Factorizations Then and Now

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Overview

- Factorizations - a second-tier categorical notion?
- History - well-known, forgotten, ignored, or overlooked papers
- Weak versus orthogonal, morphism classes versus functors
- Useful one-dimensional generalizations: cones, functors
- Enrichment and higher dimensionality
- Revisiting the fundamentals: fibrations
- Promoting strict one-sided factorization systems
- Anything left on the "To-do"-list?

Factorizations - a second-tier categorical notion?

Apparently "YES":

Ehresmann 1958	∅	Arbib-Manes 1975	⋆
Freyd 1964	∅	Manes 1976	⋆
Mitchell 1965	∅	Barr-Wells 1985	⋆⋆
Pareigis 1969/1970	∅	Adámek-Herrlich-Strecker 1990	⋆ ⋆ ⋆
Schubert 1970	∅	Borceux 1994	⋆
Mac Lane 1971	∅ (!)	Mac Lane 1997	∅
Schubert 1971/72	⋆	Awodey 2010	∅
Herrlich-Strecker 1973	⋆⋆	Leinster 2016	∅
		Grandis 2018	∅

Factorizations are in good company, though: try finding fibrations!

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Freyd-Kelly, JPAA 1972 (submitted June 1971)...

" A *factorization* $(\mathcal{E}, \mathcal{M})$ in $\mathcal{A}$ consists of two classes of morphisms in $\mathcal{A}$, *each containing the isomorphisms and closed under composition* such that

(2.2) every morphism of $\mathcal{A}$ is of the form ip, where $i \in \mathcal{M}, p \in \mathcal{E}$;

(2.3) if $vip = i'p'u$, where $i, i' \in \mathcal{M}$ and $p, p' \in \mathcal{E}$, there is a unique w rendering commutative the diagram

Since $\mathcal{E} \cap \mathcal{M}$ contains the isomorphisms, (2.3) is clearly equivalent to

(2.4) $\mathcal{E} \subseteq \mathcal{M}^{\uparrow}$ and $\mathcal{M} \subseteq \mathcal{E}^{\downarrow}$. "

The authors continue by proving $\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$, *etc.*

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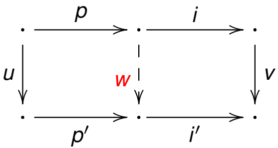

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... and now without the redundancies (AHS 1990):

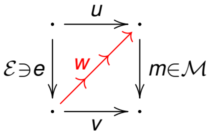
$\mathcal{A}^{\times} := \{\text{isomorphisms of } \mathcal{A}\}$

$(\mathcal{E}, \mathcal{M})$ *(orthogonal) factorization system of* $\mathcal{A}$ if

F0 $\quad \mathcal{M} \cdot \mathcal{A}^{\times} \subseteq \mathcal{M}, \;\; \mathcal{A}^{\times} \cdot \mathcal{E} \subseteq \mathcal{E}$

F1 $\quad \mathcal{A} \subseteq \mathcal{M} \cdot \mathcal{E}$

F2 $\quad \mathcal{E} \perp \mathcal{M}$



These conditions imply $\quad \mathcal{E} = {}^{\perp}\mathcal{M}, \;\; \mathcal{M} = \mathcal{E}^{\perp}$
and in particular the Freyd-Kelly *a-priori* assumptions

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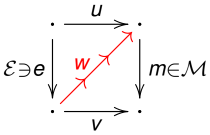

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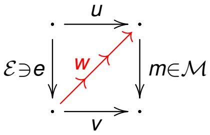

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Mac Lane's 1948/50 "bicategories"

Category $\mathcal{A}$ with two classes $\mathcal{I}$ ("*injections*"), $\mathcal{P}$ ("*projections*"), s.th.

- Identities($\mathcal{A}$) $\subseteq \mathcal{I} \cap \mathcal{P}$, $\quad \mathcal{I} \cdot \mathcal{I} \subseteq \mathcal{I}$, $\quad \mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$
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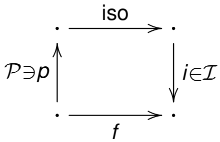

- $\mathcal{I} \cdot \mathcal{A}^{\times}$ ("*submaps*"), $\mathcal{A}^{\times} \cdot \mathcal{P}$ ("*supermaps*")
  are closed under composition
- $\forall A, B \ \exists^{\leq 1} A \longrightarrow B$ in $\mathcal{I} \cdot \mathcal{P} \cdot \ldots \cdot \mathcal{I} \cdot \mathcal{P}$ ("*idemmaps*")
- $\forall A : \mathcal{I}/A$ and $A \backslash \mathcal{P}$ are sets

Note: No a-priori epi-mono condition, but the (strange) idemmap axiom forces it!

