

Another categorical look at monoids, quantales, metrics, *etc.*

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F. W. Lawvere: Metric spaces, generalized logic, and closed categories
*Rendiconti del Seminario Matematico e Fisico di Milano** 43:135–166, 1973.
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* *Conferenza tenuta il 30 marzo 1973*

Selected references that helped me prepare this talk

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- M. Grandis: *Directed Algebraic Topology*, Cambridge U Press, 2009
- D. Hofmann, G.J. Seal, W.T. (eds.): *Monoidal Topology*, Cambridge U Press, 2014
- M. Insall, D. Luckhardt: Norms on categories and analogs of the Schröder-Bernstein Theorem, 2021
- A. Joyal, M. Tierney: *An extension of the Galois theory of Grothendieck*, AMS, 1984
- W. Kubiś: *Categories with norms*, 2018
- E. Martinelli: *Actions, injectives and injective hulls in quantale-enriched categories*, PhD thesis, 2021
- P. Perrone: *Lifting couples in Wasserstein spaces*, 2021
- I. Stubbe: “Hausdorff distance” via conical cocompletion, *Cahiers*, 2010
- W.T.: *Remarks on weighted categories and the non-symmetric Pompeiu-Hausdorff-Gromov metric*, Talk at CT 2018 (Ponta Delgada)

- 1 Monoids, actions, quantales — through a categorical lens
- 2 Passing from the terminal quantale 1 to quantale-weighted categories à la Lawvere
- 3 Some conditions on weighted categories
- 3 Two competing formulae for the Pompeiu-Hausdorff metric: which one is “right”?
- 4 Quantale-enriched cocompleteness via monoid action
- 5 A quantification of the Joyal-Tierney category of sup-lattices

What is a monoid? Some categorical answers:

It is

- a (small) one-object category (Eilenberg - Mac Lane 1945)
- a (small) discrete monoidal category (Bénabou 1963, Mac Lane 1963)
- a monoid object in the (cartesian) monoidal category **Set** (Who?)
- a (small) one-object category enriched in **Set** (Eilenberg - Kelly 1966)

et cetera, et cetera!

An action of a monoid M on a set X could then be seen as

- a functor $X : M \longrightarrow \mathbf{Set}$ (Lawvere 1963); equivalently: as a discrete cofibration $\mathbb{X} \longrightarrow M$ with $\text{ob}\mathbb{X} = X$ (Grothendieck 1960/61)
- the discrete monoidal category M acting on a (small) discrete category X
- the monoid object M acting on an object X in the monoidal category **Set**
- a functor $X : M \longrightarrow \mathbf{Set}$ of **Set**-enriched categories

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What is a quantale? Some categorical answers:

It is

- a (small) one-object 2-category \mathcal{V} whose only hom-category $\mathcal{V}(*, *)$ is a cocomplete lattice, such that all functors $\mathcal{V}(u, *), \mathcal{V}(*, u) : \mathcal{V}(*, *) \rightarrow \mathcal{V}(*, *)$ preserve colimits
- a (small) thin, skeletal, cocomplete monoidal-closed category $\mathcal{V} = (\mathcal{V}, \leq, \otimes, k)$
- a monoid \mathcal{V} in the symmetric monoidal-closed category **Sup** = $(\mathbf{Sup}, \boxtimes, 2 = \mathcal{P}1)$ of complete lattices with suprema-preserving maps (where morphisms $L \boxtimes M \rightarrow N$ classify maps $L \times M \rightarrow N$ preserving suprema in each variable; Joyal - Tierney 1984)
- a (small) one-object category enriched in **Sup** (a self-dual, monadic cat. over **Set**!)

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In down-to-earth terms:

A quantale \mathcal{V} is a complete lattice (\mathcal{V}, \leq) that comes with a monoid structure $(\mathcal{V}, \otimes, k)$, such that the monoid multiplication \otimes preserves suprema in each variable:

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i, \quad \left(\bigvee_{i \in I} v_i \right) \otimes u = \bigvee_{i \in I} (v_i \otimes u)$$

In order to avoid having to deal with two types of internal homs, throughout this talk I will assume that (the monoid) \mathcal{V} is *commutative*:

$$u \otimes v = v \otimes u, \quad u \leq [v, w] \iff u \otimes v \leq w.$$

Our first example: the terminal quantale $1 = \{*\}$ (Wow!)

