Another categorical look at monoids, quantales, metrics, *etc.*

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F. W. Lawvere: Metric spaces, generalized logic, and closed categories

*Rendiconti del Seminario Matematico e Fisico di Milano** 43:135–166, 1973.

Republished in *Reprints in Theory and Applications of Categories* 1, 2002.

* *Conferenza tenuta il 30 marzo 1973*

Selected references that helped me prepare this talk

- A. Akhvlediani, M.M. Clementino, W.T.: On the categorical meaning of Hausdorff and Gromov distances I, Topology Appl., 2010
- P. Bubenik, V. de Silva, J. Scott: Interleaving and Gromov-Hausdorff distance, 2017
- M. Grandis: Directed Algebraic Topology, Cambridge U Press, 2009
- D. Hofmann, G.J. Seal, W.T. (eds.): Monoidal Topology, Cambridge U Press, 2014
- M. Insall, D. Luckhardt: Norms on categories and analogs of the Schröder-Bernstein Theorem, 2021
- A. Joyal, M. Tierney: An extension of the Galois theory of Grothendieck, AMS, 1984
- W. Kubiś: Categories with norms, 2018
- E. Martinelli: Actions, injectives and injective hulls in quantale-enriched categories, PhD thesis, 2021
- P. Perrone: Lifting couples in Wasserstein spaces, 2021
- I. Stubbe: "Hausdorff distance" via conical cocompletion, Cahiers, 2010
- W.T.: Remarks on weighted categories and the non-symmetric Pompeiu-Hausdorff-Gromov metric, Talk at CT 2018 (Ponta Delgada)

Overview

1 Monoids, actions, quantales — through a categorical lens
2 Passing from the terminal quantale 1 to quantale-weighted categories à la Lawvere
3 Some conditions on weighted categories
3 Two competing formulae for the Pompeiu-Hausdorff metric: which one is "right"?
4 Quantale-enriched cocompleteness via monoid action
5 A quantification of the Joyal-Tierney category of sup-lattices

What is a monoid? Some categorical answers:

It is

- a (small) one-object category (Eilenberg - Mac Lane 1945)
- a (small) discrete monoidal category (Bénabou 1963, Mac Lane 1963)
- a monoid object in the (cartesian) monoidal category **Set** (Who?)
- a (small) one-object category enriched in **Set** (Eilenberg - Kelly 1966)

et cetera, et cetera!

An action of a monoid M on a set X could then be seen as

- a functor $X : M \longrightarrow$ **Set** (Lawvere 1963); equivalently: as a discrete cofibration $\mathbb{X} \longrightarrow M$ with $\mathrm{ob}\mathbb{X} = X$ (Grothendieck 1960/61)
- the discrete monoidal category M acting on a (small) discrete category X
- the monoid object M acting on an object X in the monoidal category **Set**
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What is a quantale? Some categorical answers:

It is

- a (small) one-object 2-category $\mathcal{V}$ whose only hom-category $\mathcal{V}(*,*)$ is a cocomplete lattice, such that all functors $\mathcal{V}(u,*), \mathcal{V}(*,u) : \mathcal{V}(*,*) \longrightarrow \mathcal{V}(*,*)$ preserve colimits
- a (small) thin, skeletal, cocomplete monoidal-closed category $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$
- a monoid $\mathcal{V}$ in the symmetric monoidal-closed category **Sup** $= ($**Sup**$, \boxtimes, 2 = \mathcal{P}1)$ of complete lattices with suprema-preserving maps (where morphisms $L \boxtimes M \longrightarrow N$ classify maps $L \times M \longrightarrow N$ preserving suprema in each variable; Joyal - Tierney 1984)
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In down-to-earth terms:

A quantale $\mathcal{V}$ is a complete lattice $(\mathcal{V}, \leq)$ that comes with a monoid structure $(\mathcal{V}, \otimes, \mathrm{k})$, such that the monoid multiplication $\otimes$ preserves suprema in each variable:

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i \,, \qquad (\bigvee_{i \in I} v_i) \otimes u = \bigvee_{i \in I} (v_i \otimes u)$$

In order to avoid having to deal with two types of internal homs, throughout this talk I will assume that (the monoid) $\mathcal{V}$ is *commutative*:

$$u \otimes v = v \otimes u, \qquad u \leq [v, w] \iff u \otimes v \leq w.$$

Our first example: the terminal quantale $1 = \{*\}$ (Wow!)