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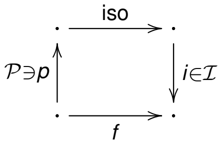

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Why one should care about “double factorization”

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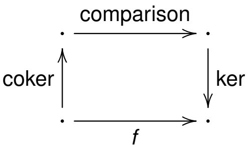

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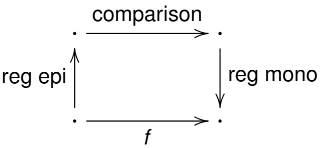

$\mathcal{A}$ with $\times, 0$:	comparison (left) is iso $\iff$ $\mathcal{A}$ Abelian
Hence:	the comparison morphism gauges Abelianess!
$\mathcal{A}$ variety (say):	$\forall f$ epi (comp. (right) is iso $\iff$ f is surjective)
$P : \mathcal{A} \longrightarrow$ Set top.:	$\{$comparison morphisms$\} = P^{-1}(\mathrm{Set}^{\times})$

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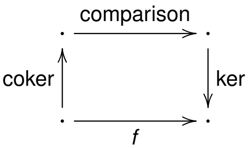

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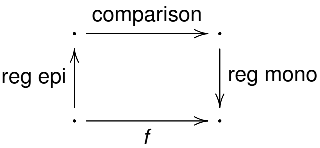

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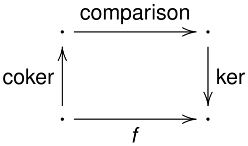

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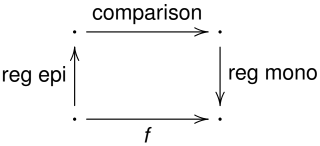

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The standard references of the “bicategory” period

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Freyd-Kelly 1972	Galois correspond. between morphism classes
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In this list, the Freyd-Kelly paper stands out in terms of clarity of exposition, and for “being light” on a-priori epi-mono conditions. But there were earlier papers, with more complete accounts of factorization systems and even lighter a-priori assumptions ...

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Overlooked: the Ehrbar-Wyler paper of 1968
Images are reflections!

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$\mathcal{M}/B \hookrightarrow \mathcal{A}/B$ *f has $\mathcal{M}$-image* $:\Leftrightarrow f : A \longrightarrow B$ has reflection into $\mathcal{M}/B$

f $\mathcal{M}$-extremal: $\Leftrightarrow$ reflection of f into $\mathcal{M}/B$ exists and is iso (in $\mathcal{A}$)
$\Leftrightarrow$ [if $1_{\mathcal{A}} \subseteq \mathcal{M}$:] $(f = ng, n \in \mathcal{M} \Rightarrow \exists! s : ns = 1, sf = g)$

$\mathcal{M} \hookrightarrow \mathcal{A}^2$ *f has strong $\mathcal{M}$-image* $\Leftrightarrow f$ has $\mathrm{cod}^{-1}(1_{\mathcal{A}})$-reflection into $\mathcal{M}$

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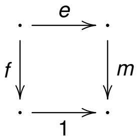
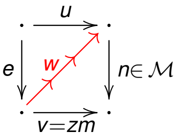
Right $\mathcal{M}$-factorizations (T 1983, Dikranjan-T 1995)

$\mathcal{M}$ *right factorization system* of $\mathcal{A}$: $\Longleftrightarrow$

RF0 $\mathcal{M} \cdot \mathcal{A}^\times \subseteq \mathcal{M}$

RF1 every morphism has a strong $\mathcal{M}$-image:
$\forall f \ \exists$ factorization $f = me$: $m \in \mathcal{M}$, $(e, m) \perp \mathcal{M}$





- Necessarily $\mathcal{A}^\times \subseteq \mathcal{M}$
- If $\mathcal{A}^\times \subseteq \mathcal{M}$, $\mathcal{A}^\times \cdot \mathcal{M} \subseteq \mathcal{M}$, then (RF1 $\Longleftrightarrow$ $\mathcal{M}$ reflective in $\mathcal{A}^2$)
- Every category with kernelpairs and their coequalizers has left RegEpi-factorizations; dually: Isbell's dominions!
- For every fibration $P : \mathcal{A} \longrightarrow \mathcal{X}$, $\mathcal{A}$ has right P-Cart-factorizations

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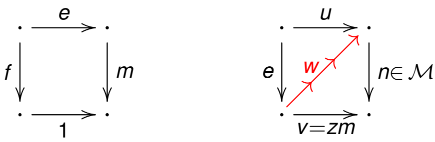

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Some Ehrbar-Wyler 1968 highlights

Lemma Assume RF1. Then:

(Right retract closure)	$gp \in \mathcal{M},\ pi = 1 \Longrightarrow g \in \mathcal{M}\cdot\mathcal{A}^{\times}$
(Left cancelation)	$wv \in \mathcal{M},\ vu \in \mathcal{M} \Longrightarrow u \in \mathcal{M}\cdot\mathcal{A}^{\times}$
(Limit closure)	$\mu : D \longrightarrow C$ pointwise in $\mathcal{M} \Longrightarrow \lim\mu \in \mathcal{M}\cdot\mathcal{A}^{\times}$
(Pullback stability)	every pullback of an $\mathcal{M}$ is in $\mathcal{M}\cdot\mathcal{A}^{\times}$

Lemma Assume RF0, RF1. Then:

(Orthogonality)	$\{\mathcal{M}\text{-extremal}\} = {}^{\perp}\mathcal{M}$

Theorem Equiv. are for $\mathcal{E}, \mathcal{M}$ with F0 ($\mathcal{M}\cdot\mathcal{A}^{\times} \subseteq \mathcal{M},\ \mathcal{A}^{\times}\cdot\mathcal{E} \subseteq \mathcal{E}$):

