Mac Lane and Factorization

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Apr 9, 93

Dear Walter

Just saw your fact rep w/ Korostenski
JPAA 85 (1993) 57

What do you mean!

Factorization systems in Isbell 1957

They went in Mac Lane

Duality for Groups BULL AMS
1950

Look it up!!!

Saunders

- Saunders Mac Lane
- Duality for groups
- Bulletin for the American Mathematical Society **56** (1950) 485-516

- Saunders Mac Lane
- Groups, categories and duality
- Bulletin of the National Academy of Sciences USA **34** (1948) 263-267)

tion, we axiomatize the terms "injection homomorphism of a subgroup into a larger group" and "projection homomorphism of a group onto a quotient group." We can then define homomorphisms onto and isomorphisms into as "supermaps" and "submaps," respectively.

DEFINITION. A *bicategory*[6] $\mathcal{C}$ is a category with two given subclasses of mappings, the classes of "injections" (κ) and "projections" (π) subject to the axioms BC-0 to BC-6 below.[7]

BC-0. A mapping equal to an injection (projection) is itself an injection (projection).

BC-1. Every identity of $\mathcal{C}$ is both an injection and a projection.

BC-2. If the product of two injections (projections) is defined, it is an injection (projection).

BC-3. (Canonical decomposition). Every mapping α of the bicategory can be represented uniquely as a product $\alpha = \kappa\theta\pi$, in which κ is an injection, θ an equivalence, and π a projection.

Any mapping of the form $\lambda = \kappa\theta$ (that is, any mapping with π equal to an identity in the canonical decomposition) is called a *submap*; any mapping of the form $\rho = \theta\pi$ is called a *supermap*.

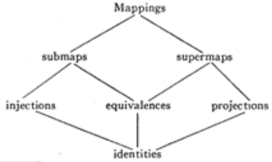
BC-4. If the product of two submaps (supermaps) is defined, it is a submap (supermap).

Any product $\kappa_1\pi_1 \cdots \kappa_n\pi_n$ of injections κ_i and projections π_i is called an *idemmap*.

BC-5. If two idemmaps have the same range and the same domain, they are equal.

BC-6. For each object A, the class of all injections with range A is a set, and the class of all projections with domain A is a set.

The inclusion relations between the various classes of mappings can be represented by the following Hasse diagram.



[6] The term "bicategory" was suggested by Professor Grace Rose.

[7] In the preliminary announcement [16], axiom BC-6 did not appear, and axiom BC-5 was present only in weaker form.

jections, projections, identities, and their products. When so formulated, it has a definite dual, but note that there may be several such formulations which lead to essentially different duals. For example, "Q is a quotient group of G" (that is, there is a projection with domain G and range Q) is equivalent to "Q is a conormal quotient group of G." The duals—"M is a subgroup of G" and "M is a normal subgroup of G"—are not equivalent.

11. **Partial order in a bicategory.** The axioms (especially axiom BC-5) suffice to introduce a relation of partial order (under "inclusion") in the objects of a bicategory. We define a mapping β to be *left cancellable* in a category if $\beta\alpha_1=\beta\alpha_2$ always implies $\alpha_1=\alpha_2$, and *left invertible* if β has a left inverse γ, with $\gamma\beta=I_{D(\beta)}$. One may readily prove, in succession, the following results.

LEMMA 11.1. *Two injections κ_1 and κ_2 such that $\kappa_1\kappa_2$ is an identity are themselves identities.*

LEMMA 11.2. *Every right factor of a submapping is a submapping.*

LEMMA 11.3. *If $\alpha\beta$ is an identity, α is a supermap and β a submap.*

LEMMA 11.4. *Every left invertible mapping is a submap, and every submap is left cancellable.*

THEOREM 11.5. *The class of objects in a bicategory is partially ordered by either of the relations*

(11.1) *$S\subset B$ if and only if there is an injection κ: $S\to B$;*

(11.1′) *$Q\leqq A$ if and only if there is a projection π: $A\to Q$.*

If $S\subset B$, we call S a *subobject* of B, while if $Q\leqq A$, Q is a *quotient-object* of A, the terms corresponding to those in group theory. By axiom BC-5 the mappings κ and π which appear in the dual definitions (11.1) and (11.1′) are unique; it is more suggestive to denote them as

(11.2) $$\kappa=[B\supset S]: S\to B; \qquad \pi=[Q\leqq A]: A\to Q.$$

Thus $[B\supset S]$ is a mapping, defined precisely when $S\subset B$ and is then an injection; every injection has this form. The notation is so chosen that

(11.3) $$[B\supset S][S\supset T]=[B\supset T], \quad [R\leqq Q][Q\leqq A]=[R\leqq A],$$

by BC-5, whenever the terms on the left are defined.