Other standards: the Boolean quantale $(2, \bot < \top, \wedge, \top)$
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The thin category $(\mathcal{V},\le)$ over $\mathbf{1}$, and applying Fam to them

$$\mathbf{1} \xrightarrow{\ \mathrm{i}\ } \mathrm{Fam}(\mathbf{1}) = \mathbf{Set} \qquad\qquad * \longmapsto 1$$

$$(\mathcal{V},\le) \xrightarrow{\ \mathrm{i}\ } \mathrm{Fam}(\mathcal{V},\le) = \mathbf{Set}//\mathcal{V}$$

$$\begin{tikzcd} (u \le v) \arrow[r, mapsto] & 1 \arrow[rr, equal] \arrow[dr, "u"'] & {} \arrow[d, phantom, "\le"] & 1 \arrow[dl, "v"] \\ & & \mathcal{V} & \end{tikzcd}$$

$\mathbf{Set}//\mathcal{V}$:

$$\begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[dr, "u=|-|"'] & {} \arrow[d, phantom, "\le"] & J \arrow[dl, "|-|=v"] \\ & \mathcal{V} & \end{tikzcd} \iff \forall\, i \in I:\ u_i \le v_{\varphi i} \iff \forall\, i \in I:\ |i| \le |\varphi i|$$

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The category of $\mathcal{V}$-weighted sets is symmetric monoidal-closed

$$I \otimes J = (\, I \times J,\ |(i,j)| = |i| \otimes |j|\,), \quad E = (1 = \{*\},\ |*| = \mathrm{k})$$

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We obtain a commutative diagram of (strict) homomorphisms of monoidal categories:

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In addition, the straight arrows preserve also the internal homs.
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Form the corresponding categories of small enriched categories...

... and their change-of-base functors:

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The category $\mathbf{Cat}/\!/\mathcal{V}$ of (small) $\mathcal{V}$-weighted categories

For a (small) category $\mathbb{X}$ to be enriched in $\mathrm{Fam}(\mathcal{V},\leq) = \mathbf{Set}/\!/\mathcal{V}$ means (without quantifiers):

$$\begin{array}{lllll} & \mathbb{X}(x,y)\otimes\mathbb{X}(y,z)\longrightarrow\mathbb{X}(x,z) & \text{and} & E\longrightarrow\mathbb{X}(x,x) & \text{live in } \mathbf{Set}/\!/\mathcal{V} \\ \iff & |f|\otimes|g| = |(f,g)| \leq |g\cdot f| & \text{and} & \mathrm{k}\leq|1_x| & \\ \iff & |\text{-}| : \mathbb{X}\longrightarrow(\mathcal{V},\otimes,\mathrm{k}) \text{ is a lax functor} & & & \end{array}$$

For a functor $F:\mathbb{X}\longrightarrow\mathbb{Y}$ to be enriched in $\mathbf{Set}/\!/\mathcal{V}$ means (without quantifiers):

$$\begin{array}{ll} & \mathbb{X}(x,y)\longrightarrow\mathbb{Y}(Fx,Fy) \quad \text{lives in } \mathbf{Set}/\!/\mathcal{V} \\ \iff & |f|\leq|Ff| \end{array}$$

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The adjunction $\mathrm{s} \dashv \mathrm{i}$, monoidal(-closed) structures, preserved by i, s

$\mathcal{V}$-**Cat** | **Cat**$//\mathcal{V}$

$$X, \quad X(x,y) \otimes X(y,z) \leq X(x,z) \qquad \overset{\mathrm{i}}{\longmapsto} \qquad \mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$$

$$\mathrm{k} \leq X(x,x) \qquad\qquad x \xrightarrow{(x,y)} y\,, \quad |(x,y)| = X(x,y)$$

$$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad \mathrm{s}\mathbb{X}(x,y) = \bigvee\{|f| \mid f : x \to y\} \qquad \overset{\mathrm{s}}{\longleftarrow\!\!\!\mid} \qquad \mathbb{X}, \quad |f| \otimes |g| \leq |g \cdot f|$$

$$\mathrm{k} \leq |1_x|$$

$X \otimes Y = X \times Y$ (as a set)

$(X \otimes Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y')$

$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}$ (as a category)

$|(f,g)| = |f| \otimes |g|$

$[X,Y] = \mathcal{V}\text{-}\mathbf{Cat}(X,Y)$ (as a set)

$$[X,Y](f,g) = \bigwedge_{x \in X} Y(fx, gx)$$

$[\mathbb{X},\mathbb{Y}] = (\mathbf{Cat}//\mathcal{V})(\mathbb{X},\mathbb{Y})$ (as a cat)

$$|\,F \xrightarrow{\alpha} G\,| = \bigwedge_{x \in \mathrm{ob}\mathbb{X}} |\alpha_x|$$

Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = (2, \bot < \top, , \wedge, \top)$

