal{X}^{\mathcal{J}} \longrightarrow \mathcal{X}^{\mathcal{I}},\ Y \mapsto YF)$$

$$\mathrm{Diag}(\mathcal{X}) = \int \mathcal{X}^{(-)}$$

Obtain the fibration

$$P_{\mathcal{X}} : \mathrm{Diag}(\mathcal{X}) \longrightarrow \mathbf{Cat},\quad \begin{array}{ccc} \mathcal{I} & \xrightarrow{\quad F \quad} & \mathcal{J} \\ & {\scriptstyle \alpha : \Longrightarrow} & \\ {\scriptstyle X} \searrow & & \swarrow {\scriptstyle Y} \\ & \mathcal{X} & \end{array} \quad \mapsto \quad F$$

$P_{\mathcal{X}}$ opfibration $\iff \forall F : \mathcal{I} \longrightarrow \mathcal{J} \quad \exists F_! \dashv F^* : \mathcal{X}^{\mathcal{J}} \longrightarrow \mathcal{X}^{\mathcal{I}}$

$\iff \forall F : \mathcal{I} \longrightarrow \mathcal{J},\ X : \mathcal{I} \longrightarrow \mathcal{X} : \quad \mathrm{Lan}_F X$ exists

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Lifting theorem for (co)limits along a bifibration

For a bifibration $P : \mathcal{E} \longrightarrow \mathcal{B}$ and any diagram category $\mathcal{D}$,
let $\mathcal{B}$ and all fibres $\mathcal{E}(B)$ $(B \in \mathcal{B})$ be $\mathcal{D}$-(co)complete.
Then $\mathcal{E}$ is also $\mathcal{D}$-(co)complete, and P preserves the $\mathcal{D}$-(co)limits.

Construction:

Step 1: Given $D : \mathcal{D} \longrightarrow \mathcal{E}$, form colimit $(\beta_d : PDd \longrightarrow B)$ in $\mathcal{B}$.

Step 2: Form a cocartesian lifting $\alpha_d : Dd \longrightarrow Ld$ of every β_d.

Step 3: Form colimit $(\lambda_d : Ld \longrightarrow E)$ in the fibre $\mathcal{E}(B)$.

Conclusion: $(\lambda_d \cdot \alpha_d : Dd \longrightarrow E)$ is a colimit of D in $\mathcal{E}$.

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Apply the lifting theorem to $P_{\mathcal{X}} : \mathrm{Diag}(\mathcal{X}) \longrightarrow \mathbf{Cat}$

$\mathcal{X}$ cocomplete (so that $P_{\mathcal{X}}$ is a bifibration), $D : \mathcal{D} \longrightarrow \mathrm{Diag}(\mathcal{X})$

$$(u : d \longrightarrow e) \quad \mapsto \quad \begin{array}{ccc} \mathcal{I}_d & \xrightarrow{\quad u_! \quad} & \mathcal{I}_e \\ & \alpha^u : \Longrightarrow & \\ X_d \searrow & & \swarrow X_e \\ & \mathcal{X} & \end{array}$$

Step 1: Form colimit $(K_d : \mathcal{I}_d \longrightarrow \mathcal{K})$ in **Cat**.

Step 2: Form

$$\begin{array}{ccc} \mathcal{I}_d & \xrightarrow{\quad K_d \quad} & \mathcal{K} \\ & \kappa_d : \Longrightarrow & \\ X_d \searrow & & \swarrow L_d = \mathrm{Lan}_{K_d} X_d \\ & \mathcal{X} & \end{array}$$

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Conclusion: $(K_d, \lambda_d K_d \cdot \kappa_d) : (\mathcal{I}_d, X_d) \longrightarrow (\mathcal{K}, X)$ is a colimit in $\mathrm{Diag}(\mathcal{X})$.

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Generalized Colimit Decomposition Formula: Proof 3

THEOREM

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$$\mathrm{colim}^{\mathcal{K}} X \;\cong\; \mathrm{colim}^{d \in \mathcal{D}}(\mathrm{colim}^{\mathcal{I}_d} X_d),$$

with $(K_d : \mathcal{K} \longrightarrow \mathcal{I}_d)$ colimit in **Cat** and $X = \mathrm{colim}^{d \in \mathcal{D}}(\mathrm{Lan}_{K_d} X_d)$ in $\mathcal{X}^{\mathcal{K}}$.