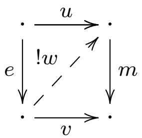
In examining prospective examples of bicategories, it is easier to formulate the axioms directly in terms of these constructions on the objects.

A brief history of factorization systems

- Mac Lane 1948/1950
- Isbell 1957/1964
- Quillen 1967
- Kennison 1968
- Kelly 1969
- Ringel 1970/1971
- Freyd-Kelly 1972
- Pumplün 1972

(Orthogonal) factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$

$e \perp m$



$$\text{(FS*1\&2)} \quad \mathcal{E} = {}^{\perp}\mathcal{M}, \mathcal{M} = \mathcal{E}^{\perp}$$
$$\text{(FS*3)} \quad \mathcal{C} = \mathcal{M} \cdot \mathcal{E}$$

$$\text{(FS*1)} \quad \mathrm{Iso} \cdot \mathcal{E} \subseteq \mathcal{E}, \mathcal{M} \cdot \mathrm{Iso} \subseteq \mathcal{M}$$
$$\text{(FS*2)} \quad \mathcal{E} \perp \mathcal{M}$$
$$\text{(FS*3)} \quad \mathcal{C} = \mathcal{M} \cdot \mathcal{E}$$

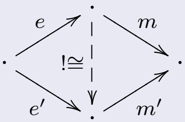
Alternative characterization

(FS1) Iso $\subseteq \mathcal{E} \cap \mathcal{M}$

(FS2) $\mathcal{E} \cdot \mathcal{E} \subseteq \mathcal{E}, \mathcal{M} \cdot \mathcal{M} \subseteq \mathcal{M}$

(FS3) $\mathcal{C} = \mathcal{M} \cdot \mathcal{E}$

(FS3!)



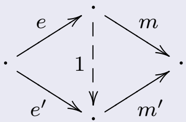
Strict factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ in $\mathcal{C}$ (M. Grandis)

(SFS1) $\mathrm{Id} \subseteq \mathcal{E}_0 \cap \mathcal{M}_0$

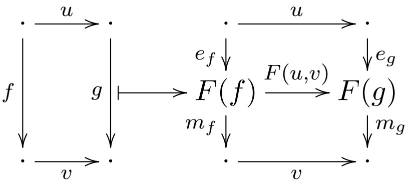
(SFS2) $\mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0$

(SFS3) $\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{E}_0$

(SFS3!)



“Higher” Justification:



- $F : \mathcal{C}^2 \to \mathcal{C} \iff$ Eilenberg-Moore structure w.r.t. $\Box^2$
- fs $\iff$ normal pseudo-algebras (Coppey, Korostenski-Tholen)
- sfs $\iff$ strict algebras (Rosebrugh-Wood)

Free structure on $\mathcal{C}^2$

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \quad = \quad \begin{array}{ccccc} \cdot & \xrightarrow{1} & \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[v]{} & \cdot & \xrightarrow[1]{} & \cdot \end{array}$$

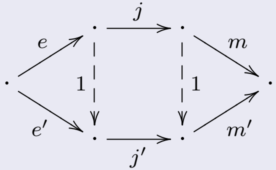
Mac Lane again:

(BC1) $\mathrm{Id} \subseteq \mathcal{E}_0 \cap \mathcal{M}_0$

(BC2) $\mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0$

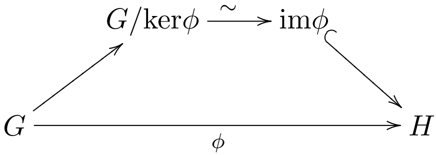
(BC3) $\mathcal{C} = \mathcal{M}_0 \cdot \mathrm{Iso} \cdot \mathcal{E}_0$

(BC3!)