Other standards: the Boolean quantale $(2, \perp < \top, \wedge, \top)$

the Lawvere quantale $([0, \infty], \geq, +, 0)$

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The thin category (\mathcal{V}, \leq) over $\mathbf{1}$, and applying Fam to them

$$\mathbf{1} \xrightarrow{i} \mathbf{Fam}(\mathbf{1}) = \mathbf{Set}$$

$$* \dashv \longrightarrow \mathbf{1}$$

$$(\mathcal{V}, \leq) \xrightarrow{i} \mathbf{Fam}(\mathcal{V}, \leq) = \mathbf{Set} // \mathcal{V}$$

$$(u \leq v) \dashv \longrightarrow \mathbf{1} \begin{array}{c} \xlongequal{\quad} \mathbf{1} \\ \swarrow u \quad \searrow v \\ \mathcal{V} \end{array}$$

Set// \mathcal{V} :

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & J \\ & \searrow \quad \swarrow & \\ & \mathcal{V} & \end{array} \quad \begin{array}{c} u = |-| \\ \leq \\ |-| = v \end{array}$$

$$\iff \forall i \in I: u_i \leq v_{\varphi i} \iff \forall i \in I: |i| \leq |\varphi i|$$

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$$\begin{array}{ccc} (I \xrightarrow{\varphi_k} J_k)_{k \in K} & \text{initial} & \iff |i| = \bigwedge_{k \in K} |\varphi_k i| \\ & \begin{array}{c} \searrow \quad \swarrow \\ \mathcal{V} \end{array} & \end{array}$$

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The category of \mathcal{V} -weighted sets is symmetric monoidal-closed

$$I \otimes J = (I \times J, |(i, j)| = |i| \otimes |j|), \quad E = (1 = \{*\}, |*| = k)$$

$$[I, J] = (\mathbf{Set}(I, J), |\varphi| = \bigwedge_{i \in I} [|i|, |\varphi i|])$$

We obtain a commutative diagram of (strict) homomorphisms of monoidal categories:

$$\begin{array}{ccc}
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 \end{array}$$

$$sI = \bigvee_{i \in I} |i|$$

In addition, the straight arrows preserve also the internal homs.

Furthermore: the left adjoint s preserves products iff \mathcal{V} is completely distributive.

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Form the corresponding categories of small enriched categories...

... and their change-of-base functors:

$$\begin{array}{ccc} \mathcal{V}\text{-Cat} & \xrightarrow{i} & \text{Fam}(\mathcal{V})\text{-Cat} =: \mathbf{Cat} // \mathcal{V} \\ \text{topological} \downarrow & \curvearrowright \begin{array}{c} \top \\ s \end{array} & \downarrow \text{topological} \\ \mathbf{Set} = \mathbf{1}\text{-Cat} & \xrightarrow{i} & \text{Fam}(\mathbf{1})\text{-Cat} = \mathbf{Cat} \\ & \curvearrowright \begin{array}{c} \top \\ s=\text{ob} \end{array} & \end{array}$$

What is $\mathbf{Cat} // \mathcal{V}$?

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The category $\mathbf{Cat} // \mathcal{V}$ of (small) \mathcal{V} -weighted categories

For a (small) category \mathbb{X} to be enriched in $\mathbf{Fam}(\mathcal{V}, \leq) = \mathbf{Set} // \mathcal{V}$ means (without quantifiers):

$$\begin{aligned} & \mathbb{X}(x, y) \otimes \mathbb{X}(y, z) \longrightarrow \mathbb{X}(x, z) \quad \text{and} \quad E \longrightarrow \mathbb{X}(x, x) \quad \text{live in } \mathbf{Set} // \mathcal{V} \\ \iff & |f| \otimes |g| = |(f, g)| \leq |g \cdot f| \quad \text{and} \quad k \leq |1_x| \\ \iff & |-| : \mathbb{X} \longrightarrow (\mathcal{V}, \otimes, k) \text{ is a lax functor} \end{aligned}$$

For a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ to be enriched in $\mathbf{Set} // \mathcal{V}$ means (without quantifiers):

$$\begin{aligned} & \mathbb{X}(x, y) \longrightarrow \mathbb{Y}(Fx, Fy) \quad \text{lives in } \mathbf{Set} // \mathcal{V} \\ \iff & |f| \leq |Ff| \\ \iff & \begin{array}{ccc} \mathbb{X} & \xrightarrow{F} & \mathbb{Y} \\ & \searrow & \swarrow \\ & \mathcal{V} & \end{array} \end{aligned}$$

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The category $\mathbf{Cat} // \mathcal{V}$ of (small) \mathcal{V} -weighted categories

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$|-\mid$ $|-\mid$

The adjunction $s \dashv i$, monoidal(-closed) structures, preserved by i, s

\mathcal{V} -Cat

$$X, \quad X(x, y) \otimes X(y, z) \leq X(x, z)$$

$$k \leq X(x, x)$$

$$s\mathbb{X} = \text{ob}\mathbb{X}, \quad s\mathbb{X}(x, y) = \bigvee \{|f| \mid f : x \rightarrow y\}$$

$$X \otimes Y = X \times Y \text{ (as a set)}$$

$$(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$$

$$[X, Y] = \mathcal{V}\text{-Cat}(X, Y) \text{ (as a set)}$$

$$[X, Y](f, g) = \bigwedge_{x \in X} Y(fx, gx)$$

Cat// \mathcal{V}

$$\xrightarrow{i}$$

$$iX, \quad \text{ob}(iX) = X$$

$$x \xrightarrow{(x, y)} y, \quad |(x, y)| = X(x, y)$$

$$\xleftarrow{s}$$

$$\mathbb{X}, \quad |f| \otimes |g| \leq |g \cdot f|$$

$$k \leq |1_x|$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$|(f, g)| = |f| \otimes |g|$$

$$[\mathbb{X}, \mathbb{Y}] = (\mathbf{Cat//}\mathcal{V})(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$|F \xrightarrow{\alpha} G| = \bigwedge_{x \in \text{ob}\mathbb{X}} |\alpha_x|$$