(i)=F1+F2	$\mathcal{A} \subseteq \mathcal{M}\cdot\mathcal{E},\ \ \mathcal{E}\perp\mathcal{M}$ $\quad (= (\mathcal{E},\mathcal{M})$ orth. fact. system of $\mathcal{A})$
(ii)	$\mathcal{M}$ satisfies RF1, $\mathcal{M}\cdot\mathcal{M} \subseteq \mathcal{M},\ \ \mathcal{E} = \{\mathcal{M}\text{-extremal}\}$
(iii)	$\mathcal{E}$ satisfies LF1=RF1^{op}, $\mathcal{E}\cdot\mathcal{E} \subseteq \mathcal{E},\ \ \mathcal{M} = \{\mathcal{E}\text{-co-extremal}\}$
(iv)	$\mathcal{M}\cdot\mathcal{M} \subseteq \mathcal{M},\ \mathcal{E}\cdot\mathcal{E} \subseteq \mathcal{E}, (\mathcal{E},\mathcal{M})$-fact. unique up to unique iso:

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(iv)	$\mathcal{M}\cdot\mathcal{M} \subseteq \mathcal{M},\ \mathcal{E}\cdot\mathcal{E} \subseteq \mathcal{E}, (\mathcal{E},\mathcal{M})$-fact. unique up to unique iso:

Some Ehrbar-Wyler 1968 highlights

Lemma Assume RF1. Then:

(Right retract closure) $gp \in \mathcal{M},\ pi = 1 \Longrightarrow g \in \mathcal{M}\cdot\mathcal{A}^{\times}$

(Left cancelation) $wv \in \mathcal{M},\ vu \in \mathcal{M} \Longrightarrow u \in \mathcal{M}\cdot\mathcal{A}^{\times}$

(Limit closure) $\mu : D \longrightarrow C$ pointwise in $\mathcal{M} \Longrightarrow \lim\mu \in \mathcal{M}\cdot\mathcal{A}^{\times}$

(Pullback stability) every pullback of an $\mathcal{M}$ is in $\mathcal{M}\cdot\mathcal{A}^{\times}$

Lemma Assume RF0, RF1. Then:

(Orthogonality) $\{\mathcal{M}\text{-extremal}\} = {}^{\perp}\mathcal{M}$

Theorem Equiv. are for $\mathcal{E}, \mathcal{M}$ with F0 ($\mathcal{M}\cdot\mathcal{A}^{\times} \subseteq \mathcal{M},\ \mathcal{A}^{\times}\cdot\mathcal{E} \subseteq \mathcal{E}$):

(i)=F1+F2 $\mathcal{A} \subseteq \mathcal{M}\cdot\mathcal{E},\ \ \mathcal{E}\perp\mathcal{M}$ $(= (\mathcal{E}, \mathcal{M})$ orth. fact. system of $\mathcal{A})$

(ii) $\mathcal{M}$ satisfies RF1, $\mathcal{M}\cdot\mathcal{M} \subseteq \mathcal{M},\ \ \mathcal{E} = \{\mathcal{M}\text{-extremal}\}$

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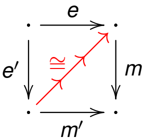

Ignored: The Ringel papers of 1970-71 (Math. Zeit.) Weak and unique diagonalization reconciled!

Motivated by

- Quillen 1967 (model categories)
- Isbell 1964 and Kennison 1968 (image factorization)
- Gabriel-Zisman 1967 (factorization through $R^{-1}(\mathrm{Iso})$),

Ringel introduces the Galois correspondences given by weak diagonalization ($l \Box r$) and unique diagonalization ($l \perp r$) and defines:

$(\mathcal{L}, \mathcal{R})$ *D-pair* $:\iff \mathcal{L} = {}^{\Box}\mathcal{R},\ \mathcal{R} = \mathcal{L}^{\Box}$

$(\mathcal{L}, \mathcal{R})$ *regular D-pair* $:\iff \mathcal{L} = {}^{\perp}\mathcal{R},\ \mathcal{R} = \mathcal{L}^{\perp}$ ("prefact. system")

and proves all standard stability and closure properties of the right class in a D-pair:

- closure under products and infinite composition (inverse chains)
- closure under retracts (in the arrow category)
- stability under pullback
- compatibility with adjunctions: $S \dashv T \Longrightarrow (S(\mathcal{X}))^{\Box} = T^{-1}(\mathcal{X}^{\Box})$

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Some Ringel 1970 highlights

Lemma	Let $\mathcal{X}$ be such that	
(Pullback)	pbs of $\mathcal{X}$-morph's exist and are in $\mathcal{X}$;	
(Section)	sections with retractions in $\mathcal{X}$ are in $\mathcal{X}$.	Then: ${}^{\square}\mathcal{X} = {}^{\perp}\mathcal{X}$

Theorem	$(\mathcal{L}, \mathcal{R})$ D-pair, $\mathcal{A}$ with pbs and pos.	Equivalent are:
(i)	$gf \in \mathcal{R}, g \in \mathcal{R} \Longrightarrow f \in \mathcal{R}$	(Weak Left Canc)
(ii)	$pi = 1, p \in \mathcal{R} \Longrightarrow i \in \mathcal{R}$	(Section)
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Ringel 1970: The "Pre-Cassidy-Hébert-Kelly Theorem"

In a finitely complete and finitely cocomplete category $\mathcal{A}$, there is a bijective correspondence between