2-Cat $=$ **Ord**

$X, \quad x \leq y \ \wedge \ y \leq z \Longrightarrow x \leq z$

$\top \Longrightarrow x \leq x$

$\stackrel{\mathrm{i}}{\longmapsto}$

Cat$//2 =$ **sCat**

$\mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$

$(\, x \xrightarrow{(x,y)} y) \ \in \mathcal{S} \Longleftrightarrow x \leq y$

$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad x \leq y \Longleftrightarrow \exists (f : x \to y) \in \mathcal{S}$

$\stackrel{\mathrm{s}}{\longleftarrow\!\!\!\!\!\!\shortmid}$

$\mathbb{X}, \mathcal{S}, \quad f, g \in \mathcal{S} \Longrightarrow g \cdot f \in \mathcal{S}$

$\top \Longrightarrow 1_x \in \mathcal{S}$

$X \otimes Y = X \times Y$

$(x, y) \leq (x', y') \Longleftrightarrow \ x \leq x' \ \wedge \ y \leq y'$

$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}$ (as a category)

$\mathcal{S}_{\mathbb{X} \otimes \mathbb{Y}} = \mathcal{S}_{\mathbb{X}} \times \mathcal{S}_{\mathbb{Y}}$

$[X, Y] =$ **Ord**(X, Y)

$f \leq g \Longleftrightarrow \forall x \in X : fx \leq gx$

$[\mathbb{X}, \mathbb{Y}] =$ **sCat**$(\mathbb{X}, \mathbb{Y})$ (as a cat)

$\alpha \in \mathcal{S}_{[\mathbb{X}, \mathbb{Y}]} \Longleftrightarrow \forall x \in \mathrm{ob}\mathbb{X} : \alpha_x \in \mathcal{S}_{\mathbb{Y}}$

Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = ([0,\infty], \geq, +, 0)$

$[0,\infty]$-**Cat** $=$ **Met**

$X, \quad d(x,y) + d(y,z) \geq d(x,z)$

$0 \geq d(x,x)$

$\xmapsto{\mathrm{i}}$

Cat$//[0,\infty] =$ **wCat**

$\mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$

$x \xrightarrow{(x,y)} y\,, \quad |(x,y)| = d(x,y)$

$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad d(x,y) = \inf_{f:x\to y} |f|$

$\xleftarrow{\mathrm{s}}$

$\mathbb{X}, \quad |f| + |g| \geq |g \cdot f|$

$0 \geq |1_x|$

$X \otimes Y = X \times Y$

$d((x,y),(x',y')) = d(x,y) + d(y,y')$

$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}$ (as a category)

$|(f,g)| = |f| + |g|$

$[X, Y] =$ **Met**(X, Y)

$d(f,g) = \sup_{x \in X} d(fx, gx)$

$[\mathbb{X}, \mathbb{Y}] =$ **wCat**$(\mathbb{X}, \mathbb{Y})$ (as a cat)

$|\, F \xrightarrow{\alpha} G \,| = \sup_{x \in \mathrm{ob}\mathbb{X}} |\alpha_x|$

Some first examples of weighted categories

We saw:

$\mathcal{V}$-categories (and their functors) are $\mathcal{V}$-weighted categories (and their functors); in fact, they are precisely the $\mathcal{V}$-weighted categories with indiscrete underlying category.

Question: May **Set** be "naturally" $[0,\infty]$-weighted?

Goal: Let $|f|$ measure the degree to which a map $f : X \to Y$ fails to be injective, as follows:

First consider $\#f := \sup_{y\in Y} \#f^{-1}y$; then, with $g : Y \to Z$, we have:

$$\#g \cdot \#f = (\sup_{z\in Z} \#g^{-1}z) \cdot (\sup_{y\in Y} \#f^{-1}y) \geq \sup_{z\in Z} \#(\bigcup_{y\in g^{-1}z} f^{-1}y) = \#(g \cdot f), \quad 1 \geq \#\mathrm{id}_X$$

Not what we wanted! But $([1,\infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0,\infty], \geq, +, 0)$ comes to the rescue:

Put $|f| := \max\{0, \log \#f\}$; then: $|g| + |f| \geq |g \cdot f|$, $0 \geq |\mathrm{id}_X|$, $(f \text{ injective} \iff |f| = 0)$

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Example of a (large) weighted category: **Lipschitz**

$\mathrm{ob}\,\mathbf{Lip} = \mathrm{ob}\,\mathbf{Met}$, $\mathbf{Lip}(X,Y) = \mathbf{Set}(X,Y)$; why call this category **Lip** ??

Recall: $f : X \to Y$ is $K(\geq 0)$-Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K\, d(x,x')$

In particular: $f : X \to Y$ is a morphism in **Met** $\iff f$ is 1-Lipschitz

Question: How far is an arbitrary map f away from being 1-Lipschitz?