Example: $(\mathcal{V}, \leq, \otimes, \mathbf{k}) = (2, \perp, \langle \top, \cdot, \wedge, \top)$

2-Cat = Ord

$$X, \quad x \leq y \wedge y \leq z \implies x \leq z$$

$$\top \implies x \leq x$$

Cat//2 = sCat

$$\xrightarrow{i} \quad iX, \quad \text{ob}(iX) = X$$

$$(x \xrightarrow{(x,y)} y) \in \mathcal{S} \iff x \leq y$$

$$s\mathbb{X} = \text{ob}\mathbb{X}, \quad x \leq y \iff \exists(f : x \rightarrow y) \in \mathcal{S}$$

$$\xleftarrow{s}$$

$$\mathbb{X}, \mathcal{S}, \quad f, g \in \mathcal{S} \implies g \cdot f \in \mathcal{S}$$

$$\top \implies 1_x \in \mathcal{S}$$

$$X \otimes Y = X \times Y$$

$$(x, y) \leq (x', y') \iff x \leq x' \wedge y \leq y'$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$\mathcal{S}_{\mathbb{X} \otimes \mathbb{Y}} = \mathcal{S}_{\mathbb{X}} \times \mathcal{S}_{\mathbb{Y}}$$

$$[X, Y] = \mathbf{Ord}(X, Y)$$

$$f \leq g \iff \forall x \in X : fx \leq gx$$

$$[\mathbb{X}, \mathbb{Y}] = \mathbf{sCat}(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$\alpha \in \mathcal{S}_{[\mathbb{X}, \mathbb{Y}]} \iff \forall x \in \text{ob}\mathbb{X} : \alpha_x \in \mathcal{S}_{\mathbb{Y}}$$

Example: $(\mathcal{V}, \leq, \otimes, \mathbf{k}) = ([0, \infty], \geq, +, 0)$

$[0, \infty]$ -**Cat** = **Met**

$$X, \quad d(x, y) + d(y, z) \geq d(x, z)$$

$$0 \geq d(x, x)$$

$$s\mathbb{X} = \text{ob}\mathbb{X}, \quad d(x, y) = \inf_{f: x \rightarrow y} |f|$$

$$X \otimes Y = X \times Y$$

$$d((x, y), (x', y')) = d(x, x') + d(y, y')$$

$$[X, Y] = \mathbf{Met}(X, Y)$$

$$d(f, g) = \sup_{x \in X} d(fx, gx)$$

Cat// $[0, \infty]$ = **wCat**

$$iX, \quad \text{ob}(iX) = X$$

$$x \xrightarrow{(x, y)} y, \quad |(x, y)| = d(x, y)$$

$$\mathbb{X}, \quad |f| + |g| \geq |g \cdot f|$$

$$0 \geq |1_x|$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$|(f, g)| = |f| + |g|$$

$$[\mathbb{X}, \mathbb{Y}] = \mathbf{wCat}(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$|F \xrightarrow{\alpha} G| = \sup_{x \in \text{ob}\mathbb{X}} |\alpha_x|$$

Some first examples of weighted categories

We saw:

\mathcal{V} -categories (and their functors) are \mathcal{V} -weighted categories (and their functors); in fact, they are precisely the \mathcal{V} -weighted categories with indiscrete underlying category.

Question: May **Set** be “naturally” $[0, \infty]$ -weighted?

Goal: Let $|f|$ measure the degree to which a map $f : X \rightarrow Y$ fails to be injective, as follows:

First consider $\#f := \sup_{y \in Y} \#f^{-1}y$; then, with $g : Y \rightarrow Z$, we have:

$$\#g \cdot \#f = \left(\sup_{z \in Z} \#g^{-1}z \right) \cdot \left(\sup_{y \in Y} \#f^{-1}y \right) \geq \sup_{z \in Z} \# \left(\bigcup_{y \in g^{-1}z} f^{-1}y \right) = \#(g \cdot f), \quad 1 \geq \#\text{id}_X$$

Not what we wanted! But $([1, \infty], \geq, \cdot, 1) \xrightarrow{\log} ([0, \infty], \geq, +, 0)$ comes to the rescue:

Put $|f| := \max\{0, \log \#f\}$; then: $|g| + |f| \geq |g \cdot f|$, $0 \geq |\text{id}_X|$, $(f \text{ injective} \iff |f| = 0)$



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Example of a (large) weighted category: **Lipschitz**

ob **Lip** = ob **Met**, **Lip**(X, Y) = **Set**(X, Y); why call this category **Lip**??

Recall: $f : X \rightarrow Y$ is $K(\geq 0)$ -Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K d(x, x')$

In particular: $f : X \rightarrow Y$ is a morphism in **Met** $\iff f$ is 1-Lipschitz

Question: How far is an **arbitrary** map f away from being 1-Lipschitz?

