# Quantale-weighted Categories

Walter Tholen

York University, Toronto, Canada

75 Jiří Rosický 75

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# Bill's metric spaces: a tribute, fifty years in the making

F. W. Lawvere: Metric spaces, generalized logic, and closed categories

*Rendiconti del Seminario Matematico e Fisico di Milano* 43:135–166, 1973.

Republished in *Reprints in Theory and Applications of Categories* 1, 2002.

This paper not only introduces metric spaces as small categories enriched in the extended real half-line (considered as a symmetric monoidal-closed category under addition), but it is also the birthplace of *normed categories*, as categories enriched in a certain symmetric monoidal category of *normed sets*.

Slogan:
Taking enriched category theory as a conceptual guide is useful not only in algebra and topology, but also in the broad area of analysis.

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# Selected references that helped me prepare this talk


- A. Akhvlediani, M.M. Clementino, W.T.: On the categorical meaning of Hausdorff and Gromov distances I, Topology Appl., 2010
- A. Aliouche and C. Simpson: Fixed points and lines in 2-metric spaces, Advances in Math., 2012
- M. Grandis: Directed Algebraic Topology, Cambridge U Press, 2009
- D. Hofmann, G.J. Seal, W.T. (eds.): Monoidal Topology, Cambridge U Press, 2014
- M. Insall, D. Luckhardt: Norms on categories and analogs of the Schröder-Bernstein Theorem, arXiv, 2021
- W. Kubiś: Categories with norms, arXiv, 2018
- P. Perrone: Lifting couples in Wasserstein spaces, arXiv, 2021
- W.T.: Remarks on weighted categories and the non-symmetric Pompeiu-Hausdorff-Gromov metric, Talk at CT 2018 (Ponta Delgada)
- W.T., J. Wang: Metagories, Topology Appl., 2020.

# Overview

1. Metric spaces: From Frechét to Lawvere
2. Passing from the terminal quantale 1 to quantale-weighted categories à la Lawvere
3. Some discussion of the axiomatics of weighted (or normed) categories
4. The connection between weighted categories and metrically enriched categories
5. Metrically enriched categories vs. metagories

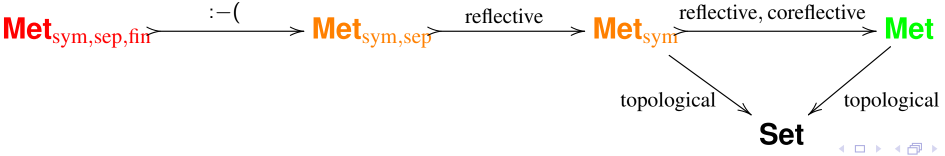
# A categorical analysis of Frechét's axioms

A (classical) metric $d : X \times X \longrightarrow [0, \infty]$ on a set $X$ satisfies

| | | |
|---|---|---|
| 0-*Selfdistances*: | $0 \geq d(x,x)$ | $1 \to X(x,x)$ |
| $\nabla$-*Inequality*: | $d(x,y) + d(y,z) \geq d(y,z)$ | $X(x,y) \times X(y,z) \to X(x,z)$ |
| *Symmetry*: | $d(x,y) = d(y,x)$ | $X(x,y) \cong X(y,x)$ |
| *Separation*: | $d(x,y) = 0 = d(y,x) \Longrightarrow x = y$ | $X(x,y) \cong 1 \cong X(x,y) \Longrightarrow x = y$ |
| *Finiteness*: | $\infty > d(x,y)$ | $\emptyset \neq X(x,y)$ |

A map $f : X \to Y$ of metric spaces is non-expansive / short / 1-Lipschitz if

| | | |
|---|---|---|
| *Contraction*: | $d(x,x') \geq d(fx, fx')$ | $X(x,x') \to Y(fx, fx')$ |



# Lawvere's early advocate: Hausdoff's *Grundzüge der Mengenlehre*

Try to extend a (classical) metric $d$ on a set $X$ to all of its subsets $A, B$:

$$d(A,B) = \sup_{x\in A} d(x,B) = \sup_{x\in A} \inf_{y\in B} d(x,y)$$

(*"the minimal effort required to evacuate every inhabitant of A to the nearest point in B"*)

The only general survivors* are the Lawvere conditions: 0-Selfdistances and $\nabla$-Inequality!