Answer: Find the least Lipschitz constant $K \geq 1$ for f (admitting $K = \infty$)

That is: $\mathrm{Lip}(f) = \max\{1, \sup_{x\neq x'} \frac{d(fx,fx')}{d(x,x')}\}$ (assuming temporarily that X be separated)

Then: $\mathrm{Lip}(g)\cdot\mathrm{Lip}(f) \geq \mathrm{Lip}(g\cdot f)$, $1 \geq \mathrm{Lip}(\mathrm{id}_X)$

No problem:

$([1,\infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0,\infty], \geq, +, 0)$, $|f| = \max\{0, \sup_{x,x'} (\log d(fx,fx') - \log d(x,x'))\}$

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On the axiomatics for weighted/normed categories

The category $\mathbb{X}$ is $\mathcal{V}$-weighted by $|\text{-}| : \mathbb{X} \longrightarrow \mathcal{V}$ if

$$\mathrm{k} \leq |1_x|$$

$$|g| \otimes |f| \leq |g \cdot f| \iff |f| \leq \bigwedge_g [|g|, |g \cdot f|] \iff |f| = \bigwedge_g [|g|, |g \cdot f|]$$

$$\iff |g| \leq \bigwedge_f [|f|, |g \cdot f|] \iff |g| = \bigwedge_f [|f|, |g \cdot f|]$$

The $\mathcal{V}$-weighted category $\mathbb{X}$ is *right/left cancellable* if

$$|f| \otimes |g \cdot f| \leq |g| \iff |f| \leq \bigwedge_g [|g \cdot f|, |g|] =: |f|^{\mathrm{R}} \quad \text{(right cancellable)}$$

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Facts (Insall-Luckhardt for $\mathcal{V} = [0, \infty]$): $\mathbb{X}$ weighted by $|\text{-}| \implies \mathbb{X}$ weighted by $|\text{-}|^{\mathrm{R}}$ and $|\text{-}|^{\mathrm{L}}$, and $|f| \leq |f|^{\mathrm{RR}}$, $|f| \leq |f|^{\mathrm{LL}}$.

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Working in **Cat**$//\mathcal{V}$ or in $\mathcal{V}$-**Cat**? The Pompeiu-Hausdorff metric

Let (X, d) be a (classical Frechét) metric space. For $A, B \subseteq X$, put

$$d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$$

(*"the minimal effort required to evacuate every inhabitant of A to the nearest point in B"*)

Then $\quad d(A, B) + d(B, C) \geq d(A, C) \quad$ and $\quad 0 \geq d(A, A) \quad$ in $[0, \infty]$, but generally we do not have $(d(A, B) < \infty)$ or $(d(A, B) = 0 = d(B, A) \Longrightarrow A = B)$ or $(d(A, B) = d(B, A))$.

The traditional rescue operation: symmetrize & restrict yourself to compact subsets, $\neq \emptyset$:

$$d_{\mathrm{sym}}(A, B) = \max\{d(A, B), d(B, A)\}.$$

But the formulae for $d(A, B)$ and $d_{\mathrm{sym}}(A, B)$ make sense for every Lawvere metric space, and each of them makes forming the power set a functor, even after trading $[0, \infty]$ for $\mathcal{V}$:

$$\mathcal{H} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}.$$

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Hausdorff à la Akhvlediani-Clementino-T, ameliorated by Stubbe

Consider moving to the presheaf category, considered as an endofunctor of $\mathcal{V}$-**Cat**:

$$\mathcal{P} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}, \quad X \longmapsto [X^{\mathrm{op}}, \mathcal{V}], \quad \mathcal{P}X(\sigma, \tau) = \bigwedge_{z \in X} [\sigma z, \tau z]$$

$$j_X : \mathcal{H}X = \{A \mid A \subseteq X\} \longrightarrow \mathcal{P}X, \quad A \longmapsto (z \mapsto X(z, A) = \bigvee_{x \in A} X(z, x)).$$

Provide $\mathcal{H}X$ with the initial (= cartesian) structure inherited from $\mathcal{P}X$ via j_X:

$$\mathcal{H}X(A, B) = \bigwedge_{z \in X} [\bigvee_{x \in A} X(z, x), \bigvee_{y \in B} X(z, y)] = \ldots = \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y).$$

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Everything seems “right”!

Theorem (Akhvlediani-Clementino-T 2009, Stubbe 2009)

*Just like $\mathcal{P}$, also $\mathcal{H}$ becomes a lax-idempotent monad of the 2-category $\mathcal{V}$-**Cat**, lifting the power-set monad of **Set**, and making $j : \mathcal{H} \longrightarrow \mathcal{P}$ a monad morphism, which induces the forgetful functor*

$$(\mathcal{V}\text{-}\mathbf{Cat})^{\mathcal{P}_{\simeq}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{consup}} \simeq (\mathcal{V}\text{-}\mathbf{Cat})^{\mathcal{H}_{\simeq}},$$

$\mathcal{V}$-$\mathbf{Cat}_{\mathrm{colim}}$:
(co)complete (= all weighted (co)limts exist) $\mathcal{V}$-categories, with cocontinous $\mathcal{V}$-functors;

$\mathcal{V}$-$\mathbf{Cat}_{\mathrm{consup}}$:
conically cocomplete (= sups exist, Yoneda preserves) $\mathcal{V}$-cats, with sup-preserving $\mathcal{V}$-funs

We’ll give some more insights to this theorem a little later!