COROLLARY (Colimit Decomposition Formula)

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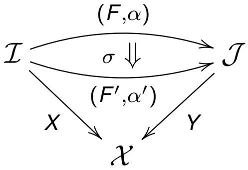
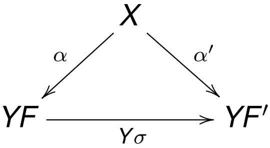
Better late than never: Diag($\mathcal{X}$) is a 2-category!





$P_{\mathcal{X}}$: Diag($\mathcal{X}$) $\longrightarrow$ **Cat** is a 2-functor!

Likewise for

$$P^*_{\mathcal{X}} : \mathrm{Diag}^*(\mathcal{X}) = (\mathrm{Diag}(\mathcal{X}^{\mathrm{op}}))^{\mathrm{co}} \longrightarrow \mathbf{Cat}.$$

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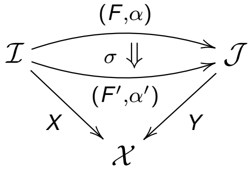

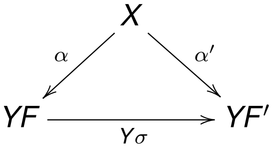

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Likewise for

$$P^*_{\mathcal{X}} : \mathrm{Diag}^*(\mathcal{X}) = (\mathrm{Diag}(\mathcal{X}^{\mathrm{op}}))^{\mathrm{co}} \longrightarrow \textbf{Cat}.$$

2-fibrations: Hermida 1999, Buckley 2014

$P : \mathcal{E} \longrightarrow \mathcal{B}$ 2-functor

1-cell $f : D \longrightarrow E$ *2-cartesian* : $\iff$

$$\begin{array}{ccc} \mathcal{E}(C,D) & \xrightarrow{\mathcal{E}(C,f)} & \mathcal{E}(C,E) \\ {\scriptstyle P_{C,D}}\downarrow & & \downarrow{\scriptstyle P_{C,E}} \\ \mathcal{B}(PC,PD) & \xrightarrow[\mathcal{B}(PC,Pf)]{} & \mathcal{B}(PC,PE) \end{array}$$

pullback in **Cat**, for all $C \in \mathcal{E}$.

2-cell $\sigma : f \longrightarrow f' : D \longrightarrow E$ *2-cartesian* : $\iff$ σ cartesian w.r.t. $P_{D,E} : \mathcal{E}(D,E) \longrightarrow \mathcal{B}(PD,PE)$.

P 2-fibration : $\iff$

(a) every 1-cell has a 2-cartesian lifting;

(b) $\forall\ D, E \in \mathcal{E}$: $P_{D,E} : \mathcal{E}(D,E) \longrightarrow \mathcal{B}(PD,PE)$ is (ordinary) fibration;

(c) 2-cartesianness of 2-cells is preserved by horizontal composition.

2-fibrations: Hermida 1999, Buckley 2014

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Setting the 2-categorical stage

$P : \mathcal{E} \longrightarrow \mathcal{B}$ *2-opfibration* $:\iff$ $P^{\mathrm{coop}} : \mathcal{E}^{\mathrm{coop}} \longrightarrow \mathcal{B}^{\mathrm{coop}}$ 2-fibration.

THEOREM

(1) $P^*_{\mathcal{X}} : \mathrm{Diag}^*(\mathcal{X}) \longrightarrow$ **Cat** is a 2-fibration (for all $\mathcal{X}$).

(2) $\mathcal{X}$ is cocomplete $\Longrightarrow P_{\mathcal{X}} : \mathrm{Diag}(\mathcal{X}) \longrightarrow$ **Cat** is a 2-opfibration.

Current and future work

- Applications of the ordinary formulae
- 2-versions of Proofs 1, 2, 3 ?
- Enriched versions?
- Higher dimensions?

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Questions?

THANKS