Standard rescue operations for the other conditions:

- Symmetrize coreflectively: $d_{\mathrm{H}}(A,B) = \max\{d(A,B), d(B,A)\}$ (Hausdorff 1914)
  or "monoidally": $d_{\mathrm{P}}(A,B) = d(A,B) + d(B,A)$ (Pompeiu 1907)
- Enforce separation by considering closed sets only
- Enforce finiteness by considering only non-empty compact sets

* ... and they survive even if we replace $[0,\infty]$ by a quantale $\mathcal{V}$, as follows!

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# What is a quantale? Some categorical answers:

It is

- a (small) one-object 2-category $\mathcal{V}$ whose only hom-category $\mathcal{V}(*,*)$ is a cocomplete lattice, such that all functors $\mathcal{V}(u,*), \mathcal{V}(*,u) : \mathcal{V}(*,*) \longrightarrow \mathcal{V}(*,*)$ preserve colimits
- a (small) thin, skeletal, cocomplete monoidal-closed category $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$
- a monoid $\mathcal{V}$ in the symmetric monoidal-closed category **Sup** $= ($**Sup**$, \boxtimes, 2 = \mathcal{P}1)$ of complete lattices with suprema-preserving maps (where morphisms $L \boxtimes M \longrightarrow N$ classify maps $L \times M \longrightarrow N$ preserving suprema in each variable; Joyal - Tierney 1984)
- a (small) one-object category enriched in **Sup** (a self-dual, monadic cat. over **Set**!)

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# In down-to-earth terms:

A quantale $\mathcal{V}$ is a complete lattice $(\mathcal{V}, \leq)$ that comes with a monoid structure $(\mathcal{V}, \otimes, \mathrm{k})$, such that the monoid multiplication $\otimes$ preserves suprema in each variable:

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i \ , \qquad (\bigvee_{i \in I} v_i) \otimes u = \bigvee_{i \in I} (v_i \otimes u)$$

In order to avoid having to deal with two types of internal homs, throughout this talk I will assume that (the monoid) $\mathcal{V}$ is *commutative*:

$$u \otimes v = v \otimes u, \qquad u \leq [v, w] \iff u \otimes v \leq w.$$

Our first example: the terminal quantale $1 = \{*\}$ (Wow!)

Other standards: the Boolean quantale $(2, \bot < \top, \wedge, \top)$; in fact: any locale;
the Lawvere quantale $([0, \infty], \geq, +, 0)$;
the free quantale $(\mathcal{P}M, \subseteq, \cdot, \{e\})$ over a (comm.) monoid $(M, \cdot, e)$.

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# The thin category $(\mathcal{V}, \leq)$ over $\mathbf{1}$, and applying $\mathrm{Fam}$ to them

$$\mathbf{1} \xrightarrow{\mathrm{i}} \mathrm{Fam}(\mathbf{1}) = \mathbf{Set} \qquad\qquad * \longmapsto 1$$

$$(\mathcal{V}, \leq) \xrightarrow{\mathrm{i}} \mathrm{Fam}(\mathcal{V}, \leq) = \mathbf{Set}/\!/\mathcal{V} \qquad\qquad (u \leq v) \longmapsto \begin{array}{ccc} 1 & = & 1 \\ & \leq & \\ {\scriptstyle u} \searrow & & \swarrow {\scriptstyle v} \\ & \mathcal{V} & \end{array}$$

$\mathbf{Set}/\!/\mathcal{V}$:

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & J \\ & \leq & \\ {\scriptstyle u = |-|} \searrow & & \swarrow {\scriptstyle |-| = v} \\ & \mathcal{V} & \end{array} \iff \forall\, i \in I: \ u_i \leq v_{\varphi i} \iff \forall\, i \in I: \ |i| \leq |\varphi i|$$

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# The category of $\mathcal{V}$-weighted sets is symmetric monoidal-closed

$$I \otimes J = (\, I \times J,\ |(i,j)| = |i| \otimes |j|\,), \qquad E = (1 = \{*\},\ |*| = \mathrm{k})$$

$$[I, J] = (\, \mathbf{Set}(I, J),\ |\varphi| = \bigwedge_{i \in I} [|i|, |\varphi i|]\,)$$

We obtain a commutative diagram of (strict) homomorphisms of monoidal categories:




$$\mathrm{s} I = \bigvee_{i \in I} |i|$$

In addition, the straight arrows preserve also the internal homs.
Furthermore: the left adjoint s preserves products iff $\mathcal{V}$ is completely distributive.