- full replete reflective subcategories $\mathcal{B}$, and
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- $R \dashv I : \mathcal{B} \hookrightarrow \mathcal{A}$ $\quad\longmapsto\quad$ $(R^{-1}(\mathcal{A}^{\times}), ({}^{\square}\mathrm{Mor}\mathcal{B})^{\square})$
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Here:

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Weak factorization systems, non-redundantly

(Beke 2000,) Adámek-Herrlich-Rosický-T 2002:

$(\mathcal{L}, \mathcal{R})$ *weak factorization system of* $\mathcal{A}$ if

WF0 $\mathcal{L}, \mathcal{R}$ closed under retracts in $\mathcal{A}^2$

WF1 $\mathcal{A} \subseteq \mathcal{R} \cdot \mathcal{L}$

WF2 $\mathcal{L} \Box \mathcal{R}$

These conditions imply $\mathcal{L} = {}^{\Box}\mathcal{R}, \ \mathcal{R} = \mathcal{L}^{\Box}$,
and in particular $\mathcal{A}^{\times} = \mathcal{L} \cap \mathcal{R}, \ \mathcal{L} \cdot \mathcal{L} \subseteq \mathcal{L}, \ \mathcal{R} \cdot \mathcal{R} \subset \mathcal{R}$

WF0 is expressed non-redundantly as

WF0⁻ (Left retract closure) $if \in \mathcal{L}, \ pi = 1 \implies f \in \mathcal{L}$
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Note: F1,2 $\Rightarrow$ WF1,2, but WF0⁻ $\Rightarrow$ F0

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Re-thinking the fundamentals: functorial factorization

Goal: eliminate choice, make things constructive!
Some big and small milestones:

Linton 1969 (forgotten!):	$\mathcal{A}^2 \longrightarrow \mathcal{A}^3, (A \longrightarrow B) \mapsto (A \longrightarrow \cdot \longrightarrow B)$
Coppey 1980 (?):	$\mathcal{A}^2 \longrightarrow \mathcal{A}$ as E-M-algebra structure wrt $(-)^2$
Korostenski-T 1993:	(not knowingly) Coppey re-invented
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Starting point: the free factorization system

$$A \xrightarrow{f} B$$

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Functorial images of the free system

Given $\quad F : \mathcal{A}^2 \longrightarrow \mathcal{A} \quad$ with $\quad (\mathcal{A} \xrightarrow{E} \mathcal{A}^2 \xrightarrow{F} \mathcal{A}) = \mathrm{Id}_{\mathcal{A}}. \quad$ Then:

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F right fact. system: $\iff (R, \overrightarrow{\lambda})$ idempotent

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Weak fact. systems functorially: Rosický-T 2002

Given **any** functorial factorization

$$\kappa = (\mathrm{dom} \xrightarrow{\lambda} F \xrightarrow{\rho} \mathrm{cod}) \quad (*)$$

(**without** insisting on $FE = \mathrm{Id}_{\mathcal{A}}$), to which extent may it differ from

$$\kappa = (\mathrm{dom} \xrightarrow{F\eta} F \xrightarrow{F\mu} \mathrm{cod}) \ ?$$

On first sight, not by much!

Evaluate $(*)$ at $(EA \xrightarrow{Ef} EB) = (1_A \xrightarrow{\eta_f} f \xrightarrow{\mu_f} 1_B)$ to obtain

$$\lambda_f = F\eta_f \cdot \lambda_{1_A} \quad \text{and} \quad \rho_f = \rho_{1_B} \cdot F\mu_f$$

So, it just depends on how you want to factor identity morphisms!

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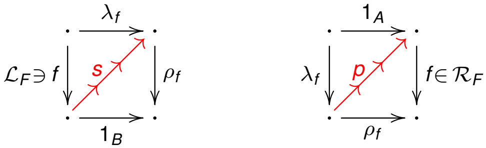
Morphism classes versus functors (Rosický-T 2002)

Given (F, λ, ρ) with $\kappa = \rho \cdot \lambda$, define $\mathcal{L}_F, \mathcal{R}_F$ more carefully by



(F, λ, ρ) *functorial realization of* wfs $(\mathcal{L}, \mathcal{R}) :\Longleftrightarrow \forall f : \lambda_f \in \mathcal{L},\ \rho_f \in \mathcal{R}$

Theorem

(1) $\forall f : \lambda_f \in \mathcal{L}_F, \rho_f \in \mathcal{R}_F \Longrightarrow (F, \lambda, \rho)$ real fun of wfs $(\mathcal{L}_F, \mathcal{R}_F)$

(2) (F, λ, ρ) real fun of wfs $(\mathcal{L}, \mathcal{R}) \Longrightarrow \mathcal{L} = \mathcal{L}_F,\ \mathcal{R} = \mathcal{R}_F$

In this case, equivalent are:
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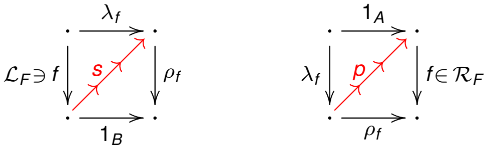

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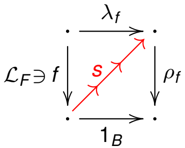