(BC4) $\mathcal{E}_0 \cdot \mathrm{Iso} \subseteq \mathrm{Iso} \cdot \mathcal{E}_0, \mathrm{Iso} \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathrm{Iso}$

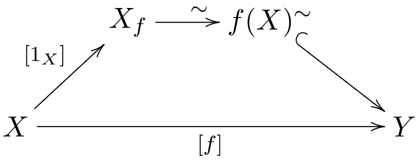
(BC5) $\left|\overline{\mathcal{M}_0 \cdot \mathcal{E}_0} \cap \mathcal{C}(A, B)\right| \leq 1$



- epimorphisms from G $\iff$ congruences on G

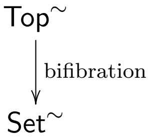
Set$^\sim$

objects:	sets X with equivalence relation $\sim_X$
morphisms:	$[f] : X \to Y$
	$x \sim_X x' \implies f(x) \sim_Y f(x')$
	$f \sim g \iff \forall x \in X : f(x) \sim_Y g(x)$
closure:	$Z \subseteq X, Z^\sim = \{x \in X \mid \exists z \in Z : x \sim_X z\}$
compare:	Freyd completion!

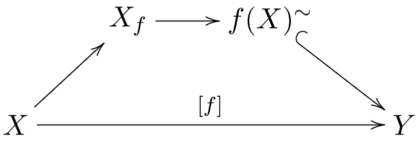


$$
\begin{aligned}
x \sim_f x' \quad &\iff \quad f(x) \sim_Y f(x') \\
\mathcal{E}_0 \quad &= \quad \{[1_X] : X \to X' \mid \ \sim_X \subseteq \sim_{X'}\} \\
\mathcal{M}_0 \quad &= \quad \{[Z \hookrightarrow Y] \mid Z^{\sim} = Z\} \\
[f] \text{ mono} \quad &\iff \quad \sim_X = \sim_f \\
[f] \text{ epi} \quad &\iff \quad f(X)^{\sim} = Y \\
\text{Epi} \cap \text{Mono} = \text{Iso} \quad &\iff \quad AC \\
&\iff \quad \text{Epi} = \text{SplitEpi}
\end{aligned}
$$

- $\mathsf{Grp}^{\sim} = \mathsf{Grp}(\mathsf{Set}^{\sim})$
- groups with a congruence relation
- homomorphisms "up to congruence"
- $\mathsf{Grp}^{\sim} \to \mathsf{Set}^{\sim}$ reflects isos



$U \subseteq X \text{open} \implies U = U^{\sim}$

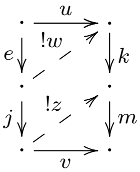


Mac Lane: $U \subseteq X_f$ open $\iff \exists V \subseteq Y \text{open} : U = f^{-1}(V)$

Better: $U \subseteq X_f$ open $\iff \exists V = V^{\sim} \subseteq Y : U = f^{-1}(V)$ open

Double factorization system $(\mathcal{E}_0, \mathcal{J}, \mathcal{M}_0)$ in $\mathcal{C}$

$(e, j) \perp (k, m)$



(DFS*1)	$\mathrm{Iso} \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathrm{Iso} \cdot \mathcal{J} \cdot \mathrm{Iso} \subseteq \mathcal{J}, \mathcal{M}_0 \cdot \mathrm{Iso} \subseteq \mathcal{M}_0$
(DFS*2)	$(\mathcal{E}_0, \mathcal{J}) \perp (\mathcal{J}, \mathcal{M}_0)$
(DFS*3)	$\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{J} \cdot \mathcal{E}_0$

$$(\mathcal{E}, \mathcal{M}) \text{ fs} \iff (\mathcal{E}, \mathrm{Iso}, \mathcal{M}) \text{ dfs}$$

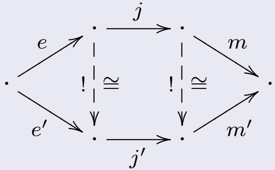
Alternative characterization

(DFS1) $\mathrm{Iso} \subseteq \mathcal{E}_0 \cap \mathcal{J} \cap \mathcal{M}_0$

(DFS2) $\mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathcal{J} \cdot \mathcal{J} \subseteq \mathcal{J}, \mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0$

(DFS3) $\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{J} \cdot \mathcal{E}_0$

(DFS3!)