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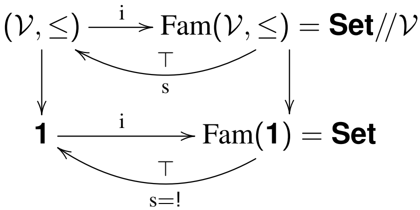


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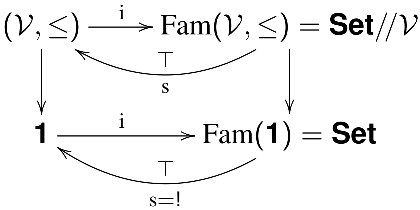


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# Form the corresponding categories of small enriched categories...

... and their change-of-base functors:

$$\begin{array}{ccc}
\mathcal{V}\text{-}\mathbf{Cat} & \underset{\underset{\mathrm{s}}{\longleftarrow}}{\overset{\mathrm{i}}{\longrightarrow}} \; (\top) & \mathrm{Fam}(\mathcal{V})\text{-}\mathbf{Cat} =: \mathbf{Cat}/\!/\mathcal{V} \\
\text{topological} \downarrow & & \downarrow \text{topological} \\
\mathbf{Set} = \mathbf{1}\text{-}\mathbf{Cat} & \underset{\underset{\mathrm{s=ob}}{\longleftarrow}}{\overset{\mathrm{i}}{\longrightarrow}} \; (\top) & \mathrm{Fam}(\mathbf{1})\text{-}\mathbf{Cat} = \mathbf{Cat}
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# The category **Cat**$//\mathcal{V}$ of (small) $\mathcal{V}$-weighted categories

For a (small) category $\mathbb{X}$ to be enriched in $\mathrm{Fam}(\mathcal{V}, \leq) = \mathbf{Set}//\mathcal{V}$ means (without quantifiers):

$$\begin{array}{lll} & \mathbb{X}(x,y) \otimes \mathbb{X}(y,z) \longrightarrow \mathbb{X}(x,z) \quad \text{and} \quad E \longrightarrow \mathbb{X}(x,x) & \text{live in } \mathbf{Set}//\mathcal{V} \\ \Longleftrightarrow & |f| \otimes |g| = |(f,g)| \leq |g \cdot f| \quad \text{and} \quad \mathrm{k} \leq |1_x| & \\ \Longleftrightarrow & |\text{-}| : \mathbb{X} \longrightarrow (\mathcal{V}, \otimes, \mathrm{k}) \text{ is a lax functor} & \end{array}$$

For a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ to be enriched in $\mathbf{Set}//\mathcal{V}$ means (without quantifiers):

$$\begin{array}{ll} & \mathbb{X}(x,y) \longrightarrow \mathbb{Y}(Fx, Fy) \quad \text{lives in } \mathbf{Set}//\mathcal{V} \\ \Longleftrightarrow & |f| \leq |Ff| \\ \Longleftrightarrow & \begin{array}{ccc} \mathbb{X} & \xrightarrow{\;F\;} & \mathbb{Y} \\ & \leq & \\ {\scriptstyle |\text{-}|} \searrow & & \swarrow {\scriptstyle |\text{-}|} \\ & \mathcal{V} & \end{array} \end{array}$$

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For a functor $F:\mathbb{X}\longrightarrow\mathbb{Y}$ to be enriched in $\mathbf{Set}/\!/\mathcal{V}$ means (without quantifiers):

$$\begin{array}{ll} & \mathbb{X}(x,y)\longrightarrow\mathbb{Y}(Fx,Fy) \quad \text{lives in } \mathbf{Set}/\!/\mathcal{V} \\ \iff & |f|\leq|Ff| \\ \iff & \mathbb{X}\xrightarrow{F}\mathbb{Y},\quad |\text{-}|:\mathbb{X}\to\mathcal{V} \;\leq\; |\text{-}|\cdot F,\quad |\text{-}|:\mathbb{Y}\to\mathcal{V} \end{array}$$

$\mathcal{V}$-**Cat** $\qquad$ **Cat**$//\mathcal{V}$

$X, \quad X(x,y) \otimes X(y,z) \leq X(x,z) \qquad \overset{\mathrm{i}}{\longmapsto} \qquad \mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$