The alternative Pompeiu-Hausdorff formula, à la Lawvere, Part 1

Given $X \in \mathcal{V}$-**Cat**, define the $\mathcal{V}$-weighted category $\mathbb{H}X \in$ **Cat**$//\mathcal{V}$ by:

objects: subsets A of X; morphisms: any maps $\varphi : A \to B$; weights: $|\varphi| = \bigwedge_{x \in A} X(x, \varphi x)$.

Indeed: $|\varphi| \otimes |\psi| \leq |\psi \cdot \varphi|$, $\mathrm{k} \leq |\mathrm{id}_A|$.

Form $\mathrm{s}(\mathbb{H}X) \in \mathcal{V}$-**Cat** and compute for $A, B \subseteq X$:

$$\mathrm{s}(\mathbb{H}X)(A,B) = \bigvee_{\varphi : A \to B} |\varphi| = \bigvee_{\varphi \in B^A} \bigwedge_{x \in A} X(x, \varphi x)$$

$$= \bigwedge_{x \in A} \bigvee_{y \in B} X(x,y) \quad \text{if } X \text{ is completely distributive!}$$

For $\mathcal{V} = [0,\infty]$: $\mathrm{s}(\mathbb{H}X)(A,B) = \inf_{\varphi \in B^A} \sup_{x \in A} d(x, \varphi x)$

$$= \sup_{x \in A} \inf_{y \in B} d(x,y);$$ back to Hausdorff: $\mathrm{s}(\mathbb{H}X) = \mathcal{H}X$!

Disappointing: $\mathbb{H}$ is not functorial!

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$$\mathrm{s}(\mathbb{H}X)(A, B) = \inf_{\varphi \in B^A} \sup_{x \in A} d(x, \varphi x)$$
$$= \sup_{x \in A} \inf_{y \in B} d(x, y);$$ back to Hausdorff: $\mathrm{s}(\mathbb{H}X) = \mathcal{H}X$!

Disappointing: $\mathbb{H}$ is not functorial!

Hausdorff à la Lawvere, Part 2.1: Which formula is “right”?

For $\mathbb{X} \in \mathbf{Cat}/\!/\mathcal{V}$, define:

$\mathrm{Fam}\mathbb{X} \in \mathbf{CAT}/\!/\mathcal{V}$:

$$\begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[dr, "x"'] & \arrow[d, phantom, "f:\Longrightarrow"] & J \arrow[dl, "y"] \\ & \mathrm{ob}\mathbb{X} & \end{tikzcd}$$

$$|(\varphi, f)| = \bigwedge_{i \in I} |\, x_i \xrightarrow{f_i} y_{\varphi i} \,|$$

Obtain: $\mathrm{Fam} : \mathbf{Cat}/\!/\mathcal{V} \longrightarrow \mathbf{CAT}/\!/\mathcal{V}, \quad (\mathbb{X} \xrightarrow{F} \mathbb{Y}) \longmapsto (\mathrm{Fam}\mathbb{X} \xrightarrow{\overline{F}} \mathrm{Fam}\mathbb{Y})$

with $\overline{F}$:

$$\begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[dr, "x"'] & \arrow[d, phantom, "f:\Longrightarrow"] & J \arrow[dl, "y"] \\ & \mathrm{ob}\mathbb{X} & \end{tikzcd} \longmapsto \begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[dr, "Fx"'] & \arrow[d, phantom, "Ff:\Longrightarrow"] & J \arrow[dl, "Fy"] \\ & \mathrm{ob}\mathbb{Y} & \end{tikzcd}$$

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Obtain: $\mathrm{Fam} : \mathbf{Cat}//\mathcal{V} \longrightarrow \mathbf{CAT}//\mathcal{V}, \quad (\mathbb{X} \xrightarrow{F} \mathbb{Y}) \longmapsto (\mathrm{Fam}\mathbb{X} \xrightarrow{\overline{F}} \mathrm{Fam}\mathbb{Y})$

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Hausdorff à la Lawvere, Part 2.2: Which formula is “right”?