Answer: Find the least Lipschitz constant $K \geq 1$ for f (admitting $K = \infty$)

That is: $\text{Lip}(f) = \max\{1, \sup_{x \neq x'} \frac{d(fx, fx')}{d(x, x')}\}$ (assuming temporarily that X be separated)

Then: $\text{Lip}(g) \cdot \text{Lip}(f) \geq \text{Lip}(g \cdot f)$, $1 \geq \text{Lip}(\text{id}_X)$

No problem:

$$([1, \infty], \geq, \cdot, 1) \xrightarrow{\log} ([0, \infty], \geq, +, 0), \quad |f| = \max\{0, \sup_{x, x'} (\log d(fx, fx') - \log d(x, x'))\}$$

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On the axiomatics for weighted/normed categories

The category \mathbb{X} is \mathcal{V} -weighted by $|-| : \mathbb{X} \rightarrow \mathcal{V}$ if

$$\mathbf{k} \leq |1_x|$$

$$\begin{aligned} |g| \otimes |f| \leq |g \cdot f| &\iff |f| \leq \bigwedge_g [|g|, |g \cdot f|] &&\iff |f| = \bigwedge_g [|g|, |g \cdot f|] \\ &\iff |g| \leq \bigwedge_f [|f|, |g \cdot f|] &&\iff |g| = \bigwedge_f [|f|, |g \cdot f|] \end{aligned}$$

The \mathcal{V} -weighted category \mathbb{X} is *right/left cancellable* if

$$|f| \otimes |g \cdot f| \leq |g| \iff |f| \leq \bigwedge_g [|g \cdot f|, |g|] =: |f|^R \quad (\text{right cancellable})$$

$$|g| \otimes |g \cdot f| \leq |f| \iff |g| \leq \bigwedge_f [|g \cdot f|, |f|] =: |g|^L \quad (\text{left cancellable; Kubiś: "norm"})$$

Facts (Insall-Luckhardt for $\mathcal{V} = [0, \infty]$): \mathbb{X} weighted by $|-| \implies \mathbb{X}$ weighted by $|-|^R$ and $|-|^L$,
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Working in $\mathbf{Cat} // \mathcal{V}$ or in $\mathcal{V}\text{-Cat}$? The Pompeiu-Hausdorff metric

Let (X, d) be a (classical Frechét) metric space. For $A, B \subseteq X$, put

$$d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$$

(“the minimal effort required to evacuate every inhabitant of A to the nearest point in B ”)

Then $d(A, B) + d(B, C) \geq d(A, C)$ and $0 \geq d(A, A)$ in $[0, \infty]$, but generally we do

not have $(d(A, B) < \infty)$ or $(d(A, B) = 0 = d(B, A) \implies A = B)$ or $(d(A, B) = d(B, A))$.

The traditional rescue operation: symmetrize & restrict yourself to compact subsets, $\neq \emptyset$:

$$d_{\text{sym}}(A, B) = \max\{d(A, B), d(B, A)\}.$$

But the formulae for $d(A, B)$ and $d_{\text{sym}}(A, B)$ make sense for every Lawvere metric space, and each of them makes forming the power set a functor, even after trading $[0, \infty]$ for \mathcal{V} :

$$\mathcal{H} : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Cat}.$$

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Consider moving to the presheaf category, considered as an endofunctor of $\mathcal{V}\text{-Cat}$:

$$\mathcal{P} : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Cat}, \quad X \longmapsto [X^{\text{op}}, \mathcal{V}], \quad \mathcal{P}X(\sigma, \tau) = \bigwedge_{z \in X} [\sigma z, \tau z]$$

$$j_X : \mathcal{H}X = \{A \mid A \subseteq X\} \longrightarrow \mathcal{P}X, \quad A \longmapsto (z \mapsto X(z, A) = \bigvee_{x \in A} X(z, x)).$$

Provide $\mathcal{H}X$ with the initial (= cartesian) structure inherited from $\mathcal{P}X$ via j_X :

$$\mathcal{H}X(A, B) = \bigwedge_{z \in X} [\bigvee_{x \in A} X(z, x), \bigvee_{y \in B} X(z, y)] = \dots = \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y).$$

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Everything seems “right”!

Theorem (Akhvlediani-Clementino-T 2009, Stubbe 2009)

*Just like \mathcal{P} , also \mathcal{H} becomes a lax-idempotent monad of the 2-category $\mathcal{V}\text{-Cat}$, lifting the power-set monad of **Set**, and making $j : \mathcal{H} \rightarrow \mathcal{P}$ a monad morphism, which induces the forgetful functor*

$$(\mathcal{V}\text{-Cat})^{\mathcal{P}\approx} \simeq \mathcal{V}\text{-Cat}_{\text{colim}} \longrightarrow \mathcal{V}\text{-Cat}_{\text{consup}} \simeq (\mathcal{V}\text{-Cat})^{\mathcal{H}\approx},$$

$\mathcal{V}\text{-Cat}_{\text{colim}}$:

(co)complete (= all weighted (co)limts exist) \mathcal{V} -categories, with cocontinuous \mathcal{V} -functors;

$\mathcal{V}\text{-Cat}_{\text{consup}}$:

conically cocomplete (= sups exist, Yoneda preserves) \mathcal{V} -cats, with sup-preserving \mathcal{V} -funs

We'll give some more insights to this theorem a little later!