$\mathrm{k} \leq X(x,x) \qquad x \xrightarrow{(x,y)} y, \quad |(x,y)| = X(x,y)$

$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad \mathrm{s}\mathbb{X}(x,y) = \bigvee\{|f| \mid f : x \to y\} \qquad \overset{\mathrm{s}}{\longleftarrow\!\!\!\mid} \qquad \mathbb{X}, \quad |f| \otimes |g| \leq |g \cdot f|$

$\mathrm{k} \leq |1_x|$

$X \otimes Y = X \times Y$ (as a set) $\qquad \mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}$ (as a category)

$(X \otimes Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y') \qquad |(f,g)| = |f| \otimes |g|$

$[X,Y] = \mathcal{V}\text{-}\mathbf{Cat}(X,Y)$ (as a set) $\qquad [\mathbb{X},\mathbb{Y}] = (\mathbf{Cat}//\mathcal{V})(\mathbb{X},\mathbb{Y})$ (as a cat)

$[X,Y](f,g) = \bigwedge_{x \in X} Y(fx,gx) \qquad |\, F \xrightarrow{\alpha} G \,| = \bigwedge_{x \in \mathrm{ob}\mathbb{X}} |\alpha_x|$

# Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = (2, \bot < \top, \wedge, \top)$

**2-Cat** $=$ **Ord** — **Cat**$//2 =$ **sCat**

$X, \quad x \leq y \ \wedge \ y \leq z \Longrightarrow x \leq z \qquad \xmapsto{\ \mathrm{i}\ } \qquad \mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$

$\top \Longrightarrow x \leq x \qquad\qquad (\, x \xrightarrow{(x,y)} y) \in \mathcal{S} \Longleftrightarrow x \leq y$

$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad x \leq y \Longleftrightarrow \exists (f : x \to y) \in \mathcal{S} \qquad \xleftarrow{\ \mathrm{s}\ } \qquad \mathbb{X}, \mathcal{S}, \quad f, g \in \mathcal{S} \Longrightarrow g \cdot f \in \mathcal{S}$

$\top \Longrightarrow 1_x \in \mathcal{S}$

$X \otimes Y = X \times Y$ — $\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}$ (as a category)

$(x, y) \leq (x', y') \Longleftrightarrow \ x \leq x' \ \wedge \ y \leq y'$ — $\mathcal{S}_{\mathbb{X} \otimes \mathbb{Y}} = \mathcal{S}_{\mathbb{X}} \times \mathcal{S}_{\mathbb{Y}}$

$[X, Y] =$ **Ord**$(X, Y)$ — $[\mathbb{X}, \mathbb{Y}] =$ **sCat**$(\mathbb{X}, \mathbb{Y})$ (as a cat)

$f \leq g \Longleftrightarrow \forall x \in X : fx \leq gx$ — $\alpha \in \mathcal{S}_{[\mathbb{X}, \mathbb{Y}]} \Longleftrightarrow \forall x \in \mathrm{ob}\mathbb{X} : \alpha_x \in \mathcal{S}_{\mathbb{Y}}$

# Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = ([0,\infty], \geq, +, 0)$

$[0,\infty]$-**Cat** $=$ **Met**

$X, \quad d(x,y) + d(y,z) \geq d(x,z)$

$0 \geq d(x,x)$

$\xrightarrow{\ \mathrm{i}\ }$

**Cat**$//[0,\infty] =$ **wCat**

$\mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$

$x \xrightarrow{(x,y)} y\ , \quad |(x,y)| = d(x,y)$

$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad d(x,y) = \inf_{f:x \to y} |f|$

$\xleftarrow{\ \mathrm{s}\ }$

$\mathbb{X}, \quad |f| + |g| \geq |g \cdot f|$

$0 \geq |1_x|$

$X \otimes Y = X \times Y$

$d((x,y),(x',y')) = d(x,y) + d(y,y')$

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$[X, Y] =$ **Met**$(X,Y)$

$d(f,g) = \sup_{x \in X} d(fx, gx)$

$[\mathbb{X}, \mathbb{Y}] =$ **wCat**$(\mathbb{X}, \mathbb{Y})$ (as a cat)

$|\ F \xrightarrow{\alpha} G\ | = \sup_{x \in \mathrm{ob}\mathbb{X}} |\alpha_x|$

# Some elementary examples of weighted categories, I

We saw:

$\mathcal{V}$-categories (and their functors) are $\mathcal{V}$-weighted categories (and their functors); in fact, they are precisely the $\mathcal{V}$ weighted categories with indiscrete underlying category.