$$\begin{array}{ccccccc}\mathcal{V}\text{-}\mathbf{Cat} & \xrightarrow{\ \mathrm{i}\ } & \mathbf{Cat}/\!/\mathcal{V} & \xrightarrow{\ \mathrm{Fam}\ } & \mathbf{CAT}/\!/\mathcal{V} & \xrightarrow{\ \mathrm{s}\ } & \mathcal{V}\text{-}\mathbf{CAT} \\ X & \longmapsto & \mathrm{i}X & \longmapsto & \mathrm{Fam}(\mathrm{i}X) & \longmapsto & \mathrm{s}(\mathrm{Fam}(\mathrm{i}X))\end{array}$$

Consider $A, B \subseteq X$ as objects $A, B \hookrightarrow X = \mathrm{ob}(\mathrm{i}X)$, compute their “distance” in $\mathrm{s}(\mathrm{Fam}(\mathrm{i}X))$:

$$\mathrm{s}(\mathrm{Fam}(\mathrm{i}X))(A, B) = \bigvee_{(\varphi,f):A\to B \text{ in } \mathrm{Fam}(\mathrm{i}X)} |(\varphi, f)| = \bigvee_{\varphi:A\to B} \bigwedge_{x\in A} X(x, \varphi x)$$

(since, given $\varphi : A \to B$, there is exactly one $f : x \to \varphi x$ in $\mathrm{i}X$, namely $f = (x, \varphi x)$).

Note: $(A \hookrightarrow X) \cong \coprod_{x\in A}(\{x\} \hookrightarrow X)$ in (ord. cat.) $\mathrm{Fam}(\mathrm{i}X)$ only if $\mathcal{V}$ is integral (*i.e.* $\mathrm{k} = \top$).

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Hausdorff á la Lawvere, Part 3.1: Which formula is "right"?

For $\mathbb{X} \in \mathbf{Cat}/\!/\mathcal{V} = \mathrm{Fam}(\mathcal{V}, \leq)$-$\mathbf{Cat}$, consider:

$$\mathbf{y}_{\mathbb{X}} : \mathbb{X} \longrightarrow \widehat{\mathbb{X}} = [\mathbb{X}^{\mathrm{op}}, \mathrm{Fam}(\mathcal{V}, \leq)] \quad \text{in} \quad \mathrm{Fam}(\mathcal{V}, \leq)\text{-}\mathbf{CAT} = \mathbf{CAT}/\!/\mathcal{V}.$$

How to compute weights in $\widehat{\mathbb{X}}$? For $\alpha : F \longrightarrow G : \mathbb{X}^{\mathrm{op}} \longrightarrow \mathrm{Fam}(\mathcal{V}, \leq)$,

$$|\alpha| = \bigwedge_{z \in \mathrm{ob}\mathbb{X}} |\, Fz \xrightarrow{\alpha_z} Gz \,| = \bigwedge_{z \in \mathrm{ob}\mathbb{X}} \bigwedge_{t \in Fz} [\,|t|, |\alpha_z(t)|\,].$$

Hausdorff á la Lawvere, Part 3.2: Which formula is “right”?

$$\mathcal{V}\text{-}\mathbf{Cat} \xrightarrow{\ \mathrm{i}\ } \mathbf{Cat}/\!/\mathcal{V} \xrightarrow{\ \widehat{(\text{-})}\ } \mathbf{CAT}/\!/\mathcal{V} \xrightarrow{\ \mathrm{s}\ } \mathcal{V}\text{-}\mathbf{CAT}$$

$$X \longmapsto \mathrm{i}X \longmapsto \widehat{\mathrm{i}X} \longmapsto \mathrm{s}(\widehat{\mathrm{i}X}) =: \mathrm{H}X$$

Associate with $A \subseteq X$ the $\mathcal{V}$-weighted functor $\overline{A} : (\mathrm{i}X)^{\mathrm{op}} \longrightarrow \mathrm{Fam}(\mathcal{V}, \leq),\ z \longmapsto (X(z,x))_{x\in A}$

Since $\mathrm{Fam}(\mathcal{V})$ is considered as a $\mathrm{Fam}(\mathcal{V})$-enriched category via its internal hom,

for $A, B \subseteq X$ one has a natural isomorphism $\mathrm{Nat}(\overline{A}, \overline{B}) \cong \mathbf{Set}(A, B)$, and obtains by Yoneda

$$\mathrm{H}X(\overline{A}, \overline{B}) = \bigvee_{\alpha:\overline{A}\to\overline{B}} |\alpha| = \bigvee_{\alpha:\overline{A}\to\overline{B}} \bigwedge_{z\in X} |\alpha_z| = \bigvee_{\varphi:A\to B} \bigwedge_{z\in X} \bigwedge_{x\in A} [X(z,x), X(z,\varphi x)] = \bigvee_{\varphi} \bigwedge_{x\in A} X(x, \varphi x).$$

Note: One has $\overline{A} \cong \coprod_{x\in A} \mathrm{i}X(-,x)$ in the $\mathrm{Fam}(\mathcal{V})$-enriched category $\widehat{\mathrm{i}X}$.