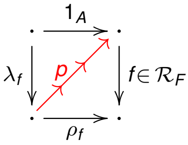

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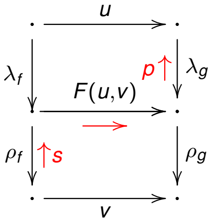
Making liftings constructive: Grandis-T 2006



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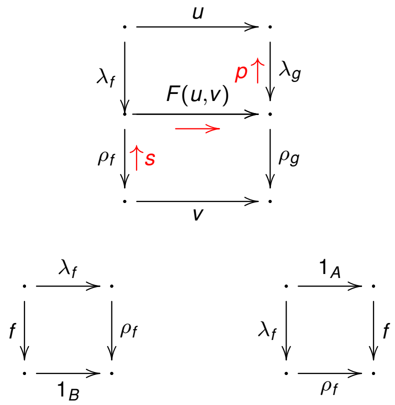

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Natural/algebraic weak factorization systems

In an *algebraic wfs*,
$R : \mathcal{A}^2 \longrightarrow \mathcal{A}^2$ carries a monad structure with unit $\overrightarrow{\lambda}$,
$L : \mathcal{A}^2 \longrightarrow \mathcal{A}^2$ carries a comonad structure with counit $\underrightarrow{\rho}$,
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The morphisms p and s needed for the construction of a lifting (as above) are R-algebra and L-coalgebra structures on g and f, respectively.

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Natural/algebraic weak factorization systems

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Monadic and double-cat perspectives of AWFSs

Bourke - Garner 2016 (JPAA 220:108-147, 148-174)

- Thorough use of monad theory for AWFSs, incl. Dubuc's theorem
- Comprehensive study of AWFSs and their induced double cats
- Various old and new groups of examples exhibited as AWFSs
- Left and right weak maps associated with an AWFS

Two specifics:

- Depending on an L-coalgebra structure s on f, and an R-algebra structure p on g, recognize the "lifting" $p \cdot F(u, v) \cdot s$ for $(u, v) : f \longrightarrow g$ as the value of a natural transformation of functors $L\text{-CoAlg}^{op} \times R\text{-Alg} \longrightarrow \text{Set}$
- Restrict $R\,(L)$ to $\mathcal{A}/1 \cong \mathcal{A}$ $(\cong 0\backslash\mathcal{A})$ to obtain a (co)monad on $\mathcal{A}$
  Its (co)Kleisli cat gives the category of *right (left) weak maps* of $\mathcal{A}$
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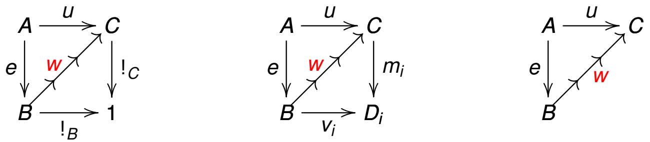
Orthogonality and injectivity via diagonalization



$e \perp !_C$	$e \perp (C, m_i)_I$	$e \perp C$
$e \square !_C$	$e \square (C, m_i)_I$	$e \square C$

No surprise then:

- Orthogonality/injectivity classes enjoy the "same" closure properties as the right factorization classes!
- For I fixed, investigate orthogonal/weak $(\mathcal{E}, \mathbb{M})$ facts of I-cones
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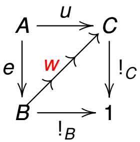

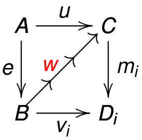

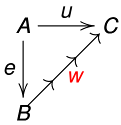

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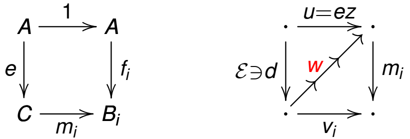
Left $\mathcal{E}$-factorizations for all cones (T 1979)

Equivalent are for any class $\mathcal{E}$ in $\mathcal{A}$:

(i) LSF0 $\mathcal{A}^{\times} \cdot \mathcal{E} \subseteq \mathcal{E}$

LSF1 every source has a strong $\mathcal{E}$-coimage:

$\forall (f_i : A \longrightarrow B_i)_I \ \exists$ factorization $f_i = m_i e : \ e \in \mathcal{E}, \ \mathcal{E} \perp (e, m_i)_I$



(ii) $\mathcal{A}$ is $\mathcal{E}$-*cocomplete*, that is:

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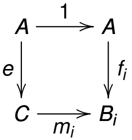

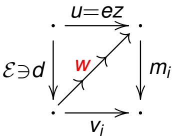

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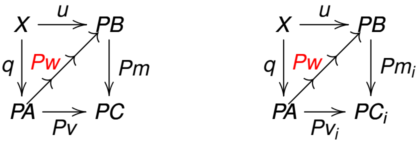
Throwing a functor into the mix (Herrlich 1971, T 1976)

$P : \mathcal{A} \longrightarrow \mathcal{X}$



$(q, A) \perp m$ $\qquad$ $(q, A) \perp (B, m_i)_I$

Examples:

- $\{P$-universal arrows$\} \perp \{$all morphisms/cones$\}$
- $\{$all P-morphisms$\} \perp \{$limit cones mapped by P to monic cones$\}$
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P fibration $\iff$ every P-morphism factors (P-Vert, P-Cart)
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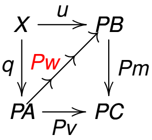