(DFS4) $\mathcal{J} \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathcal{J}, \mathcal{E}_0 \cdot \mathcal{J} \subseteq \mathcal{J} \cdot \mathcal{E}_0$

$$(\mathcal{E}_0, \mathcal{J}, \mathcal{M}_0) \text{ dfs} \iff (\mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{J}), (\mathcal{J} \cdot \mathcal{E}_0, \mathcal{M}_0) \text{ fs}$$
$$\mathcal{J} = \mathcal{J} \cdot \mathcal{E}_0 \cap \mathcal{M}_0 \cdot \mathcal{J}$$

Free structure on $\mathcal{C}^3$:

$$
\begin{array}{ccc}
\cdot & \xrightarrow{u} & \cdot \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_1} \\
\cdot & \xrightarrow{v} & \cdot \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_2} \\
\cdot & \xrightarrow[w]{} & \cdot
\end{array}
\quad = \quad
\begin{array}{ccccccc}
\cdot & \xrightarrow{1} & \cdot & \xrightarrow{1} & \cdot & \xrightarrow{u} & \cdot \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle v f_1}\downarrow & & \downarrow{\scriptstyle g_1} \\
\cdot & \xrightarrow{1} & \cdot & \xrightarrow{v} & \cdot & \xrightarrow{1} & \cdot \\
{\scriptstyle f_2}\downarrow & & {\scriptstyle w f_2}\downarrow & & {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle g_2} \\
\cdot & \xrightarrow[w]{} & \cdot & \xrightarrow[1]{} & \cdot & \xrightarrow[1]{} & \cdot
\end{array}
$$

$$
\begin{array}{rcl}
(\mathcal{E}_0, \mathcal{J}, \mathcal{M}_0) & \leftrightarrow & (\mathcal{E}, \mathcal{W}, \mathcal{M}) \\
\mathcal{E}_0 = \mathcal{E} \cap \mathcal{W} & & \mathcal{E} = \mathcal{J} \cdot \mathcal{E}_0 \\
\mathcal{J} = \mathcal{E} \cap \mathcal{M} & & \mathcal{W} = \mathcal{M}_0 \cdot \mathcal{E}_0 \\
\mathcal{M}_0 = \mathcal{M} \cap \mathcal{W} & & \mathcal{M} = \mathcal{M}_0 \cdot \mathcal{J}_0
\end{array}
$$

- $\mathcal{W}$ is closed under retracts in $\mathcal{C}^3$.
- When does $\mathcal{W}$ have the 2-out-of-3 property?

Double factorization systems $(\mathcal{E}_0, \mathcal{J}, \mathcal{M}_0)$:

$(\mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{J}), (\mathcal{J} \cdot \mathcal{E}_0, \mathcal{M}_0)$ fs,
$\mathcal{E}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathcal{E}_0$,
$ej \in \mathcal{E}_0, e \in \mathcal{E}_0, j \in \mathcal{J} \implies j$ iso,
$jm \in \mathcal{M}_0, m \in \mathcal{M}_0, j \in \mathcal{J} \implies j$ iso.

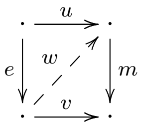
"Quillen factorization systems" $(\mathcal{E}, \mathcal{W}, \mathcal{M})$:

$(\mathcal{E} \cap \mathcal{W}, \mathcal{M}), (\mathcal{E}, \mathcal{M} \cap \mathcal{W})$ fs,
$\mathcal{W}$ has 2-out-of-3 property.

(Pultr-Tholen 2002)

Weak factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$

$e \square m$



$$
\begin{array}{rl}
\text{(WFS*1\&2)} & \mathcal{E} = {}^{\square}\mathcal{M}, \mathcal{M} = \mathcal{E}^{\square} \\
\text{(WFS*3)} & \mathcal{C} = \mathcal{M} \cdot \mathcal{E}
\end{array}
$$

$$
\begin{array}{rl}
\text{(WFS*1a)} & gf \in \mathcal{E}, g \text{ split mono} \implies f \in \mathcal{E} \\
\text{(WFS*1b)} & gf \in \mathcal{M}, f \text{ split epi} \implies g \in \mathcal{M} \\
\text{(WFS*2)} & \mathcal{E} \square \mathcal{M} \\
\text{(WFS*3)} & \mathcal{C} = \mathcal{M} \cdot \mathcal{E}
\end{array}
$$

(Mono,Epi) in Set

- (Mono,Mono$^{\square}$) wfs in $\mathcal{C}$ with binary products and enough injectives
- ($\coprod$, SplitEpi) wfs in every lextensive category $\mathcal{C}$

fs $\implies$ wfs

$\mathcal{E}^{\Box}$: closed under composition, direct products
stable under pullback, intersection