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Let's recall that ...

... we already (somehow) saw that

- $(\mathcal{V}, \otimes, \mathrm{k})$ is a comm. monoid in the symmetric monoidal-closed category $(\mathbf{Sup}, \boxtimes, 2)$
- $(\mathcal{V}, \leq)$ is a small symmetric monoidal-closed category, and therefore gives
- $\mathcal{V}$-**Cat**: objects are sets X, structured by a monoid in the thin m.-c. cat. $\mathcal{V}$-**Rel**(X, X)
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- $(\mathcal{V}, \otimes, \mathrm{k})$ is a monoid in $\mathcal{V}$-**Cat**, where $\mathcal{V}$ is a $\mathcal{V}$-category qua its internal hom [-,-]:

$$\otimes : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \quad \mathcal{V}\text{-functor:} \quad [u, u'] \otimes [v, v'] \leq [u \otimes v,\ u' \otimes v']$$

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What is an action of the monoid $(\mathcal{V}, \otimes, \mathrm{k})$ in **Sup**?

It's a (co)complete lattice X equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ such that:

(1$^+$) $\quad \mathrm{k} \odot x = x$

(2$^+$) $\quad (u \otimes v) \odot x = u \odot (v \odot x)$

and $\odot$ preserves suprema in each variable; that is:

(3$^+$) $\quad (\bigvee_{i \in I} u_i) \odot x = \bigvee_{i \in I} (u_i \odot x)$

(4$^+$) $\quad u \odot (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (u \odot x_i)$

Replacing **Sup** $=$ **Ord**$_{\mathrm{sep,sup}}$ by the category **Ord**$_{\mathrm{sup}}$ of (co)complete preordered sets and sup-maps, an action X would satisfy the conditions (1)-(4), with (i) obtained from (i$^+$) by replacing $=$ by $\simeq$ (order equivalence); that is: X would be just a pseudo-action.

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Starting with an even weaker set of conditions

Let X be just a preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying

(1) $\mathrm{k} \odot x \simeq x$

(2) $(u \otimes v) \odot x \simeq u \odot (v \odot x)$

(3) $(\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I}(u_i \odot x)$ (with the RHS $\bigvee$ existing in X, as part of the condition)

(4^-) $x \leq y \Longrightarrow u \odot x \leq u \odot y$

Then, for every $x \in X$, the map $- \odot x : \mathcal{V} \longrightarrow X$ has a right adjoint $X(x, -)$, defined by

$$X(x, y) = \bigvee \{u \mid u \otimes x \leq y\},$$

making X a $\mathcal{V}$-category, whose underlying preorder is the given one and, by adjunction, satisfies

$$X(u \odot x, y) = [u, X(x, y)],$$

making X a *tensored* $\mathcal{V}$-category.

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Tensored $\mathcal{V}$-categories via half-cocontinuous action

Theorem (Martinelli 2021)

There is a 2-equivalence

$$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{tensor}} \simeq \mathbf{Ord}^{\mathcal{V}}_{\frac{1}{2}\mathrm{cocts}}$$

$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{tensor}}$:
small tensored $\mathcal{V}$-categories, with tensor-preserving $\mathcal{V}$-functors

$\mathbf{Ord}^{\mathcal{V}}_{\frac{1}{2}\mathrm{cocts}}$:
preordered sets on which $\mathcal{V}$ acts satisfying conditions $(1), (2), (3), (4^{-})$, with monotone and pseudo-equivariant maps.

Cocomplete $\mathcal{V}$-categories via cocontinuous action

Let now X be a (co)complete preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying (1), (2), (3), (4). Then (4) (= sup-preservation of every $u \odot - : X \longrightarrow X$) makes the (existing) sups in X conical colimits:

$$X(\bigvee_{i \in I} x_i, y) = \bigwedge_{i \in I} X(x_i, y).$$

Combine this with *the* two fundamental enriched colimit formulae one must not forget:

$$(\mathrm{colim}^{\omega} h)(w) \simeq \bigvee_{z} \omega(z, w) \odot h(z) \qquad (h : Z \to X,\ \omega : Z^{\mathrm{op}} \otimes W \to \mathcal{V})$$

$$X(\mathrm{colim}^{\omega} 1_X, x) \simeq [X^{\mathrm{op}}, \mathcal{V}](\omega, \mathbf{y}_X x) \qquad (\omega : X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \to \mathcal{V}),\ \text{saying}\ \mathrm{colim}^{(\text{-})} \dashv \mathbf{y}_X,$$

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Cocomplete $\mathcal{V}$-categories via cocontinuous action: the Theorem

Theorem (Folklore 19??)