Question: May **Set** be "naturally" $[0,\infty]$-weighted?

Goal 1: Let $|f|$ measure the degree to which a map $f : X \to Y$ fails to be surjective.

Simply put $\quad |f| := \#(Y \setminus f(X)) \in \mathbb{N} \cup \{\infty\} \subseteq [0,\infty]$.

Then: $0 \geq |\mathrm{id}_X|$, and with $g : Y \to Z$ we have $|f| + |g| \geq |g \cdot f|$

since (assuming Choice and $Y \cap Z = \emptyset$) there is an injective map

$$Z \setminus (g(f(X))) \longrightarrow (Y \setminus f(X)) + (Z \setminus g(Y)).$$

Note: $\quad f$ surjective $\iff |f| = 0$.

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# Some elementary examples of weighted categories, II

Question: May something similar be done for injectivity? That is:

Goal 2: Let $|f|$ measure the degree to which a map $f : X \to Y$ fails to be injective.

First consider $\#f := \sup_{y \in Y} \#f^{-1}y$; then, with $g : Y \to Z$, we have:

$$\#g \cdot \#f = (\sup_{z\in Z} \#g^{-1}z) \cdot (\sup_{y \in Y} \#f^{-1}y) \geq \sup_{z \in Z} \#(\bigcup_{y \in g^{-1}z} f^{-1}y) = \#(g \cdot f), \quad 1 \geq \#\mathrm{id}_X$$

Not what we wanted! But $([1,\infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0,\infty], \geq, +, 0)$ comes to the rescue:

Put $|f| := \max\{0, \log \#f\}$; then: $|g| + |f| \geq |g \cdot |f|.$ $0 \geq |\mathrm{id}_X|$.

Note: $f$ injective $\iff |f| = 0$.

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# A more interesting example of a (large) weighted category: **Lipschitz**

$\mathrm{ob}\,\mathbf{Lip} = \mathrm{ob}\,\mathbf{Met}$, $\mathbf{Lip}(X, Y) = \mathbf{Set}(X, Y)$; why call this category **Lip** ??

Recall: $f : X \to Y$ is $K(\geq 0)$-Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K\, d(x, x')$

In particular: $f : X \to Y$ is a morphism in **Met** $\iff$ $f$ is 1-Lipschitz

Question: How far is an arbitrary map $f$ away from being 1-Lipschitz?

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That is: $\mathrm{Lip}(f) = \max\{1, \sup_{x \neq x'} \frac{d(fx, fx')}{d(x, x')}\}$ (assuming temporarily that $X$ be separated)

Then: $\mathrm{Lip}(g) \cdot \mathrm{Lip}(f) \geq \mathrm{Lip}(g \cdot f), \quad 1 \geq \mathrm{Lip}(\mathrm{id}_X)$

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$([1, \infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0, \infty], \geq, +, 0)\ , \quad |f| = \max\{0, \sup_{x,x'} (\log d(fx, fx') - \log d(x, x'))\}$

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# On the axiomatics for weighted/normed categories

The category $\mathbb{X}$ is $\mathcal{V}$-weighted by $|\text{-}| : \mathbb{X} \longrightarrow \mathcal{V}$ if

$$\mathrm{k} \leq |1_x|$$

$$|g| \otimes |f| \leq |g \cdot f| \iff |f| \leq \bigwedge_g [|g|, |g \cdot f|] \iff |f| = \bigwedge_g [|g|, |g \cdot f|]$$

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The $\mathcal{V}$-weighted category $\mathbb{X}$ is *right/left cancellable* if

$$|f| \otimes |g \cdot f| \leq |g| \iff |f| \leq \bigwedge_g [|g \cdot f|, |g|] =: |f|^{\mathrm{R}} \quad \text{(right cancellable)}$$

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Facts (Insall-Luckhardt for $\mathcal{V} = [0, \infty]$): $\mathbb{X}$ weighted by $|\text{-}| \implies \mathbb{X}$ weighted by $|\text{-}|^{\mathrm{R}}$ and $|\text{-}|^{\mathrm{L}}$, and $|f| \leq |f|^{\mathrm{RR}}$, $|f| \leq |f|^{\mathrm{LL}}$.