If $\mathcal{C}$ has kernelpairs, any of the following will make a wfs $(\mathcal{E}, \mathcal{M})$ an fs:

- $\mathcal{M}$ closed under any type of limit
- $gf \in \mathcal{M}, g \in \mathcal{M} \implies f \in \mathcal{M}$
- $gf = 1, g \in \mathcal{M} \implies f \in \mathcal{M}$

Cassidy-Hébert-Kelly (1985), Ringel (1970)

$\mathcal{C}$ finitely well-complete

- reflective subcategories of $\mathcal{C}$ (full, replete)
- factorization systems $(\mathcal{E}, \mathcal{M})$ with $gf \in \mathcal{E}, g \in \mathcal{E} \implies f \in \mathcal{E}$

$$(\mathcal{E}, \mathcal{M}) \mapsto \mathcal{F}(\mathcal{M}) = \{B \in \mathcal{C} \mid (B \to 1) \in \mathcal{M}\}$$

$\mathcal{F}$ reflective in finitely complete $\mathcal{C}$ with reflection $\rho : 1 \to R$

$$(\mathcal{E}, \mathcal{M}) = \big(R^{-1}(\mathrm{Iso}), \mathrm{Cart}(R, \rho)\big) \text{ fs} \iff \begin{array}{c} \forall f : A \to B \ : \\ \big(A \xrightarrow{(\rho_A, f)} RA \times_{RB} B\big) \in \mathcal{E} \end{array}$$

$$\Uparrow$$

$$\mathcal{E} \text{ stable under pb along } \mathcal{M} \iff \mathcal{F} = \mathcal{F}(\mathcal{M}) \text{ semilocalization}$$

$$\Uparrow$$

$$\mathcal{E} \text{ stable under pullback} \iff \mathcal{F} = \mathcal{F}(\mathcal{M}) \text{ localization}$$

$\mathcal{C}$ with 0

$$(\mathcal{E},\mathcal{M}) \text{ torsion theory} \iff (\mathcal{E},\mathcal{M}) \text{ fs, } \mathcal{E},\mathcal{M} \text{ have 2-out-of-3 property}$$

$$\begin{aligned} \mathcal{F}(\mathcal{M}) &= \{B \mid (B \to 0) \in \mathcal{M}\} \\ \mathcal{T}(\mathcal{E}) &= \{A \mid (0 \to A) \in \mathcal{E}\} \end{aligned}$$

$\mathcal{C}$ with kernels and cokernels

$$
\begin{array}{ccccc}
SKC \cong SC & \xrightarrow{1} & SC & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \sigma_{KC} \cong \alpha_C} & & \Big\downarrow{\scriptstyle \sigma_C} & & \Big\downarrow \\
KC & \xrightarrow{\kappa_C} & C & \xrightarrow{\pi_C} & QC \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_C} & & \Big\downarrow{\scriptstyle \beta_C \cong \rho_{QC}} \\
0 & \longrightarrow & RC & \xrightarrow[1]{} & RC \cong RQC
\end{array}
$$

$$
\begin{aligned}
C \in \mathcal{F}(\mathcal{M}) &\iff SC = 0 \iff KC = 0 \\
C \in \mathcal{T}(\mathcal{E}) &\iff RC = 0 \iff QC = 0
\end{aligned}
$$

$$\alpha_C \text{ iso} \iff \beta_C \text{ iso} \iff \pi_C \kappa_C = 0$$
$$(\mathcal{E}, \mathcal{M}) \text{ simple} \implies (\mathcal{E}, \mathcal{M}) \text{ normal}$$

$\mathcal{C}$ homological, $\mathcal{C}^{\mathrm{op}}$ homological:
normal torsion theories $(\mathcal{E}, \mathcal{M}) \iff$ standard torsion theories $(\mathcal{T}, \mathcal{F})$

$$0 \to T \to C \to F \to 0$$
$$\mathcal{C}(\mathcal{T}, \mathcal{F}) = 0$$

Example

$\mathcal{C}$: abelian groups with $(4x = 0 \implies 2x = 0)$
$\mathcal{F}$: abelian groups with $2x = 0$

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow \\
\mathbb{Z} \cong 2\mathbb{Z} & \overset{\kappa}{\hookrightarrow} & \mathbb{Z} & \xrightarrow{\pi = 1} & \mathbb{Z} \\
\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2
\end{array}
$$