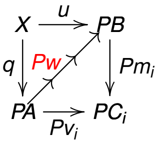

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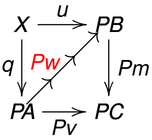

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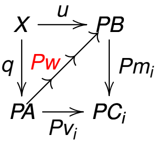

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Left $\mathcal{Q}$-factorizations for all P-sources (T 1979)

Equivalent are for $P : \mathcal{A} \longrightarrow \mathcal{X}$ transportable ("*P* is *solid*"):

(i) $\exists \mathcal{Q} \subseteq \mathcal{X}{\downarrow}P : \ \{P\text{-isos}\} \subseteq \mathcal{Q},\ P(\mathcal{A}^{\times}) \cdot \mathcal{Q} \subseteq \mathcal{Q}$

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Enriched orthogonality

Selected milestones: B. Day (Springer LNM 420) 1974
Kelly 1982
Anghel (PhD thesis) 1987, (Comm. Alg.) 1990
Lucyshyn-Wright (PhD thesis) 2013, (TAC) 2014

Basic idea:

$\mathcal{A}$ $\mathcal{V}$-enriched ($\mathcal{V}$ symm. monoidal-cl.), $e : A \longrightarrow B$, $m : C \longrightarrow D$ in $\mathcal{A}$

$e \perp_{\mathcal{V}} m$ | $\mathcal{A}$ tensored | $\mathcal{A}$ cotensored

$w : X \longrightarrow \mathcal{A}(B, C)$ | $w^\sharp : X \otimes B \longrightarrow C$ | $w^\flat : B \longrightarrow X \pitchfork C$

$X = I$ $\otimes$-neutral: $e \perp_{\mathcal{V}} m \Longrightarrow e \perp m$ in $\mathcal{A}_o$

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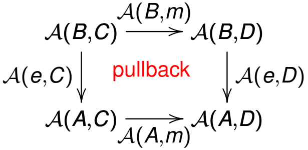

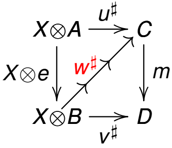

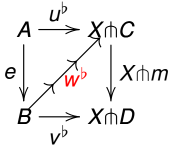

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Stabilty properties of $\mathcal{M} = \mathcal{X}^{\perp_\mathcal{V}}$:

- contains the isomorphisms
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But there is then a range of variations: strict, pseudo, lax, ... ?

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Orthogonal factorization in 2-categories

"Millennium cluster"	Kasangian-Vitale 2000:	all 2-cells are iso
	Dupont 2001	(*Mém. de Licence*)
	Milius 2001	lax orthogonality
	Dupont-Vitale 2002	pseudo-orthogonality

$$\begin{array}{ccc} \mathcal{A}(B,C) & \xrightarrow{\mathcal{A}(B,m)} & \mathcal{A}(B,D) \\ {\scriptstyle \mathcal{A}(e,C)}\downarrow & \text{bipullback} & \downarrow{\scriptstyle \mathcal{A}(e,D)} \\ \mathcal{A}(A,C) & \xrightarrow[\mathcal{A}(A,m)]{} & \mathcal{A}(A,D) \end{array} \qquad \begin{array}{ccc} \mathcal{A}^2(e,m) & \xrightarrow{\text{cod}} & \mathcal{A}(B,D) \\ {\scriptstyle \text{dom}}\downarrow & \text{bipullback} & \downarrow{\scriptstyle \mathcal{A}(e,D)} \\ \mathcal{A}(A,C) & \xrightarrow[\mathcal{A}(A,m)]{} & \mathcal{A}(A,D) \end{array}$$

What does this mean for e and m ?

Orthogonal factorization in 2-categories

"Millennium cluster"	Kasangian-Vitale 2000:	all 2-cells are iso
	Dupont 2001	(*Mém. de Licence*)
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\end{array}
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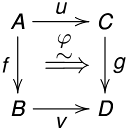
What does this mean for e and m ?

The 2-category $\mathcal{A}^2$ when $\mathcal{A}$ is a 2-category

Objects: objects of $\mathcal{A}$

Arrows: $(u, \varphi, v) : f \longrightarrow g$, φ inv'ble



$(*)$

2-cells: $(\sigma, \tau) : (u, \varphi, v) \Rightarrow (x, \psi, y)$ $\quad \sigma : u \Rightarrow x, \ \tau : v \Rightarrow y$

$g\sigma \cdot \varphi = \psi \cdot \tau f$

Fill-ins for $(*)$: (α, w, β), α, β invertible

$\varphi \cdot \beta f = g\alpha$

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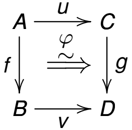

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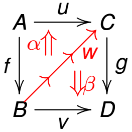

$\varphi \cdot \beta f = g\alpha$

Pseudo-orthogonality

$f \curlywedge g$ — *"f pseudo-orthogonal to g"*

$\iff$ $\mathcal{A}(B,C) \longrightarrow \mathcal{A}^2(f,g)$, $w \mapsto (wf, wg)$ — equivalence of categories

$\iff$
1. $\forall (u,\varphi,g) : f \Rightarrow g$ — $\exists$ fill-in (α, w, β),
2. $\forall (\sigma,\tau) : (u,\varphi,v) \Rightarrow (x,\psi,y)$ — with fill-in (γ, z, δ)
   $\exists!\ \xi : w \Longrightarrow z :$ — $\gamma \cdot \xi f = \sigma \cdot \beta, \delta \cdot g\xi = \tau \cdot \beta$