The alternative Pompeiu-Hausdorff formula, à la Lawvere, Part 1

Given $X \in \mathcal{V}\text{-Cat}$, define the \mathcal{V} -weighted category $\mathbb{H}X \in \mathbf{Cat} // \mathcal{V}$ by:

objects: subsets A of X ; morphisms: **any** maps $\varphi : A \rightarrow B$; weights: $|\varphi| = \bigwedge_{x \in A} X(x, \varphi x)$.

Indeed: $|\varphi| \otimes |\psi| \leq |\psi \cdot \varphi|$, $\mathbf{k} \leq |\text{id}_A|$.

Form $s(\mathbb{H}X) \in \mathcal{V}\text{-Cat}$ and compute for $A, B \subseteq X$:

$$\begin{aligned} s(\mathbb{H}X)(A, B) &= \bigvee_{\varphi: A \rightarrow B} |\varphi| = \bigvee_{\varphi \in B^A} \bigwedge_{x \in A} X(x, \varphi x) \\ &= \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y) \quad \text{if } X \text{ is completely distributive!} \end{aligned}$$

For $\mathcal{V} = [0, \infty]$: $s(\mathbb{H}X)(A, B) = \inf_{\varphi \in B^A} \sup_{x \in A} d(x, \varphi x)$
 $= \sup_{x \in A} \inf_{y \in B} d(x, y)$; back to Hausdorff: $s(\mathbb{H}X) = \mathcal{H}X!$

Disappointing: \mathbb{H} is not functorial!

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Hausdorff à la Lawvere, Part 2.1: Which formula is “right”?

For $\mathbb{X} \in \mathbf{Cat} // \mathcal{V}$, define:

$\mathbf{Fam} \mathbb{X} \in \mathbf{CAT} // \mathcal{V}$:

$$\begin{array}{ccc}
 I & \xrightarrow{\varphi} & J \\
 & \searrow x & \swarrow y \\
 & \text{ob} \mathbb{X} &
 \end{array}
 \quad f: \Rightarrow$$

$$|(\varphi, f)| = \bigwedge_{i \in I} |x_i \xrightarrow{f_i} y_{\varphi i}|$$

Obtain: $\mathbf{Fam} : \mathbf{Cat} // \mathcal{V} \rightarrow \mathbf{CAT} // \mathcal{V}$, $(\mathbb{X} \xrightarrow{F} \mathbb{Y}) \mapsto (\mathbf{Fam} \mathbb{X} \xrightarrow{\bar{F}} \mathbf{Fam} \mathbb{Y})$

with $\bar{F} :$

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$$|(\varphi, f)| = \bigwedge_{i \in I} |x_i \xrightarrow{f_i} y_{\varphi i}|$$

Obtain: $\mathbf{Fam} : \mathbf{Cat} // \mathcal{V} \rightarrow \mathbf{CAT} // \mathcal{V}$, $(\mathbb{X} \xrightarrow{F} \mathbb{Y}) \mapsto (\mathbf{Fam} \mathbb{X} \xrightarrow{\bar{F}} \mathbf{Fam} \mathbb{Y})$

with $\bar{F} :$

$$\begin{array}{ccc}
 I & \xrightarrow{\varphi} & J \\
 & \searrow x & \swarrow y \\
 & \text{ob} \mathbb{X} &
 \end{array}
 \quad f: \Rightarrow
 \quad \mapsto \quad
 \begin{array}{ccc}
 I & \xrightarrow{\varphi} & J \\
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Hausdorff à la Lawvere, Part 2.2: Which formula is “right”?

$$\mathcal{V}\text{-Cat} \xrightarrow{i} \mathbf{Cat} // \mathcal{V} \xrightarrow{\text{Fam}} \mathbf{CAT} // \mathcal{V} \xrightarrow{s} \mathcal{V}\text{-CAT}$$

$$X \vdash \longrightarrow iX \vdash \longrightarrow \text{Fam}(iX) \vdash \longrightarrow s(\text{Fam}(iX))$$

Consider $A, B \subseteq X$ as objects $A, B \hookrightarrow X = \text{ob}(iX)$, compute their “distance” in $s(\text{Fam}(iX))$:

$$s(\text{Fam}(iX))(A, B) = \bigvee_{(\varphi, f): A \rightarrow B \text{ in Fam}(iX)} |(\varphi, f)| = \bigvee_{\varphi: A \rightarrow B} \bigwedge_{x \in A} X(x, \varphi x)$$

(since, given $\varphi : A \rightarrow B$, there is exactly one $f : x \rightarrow \varphi x$ in iX , namely $f = (x, \varphi x)$).

Note: $(A \hookrightarrow X) \cong \coprod_{x \in A} (\{x\} \hookrightarrow X)$ in (ord. cat.) $\text{Fam}(iX)$ only if \mathcal{V} is integral (i.e. $k = \top$).