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# The underlying ordinary category $\mathbb{X}_0$ of a $\mathcal{V}$-weighted category $\mathbb{X}$

Note:

An isomorphism $f$ in $\mathbb{X}$ may not satisfy $\mathrm{k} \leq |f|$, and even when it does, we may not have $\mathrm{k} \leq |f^{-1}|$ (unless the weight is left/right cancellable). Still, in many of the examples with $\mathcal{V} = [0, \infty]$ considered in the literature, morphisms $f$, and especially isomorphisms, of norm 0 play an important role. They are called "modulators" by Insall-Luckhardt.

Question:

What is the "enriched significance" of considering morphisms $f$ with $\mathrm{k} \leq |f|$?

Answer:

These are precisely the morphisms of the underlying ordinary category $\mathbb{X}_0$ of the $(\mathbf{Set}/\!/\mathcal{V})$-enriched category $\mathbb{X}$.

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Recall: groups $(X, -, 0)$ in subtractive notation:

$$x - 0 = x,\ x - x = 0,\ (x - y) - (z - y) = x - z$$

Write $\mathcal{V}$-**Met** for $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sym}}$: "$\mathcal{V}$-metric spaces" = $\mathcal{V}$-categories $X$ with $X(x, y) = X(y, x)$

Form the category $\mathcal{V}$-**MetGrp** of "$\mathcal{V}$-metric groups":

objects are $\mathcal{V}$-metric spaces $X$ with a group structure that makes distances invariant under translations:

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morphisms are $\mathcal{V}$-contractive homomorphisms.

$\mathcal{V}$-**MetGrp** inherits its symmetric monoidal structure from $\mathcal{V}$-**Cat** and the cartesian cat **Grp**.

# $\mathcal{V}$-weighted categories *vs.* $\mathcal{V}$-metrically enriched cats: syntax prep

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# $\mathcal{V}$-metric groups as $\mathcal{V}$-weighted groups

The category **Grp**$//\mathcal{V}$ has as

objects: $\mathcal{V}$-weighted sets $(X, |\text{-}|)$ with a group structure such that

$$\mathrm{k} \leq |0|, \quad |x| \otimes |y| \leq |x - y|;$$

morphisms live in both, **Set**$//\mathcal{V}$ and **Grp**.

Obtain:

$$\mathbf{Grp}//\mathcal{V} \xleftarrow{\quad\cong\quad} \mathcal{V}\text{-}\mathbf{MetGrp}$$

$$X \longmapsto X(x,y) = |x-y|$$

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# $\mathcal{V}$-weighted cats *vs.* $\mathcal{V}$-metrically enriched cats via change of base

$$\begin{array}{ccccc} & & \mathbf{Grp}/\!/\mathcal{V} & \xleftrightarrow{\;\cong\;} & \mathcal{V}\text{-}\mathbf{MetGrp} \\ & & \downarrow & & \downarrow \\ \mathcal{V} & \overset{i}{\rightarrowtail} & \mathbf{Set}/\!/\mathcal{V} & & \mathcal{V}\text{-}\mathbf{Met} \end{array}$$

$$\begin{array}{ccccc} & & (\mathbf{Grp}/\!/\mathcal{V})\text{-}\mathbf{Cat} & \xleftrightarrow{\;\cong\;} & (\mathcal{V}\text{-}\mathbf{MetGrp})\text{-}\mathbf{Cat} \\ & & \downarrow & & \downarrow \\ \mathcal{V}\text{-}\mathbf{Cat} & \overset{i}{\rightarrowtail} & (\mathbf{Set}/\!/\mathcal{V})\text{-}\mathbf{Cat} = \mathbf{Cat}/\!/\mathcal{V} & & (\mathcal{V}\text{-}\mathbf{Met})\text{-}\mathbf{Cat} \end{array}$$

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# Completing the picture: $\mathcal{V}$-metrically approximate categories

$$\begin{array}{ccccccc}
 & & & & (\mathbf{Grp}/\!/\mathcal{V})\text{-}\mathbf{Cat} & \overset{\cong}{\longleftrightarrow} & (\mathcal{V}\text{-}\mathbf{MetGrp})\text{-}\mathbf{Cat} \\
 & & & & \downarrow & & \downarrow \\
\mathcal{V}\text{-}\mathbf{Met} & \rightarrowtail & \mathcal{V}\text{-}\mathbf{Cat} & \overset{\mathrm{i}}{\rightarrowtail} & (\mathbf{Set}/\!/\mathcal{V})\text{-}\mathbf{Cat} = \mathbf{Cat}/\!/\mathcal{V} & & (\mathcal{V}\text{-}\mathbf{Met})\text{-}\mathbf{Cat} \rightarrowtail \mathcal{V}\text{-}\mathbf{Metag}
\end{array}$$