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$\iff$	1. $\forall (u, \varphi, g) : f \Rightarrow g$ 2. $\forall (\sigma, \tau) : (u, \varphi, v) \Rightarrow (x, \psi, y)$ $\exists !\ \xi : w \Longrightarrow z$:	$\exists$ fill-in (α, w, β), with fill-in (γ, z, δ) $\gamma \cdot \xi f = \sigma \cdot \beta, \delta \cdot g\xi = \tau \cdot \beta$

Pseudo-orthogonal factorization system $(\mathcal{E}, \mathcal{M})$

Definition

$\mathcal{A}^{\ltimes} := \{$equivalences in $\mathcal{A}\}$

PF0a $\mathcal{A}^{\ltimes} \cdot \mathcal{E} \subseteq \mathcal{E}, \quad \mathcal{M} \cdot \mathcal{A}^{\ltimes} \subseteq \mathcal{M}$

PF0b $f \cong e \in \mathcal{E} \Longrightarrow f \in \mathcal{E}, \quad f \cong m \in \mathcal{M} \Longrightarrow f \in \mathcal{M}$

PF1 $\forall f\ \exists m \cdot e \cong f,\ e \in \mathcal{E},\ m \in \mathcal{M}$

PF2 $\mathcal{E} \curlywedge \mathcal{M}$

Properties:

- $\mathcal{E} \cap \mathcal{M} = \mathcal{A}^{\ltimes}$
- $\mathcal{E} = {}^{\curlywedge}\mathcal{M},\ \mathcal{M} = \mathcal{E}^{\curlywedge}$ and, hence, closed under composition, ...

Important consequence:
$(\mathrm{Hot}(\mathcal{E}), \mathrm{Hot}(\mathcal{M}))$ is a weak factorization system of $\mathrm{Hot}(\mathcal{A})$

Enriched and 2-categorical functorial factorization

Enriched functorial factorization:

- Riehl 2016 *Categorical Homotopy Theory*, preceded by Riehl 2011 (NY J. Math), Riehl 2013 (JPAA)
- Feature: enriching Garner's Small Object Argument

Lax-orthogonal functorial factorization:

- Clementino - López Franco 2016 (Adv. Math.), 2017 (LMCS)
- Idea of lax orth: for $f : A \longrightarrow B, g : C \longrightarrow D$ in a 2-cat $\mathcal{A}$: let $\mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C) \times_{\mathcal{A}(A,D)} \mathcal{A}(B, D)$ be rari (Gray 1966) (= existence of least diagonal fill-ins in the ordered case)
- Example: Gray's (lali,cofibration)-factorization in Cat
- Equivalently: AWFS (L, R) with L and/or R lax idempotent
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Fibrations

$$\begin{array}{rl} & P : \mathcal{A} \longrightarrow \mathcal{C} \text{ fibration} \\ \iff & \forall B \in \mathcal{C} : P^B : \mathcal{A}/B \longrightarrow \mathcal{C}/PB \ \text{ has rari} \\ \iff & \tilde{P} : \mathcal{A}^2 = \mathcal{A}{\downarrow}\mathcal{A} \longrightarrow \mathcal{C}{\downarrow}P \ \text{ has rari} \end{array}$$

P fib $\Rightarrow$ (P cof $\iff$ $\forall f : X \longrightarrow Y$ in $\mathcal{C} : f^* : P^{-1}Y \longrightarrow P^{-1}X$ has l.a. $f_!$)
P cof $\Rightarrow$ (P fib $\iff$ $\forall f : X \longrightarrow Y$ in $\mathcal{C} : f_! : P^{-1}X \longrightarrow P^{-1}Y$ has r.a. f^*)

Consider any $\mathcal{M}_0$ in $\mathcal{C}$ with $1_{\mathcal{C}} := \{1_X \mid X \in \mathcal{C}\} \subseteq \mathcal{M}_0$ and let cod: $\mathcal{M}_0 \hookrightarrow \mathcal{C}^2 \longrightarrow \mathcal{C}$ be the codomain functor. Key observation:

$$\begin{array}{ccc} M & \xrightarrow{e} & N \\ {\scriptstyle m}\downarrow & (*) & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

cod-cartesian $\iff$ $(*)$ pullback diagram
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When is cod a bifibration?

Theorem (adaptation from T 1983, Dikranjan-T 1995)

Equivalent for $\mathcal{M}_0$ in $\mathcal{C}$ with $1_{\mathcal{C}} \subseteq \mathcal{M}_0$:

(i) $\mathcal{C}$ has $\mathcal{M}_0$-pullbacks and strong $\mathcal{M}_0$-images

(ii) $\mathcal{C}$ has $\mathcal{M}_0$-pullbacks, $\forall f$ in $\mathcal{C}$: $f^* : \mathcal{M}_0/Y \longrightarrow \mathcal{M}_0/X$ has l.a. $f_!$