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Hausdorff á la Lawvere, Part 3.1: Which formula is “right”?

For $\mathbb{X} \in \mathbf{Cat} // \mathcal{V} = \mathbf{Fam}(\mathcal{V}, \leq)\text{-}\mathbf{Cat}$, consider:

$$\mathbf{y}_{\mathbb{X}} : \mathbb{X} \longrightarrow \widehat{\mathbb{X}} = [\mathbb{X}^{\text{op}}, \mathbf{Fam}(\mathcal{V}, \leq)] \quad \text{in} \quad \mathbf{Fam}(\mathcal{V}, \leq)\text{-}\mathbf{CAT} = \mathbf{CAT} // \mathcal{V}.$$

How to compute weights in $\widehat{\mathbb{X}}$? For $\alpha : F \longrightarrow G : \mathbb{X}^{\text{op}} \longrightarrow \mathbf{Fam}(\mathcal{V}, \leq)$,

$$|\alpha| = \bigwedge_{z \in \text{ob} \mathbb{X}} |Fz \xrightarrow{\alpha_z} Gz| = \bigwedge_{z \in \text{ob} \mathbb{X}} \bigwedge_{t \in Fz} [|t|, |\alpha_z(t)|].$$

Hausdorff á la Lawvere, Part 3.2: Which formula is “right”?

$$\mathcal{V}\text{-Cat} \xrightarrow{i} \mathbf{Cat} // \mathcal{V} \xrightarrow{\widehat{(-)}} \mathbf{CAT} // \mathcal{V} \xrightarrow{s} \mathcal{V}\text{-CAT}$$

$$X \vdash \longrightarrow iX \vdash \longrightarrow \widehat{iX} \vdash \longrightarrow s(\widehat{iX}) =: \mathbf{HX}$$

Associate with $A \subseteq X$ the \mathcal{V} -weighted functor $\bar{A} : (iX)^{\text{op}} \rightarrow \mathbf{Fam}(\mathcal{V}, \leq)$, $z \mapsto (X(z, x))_{x \in A}$

Since $\mathbf{Fam}(\mathcal{V})$ is considered as a $\mathbf{Fam}(\mathcal{V})$ -enriched category via its internal hom,

for $A, B \subseteq X$ one has a natural isomorphism $\mathbf{Nat}(\bar{A}, \bar{B}) \cong \mathbf{Set}(A, B)$, and obtains by Yoneda

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Note: One has $\bar{A} \cong \coprod_{x \in A} iX(-, x)$ in the $\mathbf{Fam}(\mathcal{V})$ -enriched category \widehat{iX} .

Hausdorff á la Lawvere, Part 3.2: Which formula is “right”?

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Let's recall that ...

... we already (somehow) saw that

- $(\mathcal{V}, \otimes, \mathbf{k})$ is a comm. monoid in the symmetric monoidal-closed category $(\mathbf{Sup}, \boxtimes, 2)$
- (\mathcal{V}, \leq) is a small symmetric monoidal-closed category, and therefore gives
- $\mathcal{V}\text{-Cat}$: objects are sets X , structured by a monoid in the thin m.-c. cat. $\mathcal{V}\text{-Rel}(X, X)$
- $(\mathcal{V}\text{-Cat}, \otimes, E)$ is a symmetric monoidal-closed category, and its object
- $(\mathcal{V}, \otimes, \mathbf{k})$ is a monoid in $\mathcal{V}\text{-Cat}$, where \mathcal{V} is a \mathcal{V} -category qua its internal hom $[-, -]$:

$$\otimes : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \quad \mathcal{V}\text{-functor:} \quad [u, u'] \otimes [v, v'] \leq [u \otimes v, u' \otimes v']$$

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One should obviously study the actions of the monoid \mathcal{V} in its various incarnations!

What is an action of the monoid $(\mathcal{V}, \otimes, k)$ in **Sup**?

It's a (co)complete lattice X equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ such that:

$$(1^+) \quad k \odot x = x$$

$$(2^+) \quad (u \otimes v) \odot x = u \odot (v \odot x)$$

and \odot preserves suprema in each variable; that is:

$$(3^+) \quad (\bigvee_{i \in I} u_i) \odot x = \bigvee_{i \in I} (u_i \odot x)$$

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Replacing **Sup** = **Ord**_{sep,sup} by the category **Ord**_{sup} of (co)complete preordered sets and sup-maps, an action X would satisfy the conditions (1)-(4), with (i) obtained from (i⁺) by replacing = by \simeq (order equivalence); that is: X would be just a pseudo-action.

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Starting with an even weaker set of conditions

Let X be just a preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying

$$(1) \quad \mathbf{k} \odot x \simeq x$$

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$$(3) \quad (\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I} (u_i \odot x) \text{ (with the RHS } \bigvee \text{ existing in } X, \text{ as part of the condition)}$$

$$(4^-) \quad x \leq y \implies u \odot x \leq u \odot y$$

Then, for every $x \in X$, the map $- \odot x : \mathcal{V} \longrightarrow X$ has a right adjoint $X(x, -)$, defined by

$$X(x, y) = \bigvee \{u \mid u \odot x \leq y\},$$

making X a \mathcal{V} -category, whose underlying preorder is the given one and, by adjunction, satisfies

$$X(u \odot x, y) = [u, X(x, y)],$$

making X a *tensor*ed \mathcal{V} -category.