There are 2-equivalences

$$\mathcal{V}\text{-}\mathbf{Cat}^{\mathcal{P}\simeq} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}} \simeq (\mathbf{Ord}_{\mathrm{sup}})^{\mathcal{V}}$$

$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}}$*:*
(co)complete $\mathcal{V}$-categories, with cocontinuous $\mathcal{V}$-functors

$\mathbf{Ord}^{\mathcal{V}}_{\mathrm{sup}}$*:*
(co)complete *preordered sets on which $\mathcal{V}$ acts satisfying conditions* (1), (2), (3), (4)*, with sup-preserving and pseudo-equivariant maps*

Corollary

There are 2-equivalences

$$(\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep}})^{\mathcal{P}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,\,colim}} \simeq \mathbf{Sup}^{\mathcal{V}}$$

Summary diagram

$$
\begin{tikzcd}
 & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{tensor}} \arrow[dl] \arrow[dd, "?\mathrm{ps{-}mon.}?"] & & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}} \arrow[ll, tail] \arrow[rr, tail] \arrow[dl] \arrow[dd, "\mathrm{ps{-}mon.}"] & & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{consup}} \arrow[dl] \arrow[dd, "\mathrm{ps{-}mon.}"] \\
\mathbf{Ord}_{\perp} \arrow[dd, "\mathrm{ps{-}mon.}"] & & \mathbf{Ord}_{\mathrm{sup}} \arrow[ll, tail] \arrow[rr, equal] \arrow[dd, "\mathrm{ps{-}mon.}"] & & \mathbf{Ord}_{\mathrm{sup}} \arrow[dd, "\mathrm{ps{-}mon.}"] & \\
 & \mathcal{V}\text{-}\mathbf{Cat} \arrow[dl] \arrow[rr, equal] & & \mathcal{V}\text{-}\mathbf{Cat} \arrow[dl] \arrow[rr, equal] & & \mathcal{V}\text{-}\mathbf{Cat} \arrow[dl] \\
\mathbf{Ord} \arrow[rr, equal] & & \mathbf{Ord} \arrow[rr, equal] & & \mathbf{Ord} &
\end{tikzcd}
$$

Celebrating **Sup**

- Monadicity:

$$\begin{tikzcd} \mathbf{Sup} \arrow[r, "\text{monadic}"] \arrow[rrr, bend left, "\text{monadic}"] & \mathbf{Pos} \arrow[r, "\text{monadic}"] & \mathbf{Ord} \arrow[r, "\text{topological}"] & \mathbf{Set} \end{tikzcd}$$

- Self-duality:

$$\mathbf{Sup} \xrightarrow{\cong} \mathbf{Sup}^{\mathrm{op}}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "\dashv"] & \longmapsto & X^{\mathrm{op}} \\ Y \arrow[u, bend right, "f_*"'] & \longmapsto & Y^{\mathrm{op}} \arrow[u, "(f_*)^{\mathrm{op}}"'] \end{tikzcd}$$

- Symmetric monoidal-closed structure: $\boxtimes$ classifies "**Sup**-bimorphisms"

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$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "\dashv"] & \longmapsto & X^{\mathrm{op}} \\ Y \arrow[u, bend right=60, "f_*"'] & \longmapsto & Y^{\mathrm{op}} \arrow[u, "(f_*)^{\mathrm{op}}"'] \end{tikzcd}$$

- Symmetric monoidal-closed?

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- Symmetric monoidal-closed?

$\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sep,colim}}$ is symmetric monoidal closed!

Having an equational presentation of separated cocomplete $\mathcal{V}$-categories, we construct the tensor product classifying "bimorphisms" in a standard manner:

Given objects X, Y, form the free object $\mathcal{P}(X \times Y)$ (with the $\mathcal{V}$-powerset monad of **Set**) and then put

$$X \boxtimes Y = \mathcal{P}(X \times Y)/\sim$$

with the least congruence relation $\sim$ making the Yoneda map $\mathbf{y} : X \times Y \longrightarrow \mathcal{P}(X \times Y)/\sim$ a bimorphism; so, $\sim$ is generated by:

$$\mathbf{y}(u \odot x, y) \sim u \odot \mathbf{y}(x, y) \sim \mathbf{y}(x, u \odot y),$$

$$\mathbf{y}(\bigvee_{i \in I} x_i, y) \sim \bigvee_{i \in I} \mathbf{y}(x_i, y), \qquad \mathbf{y}(x, \bigvee_{i \in I} y_i) \sim \bigvee_{i \in I} \mathbf{y}(x, y_i)$$

Now we can start all over again!

Theorem

The category $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sep,colim}}$ *is*

- *monadic over* **Set**,
- *self-dual, and*
- *symmetric monoidal-closed.*

Grazie - Obrigado - Danke!