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In that case,
$\mathcal{M} := \mathcal{M}_0 \cdot \mathcal{C}^{\times}$ is a right factorization system of $\mathcal{C}$,
${}^{\perp}\mathcal{M}_0 = {}^{\perp}\mathcal{M} = \{f \mid f_!(1_X) \cong 1_Y\} =: \mathcal{E}$,
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$\mathcal{M}_0$ *strict right fact system* in $\mathcal{C}$ if

RF0⁻ $1_{\mathcal{C}} \subseteq \mathcal{M}_0$

RF1 $\mathrm{cod} : \mathcal{M}_0 \longrightarrow \mathcal{C}$ cofibration

Then

$$\mathrm{cod} \dashv I \dashv \mathrm{dom}$$

with $I : \mathcal{C} \longrightarrow \mathcal{M}_0,\ X \mapsto 1_X,$ being rari of cod and lari of dom, s. th. the counits

$$\varepsilon_m : I\,\mathrm{dom}(m) \longrightarrow m \qquad \begin{array}{ccc} M & \xrightarrow{1_M} & M \\ {\scriptstyle 1_M}\downarrow & \varepsilon_m & \downarrow{\scriptstyle m} \\ M & \xrightarrow[m]{} & X \end{array}$$

are cod-cocartesian

Strict right factorization systems

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Fibrational Characterization Theorem (after Hughes-Jacobs 2003)

Strict right factorization systems in a category $\mathcal{C}$ are equivalently described by double adjunctions

$$P \dashv J \dashv Q : \mathcal{A} \longrightarrow \mathcal{C}$$

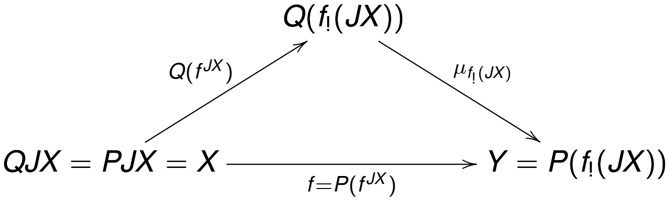
such that

- P is a cofibration
- J is rari of P and lari of Q
- the counits $\varepsilon_A : JQA \longrightarrow A$ $(A \in \mathcal{A})$ are P-cocartesian.

The corresponding strict right factorization system of $\mathcal{C}$ is

$$\mathcal{M}_0 = \{\mu_A := P\varepsilon_A : QA \longrightarrow PA \mid A \in \mathcal{A}\}.$$

Construction of factorizations from (P, J, Q, ε)



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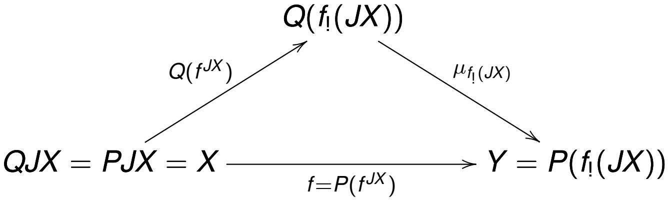

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The global categorical equivalence

- Objects: strict right factorization systems $\mathcal{M}_0 \subseteq \mathcal{C}^2$
  Morphisms: $\mathcal{M}_0 \hookrightarrow \mathcal{N}_0$
- Objects: "factorization cofibrations" $(P : \mathcal{A} \longrightarrow \mathcal{C}, J, Q, \varepsilon)$
  Morphisms: $F : (P, J, Q, \varepsilon) \longrightarrow (R : \mathcal{B} \longrightarrow \mathcal{C}, K, S, \delta)$ with
  $F : \mathcal{A} \longrightarrow \mathcal{B},\ RF = P,\ FJ = K,\ SF = Q,\ F\varepsilon = \delta F$
- Trivially: $\mathcal{M}_0 \mapsto (\mathrm{cod}_{\mathcal{M}_0}, I, \mathrm{dom}_{\mathcal{M}_0}, \varepsilon) \mapsto \mathcal{M}_0$
- Non-trivially: given (P, J, Q, ε), the "comparison functor"

$$F : \mathcal{A} \longrightarrow \{\mu_A \,|\, A \in \mathcal{A}\}, \quad A \mapsto \mu_A,$$

  is fully faithful precisely because each ε_A is P-cocartesian.
- $F : (P, J, Q, \varepsilon) \longrightarrow (\mathrm{cod}_{\{\mu_A\}}, I, \mathrm{dom}_{\{\mu_A\}}, \varepsilon)$ has a quasi-inverse

Fibrational Characterization Theorem continued

$\mathcal{M} := \mathcal{M}_0 \cdot \mathcal{C}^{\times}$ belongs to an orthogonal factorization system $\iff$ $\forall A, B \in \mathcal{A}$ $(PB = QA \Longrightarrow Q(P\text{-cocart. lift of } \mu_A : PB \longrightarrow PA) \text{ is iso})$ of cats

If $\mathcal{C}$ has pullbacks:
$\mathcal{M}$ belongs to a stable orthogonal factorization system $\iff$
P is a bifibration satisfying Beck-Chevalley, that is:

$\Longrightarrow \quad u_! \cdot f^* = g^* \cdot v_!$

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$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\downarrow & pb & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \qquad \Longrightarrow \qquad u_! \cdot f^* = g^* \cdot v_!$$

To do?

Take strict one-sided factorization systems to the “next level”:

- enriched
- 2-categorical
- bicategorical
- ...,

and to say ...

“Thank you !”