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Theorem (Martinelli 2021)

There is a 2-equivalence

$$\mathcal{V}\text{-Cat}_{\text{tensor}} \simeq \mathbf{Ord}_{\frac{1}{2}\text{cocts}}^{\mathcal{V}}$$

$\mathcal{V}\text{-Cat}_{\text{tensor}}$:

small tensored \mathcal{V} -categories, with tensor-preserving \mathcal{V} -functors

$\mathbf{Ord}_{\frac{1}{2}\text{cocts}}^{\mathcal{V}}$:

preordered sets on which \mathcal{V} acts satisfying conditions (1), (2), (3), (4⁻), with monotone and pseudo-equivariant maps.

Cocomplete \mathcal{V} -categories via cocontinuous action

Let now X be a (co)complete preordered set equipped with a map $\odot : \mathcal{V} \times X \rightarrow X$ satisfying (1), (2), (3), (4). Then (4) (= sup-preservation of every $u \odot - : X \rightarrow X$) makes the (existing) sups in X conical colimits:

$$X\left(\bigvee_{i \in I} x_i, y\right) = \bigwedge_{i \in I} X(x_i, y).$$

Combine this with *the* two fundamental enriched colimit formulae one must not forget:

$$(\operatorname{colim}^{\omega} h)(w) \simeq \bigvee_z \omega(z, w) \odot h(z) \quad (h : Z \rightarrow X, \omega : Z^{\operatorname{op}} \otimes W \rightarrow \mathcal{V})$$

$$X(\operatorname{colim}^{\omega} 1_X, x) \simeq [X^{\operatorname{op}}, \mathcal{V}](\omega, \mathbf{y}_X x) \quad (\omega : X^{\operatorname{op}} \cong X^{\operatorname{op}} \otimes E \rightarrow \mathcal{V}), \text{ saying } \operatorname{colim}^{(-)} \dashv \mathbf{y}_X,$$

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Cocomplete \mathcal{V} -categories via cocontinuous action: the Theorem

Theorem (Folklore 19??)

There are 2-equivalences

$$\mathcal{V}\text{-Cat}^{\mathcal{P}\simeq} \simeq \mathcal{V}\text{-Cat}_{\text{colim}} \simeq (\mathbf{Ord}_{\text{sup}})^{\mathcal{V}}$$

$\mathcal{V}\text{-Cat}_{\text{colim}}$:

(co)complete \mathcal{V} -categories, with cocontinuous \mathcal{V} -functors

$\mathbf{Ord}_{\text{sup}}^{\mathcal{V}}$:

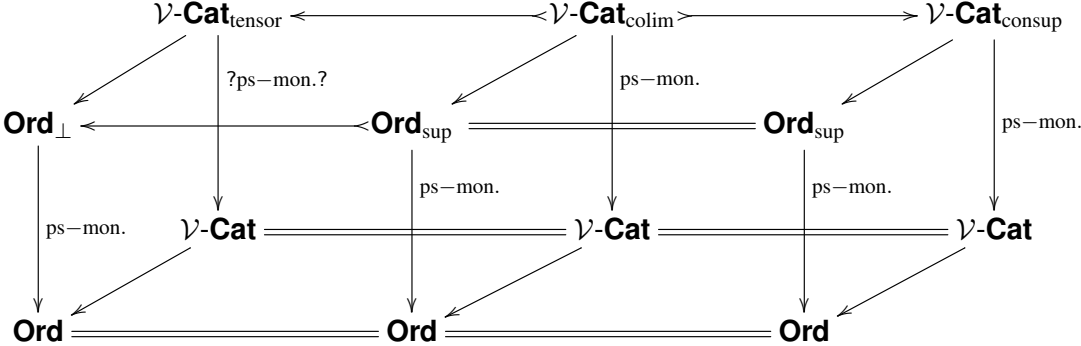
(co)complete preordered sets on which \mathcal{V} acts satisfying conditions (1), (2), (3), (4), with sup-preserving and pseudo-equivariant maps

Corollary

There are 2-equivalences

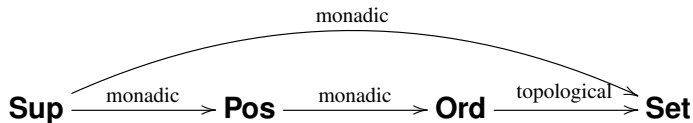
$$(\mathcal{V}\text{-Cat}_{\text{sep}})^{\mathcal{P}} \simeq \mathcal{V}\text{-Cat}_{\text{sep, colim}} \simeq \mathbf{Sup}^{\mathcal{V}}$$

Summary diagram

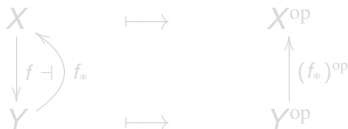


Celebrating **Sup**

- Monadicity:



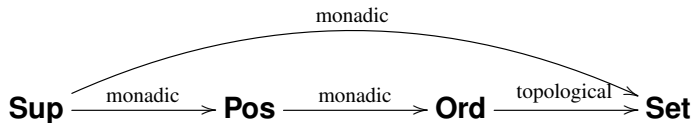
- Self-duality:



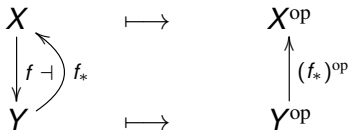
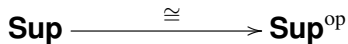
- Symmetric monoidal-closed structure: \boxtimes classifies “**Sup**-bimorphisms”

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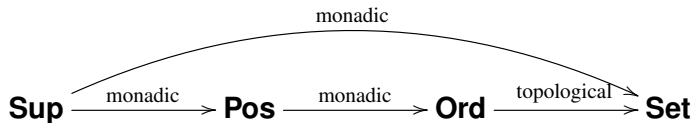
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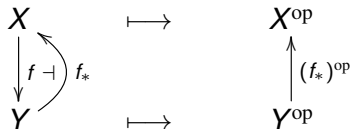
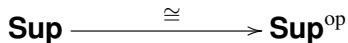
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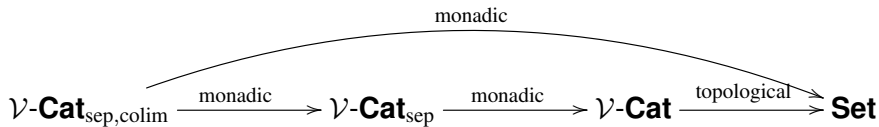
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\mathcal{V} -Cat_{sep,colim} quantification of Sup?

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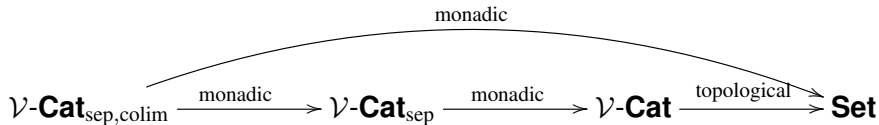
$$\mathcal{V}\text{-Cat}_{\text{sep,colim}} \xrightarrow{\cong} (\mathcal{V}\text{-Cat}_{\text{sep,colim}})^{\text{op}}$$

A diagram illustrating the correspondence between a monad and its adjoint. On the left, a monad is represented by a vertical arrow $f \dashv$ from X to Y , with a curved arrow f_* from Y back to X . On the right, the adjoint is represented by a vertical arrow $(f_*)^{\text{op}}$ from Y^{op} to X^{op} . Two horizontal double arrows \iff connect the two structures, indicating an equivalence between the monad and its adjoint.

- Symmetric monoidal-closed?

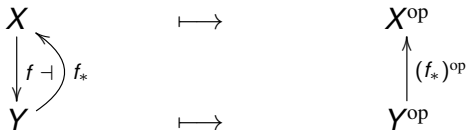
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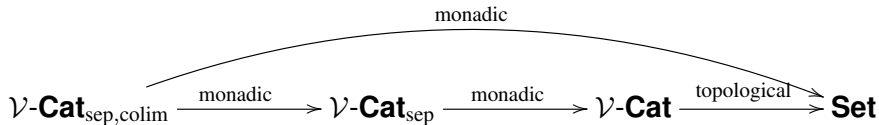
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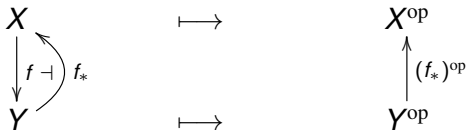
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- Symmetric monoidal-closed?

$\mathcal{V}\text{-Cat}_{\text{sep, colim}}$ is symmetric monoidal closed!

Having an equational presentation of separated cocomplete \mathcal{V} -categories, we construct the tensor product classifying “bimorphisms” in a standard manner:

Given objects X, Y , form the free object $\mathcal{P}(X \times Y)$ (with the \mathcal{V} -powerset monad of **Set**) and then put

$$X \boxtimes Y = \mathcal{P}(X \times Y) / \sim$$

with the least congruence relation \sim making the Yoneda map $\mathbf{y} : X \times Y \longrightarrow \mathcal{P}(X \times Y) / \sim$ a bimorphism; so, \sim is generated by:

$$\begin{aligned} \mathbf{y}(u \odot x, y) &\sim u \odot \mathbf{y}(x, y) \sim \mathbf{y}(x, u \odot y), \\ \mathbf{y}\left(\bigvee_{i \in I} x_i, y\right) &\sim \bigvee_{i \in I} \mathbf{y}(x_i, y), & \mathbf{y}\left(x, \bigvee_{i \in I} y_i\right) &\sim \bigvee_{i \in I} \mathbf{y}(x, y_i) \end{aligned}$$

Now we can start all over again!

Theorem

The category $\mathcal{V}\text{-Cat}_{\text{sep, colim}}$ is

- monadic over **Set**,
- self-dual, and
- symmetric monoidal-closed.

Grazie - Obrigado - Danke!