A $\mathcal{V}$-metagory $\mathbb{X}$ is a graph with distinguished loops $1_x \in \mathbb{X}(x,x)$ that comes with functions

$$\delta_{x,y,z} : \mathbb{X}(x,y) \times \mathbb{X}(y,z) \times \mathbb{X}(x,z) \longrightarrow \mathcal{V}$$

which assign to every triangle




an "area"-value in $\mathcal{V}$, satisfying so-called tetrahedral inequalities which mimic lax identity and associativity laws. Morphisms are $\mathcal{V}$-contractive morphisms of graphs.

# Completing the picture: $\mathcal{V}$-metrically approximate categories

$$\begin{array}{ccccccc}
 & & (\mathbf{Grp}/\!/\mathcal{V})\text{-}\mathbf{Cat} & \xleftrightarrow{\;\cong\;} & (\mathcal{V}\text{-}\mathbf{MetGrp})\text{-}\mathbf{Cat} & & \\
 & & \downarrow & & \downarrow & & \\
\mathcal{V}\text{-}\mathbf{Met} \rightarrowtail \mathcal{V}\text{-}\mathbf{Cat} & \xrightarrow{\;\mathrm{i}\;} & (\mathbf{Set}/\!/\mathcal{V})\text{-}\mathbf{Cat} = \mathbf{Cat}/\!/\mathcal{V} & & (\mathcal{V}\text{-}\mathbf{Met})\text{-}\mathbf{Cat} & \rightarrowtail & \mathcal{V}\text{-}\mathbf{Metag}
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$$\begin{array}{ccc}
 & y & \\
{\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\
x & \xrightarrow{\quad a \quad} & z
\end{array}$$

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# Two expected facts and a surprising theorem for $\mathcal{V}$-metagories

- Every $\mathcal{V}$-**Met**-enriched category becomes $\mathcal{V}$-metagory via

$$\delta(f, g, a) = d(g \cdot f, a)$$

giving a full (reflective) embedding ($\mathcal{V}$-**Met**)-**Cat** $\longrightarrow$ $\mathcal{V}$-**Metag.**

- Every $\mathcal{V}$-metagory becomes a $\mathcal{V}$-**Met**-enriched graph via

$$d(f, f') = \delta(f, 1_y, f') = \delta(1_x, f, f')$$

giving a forgetful functor $\mathcal{V}$-**Metag** $\longrightarrow$ ($\mathcal{V}$-**Met**)-**Gph.**

- THEOREM (W.T., J. Wang) $\mathcal{V}$-**Metag** is symmetric monoidal-closed.

Moreover: $\mathcal{V}$-**Metag** is enriched in $\mathcal{V}$-**Metag**; hence, one has a composition (!) law

$$[\mathbb{X}, \mathbb{Y}] \otimes [\mathbb{Y}, \mathbb{Z}] \longrightarrow [\mathbb{X}, \mathbb{Z}].$$

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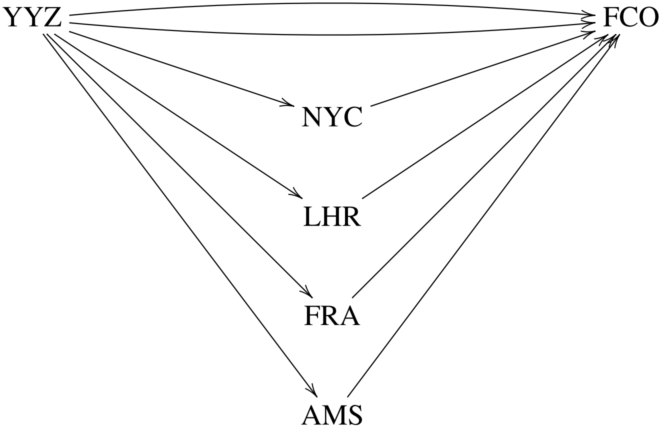
# You need your personalized metagory when searching for flights!




... and many more!

Happy $75 \longrightarrow 100$, *etc*, Jirka!