A Personal Glance at George's Category Theory

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1974	Diploma	Tbilisi State University
1978	Ph.D.	Tbilisi State University
1992	D.Sc.	St.-Petersburg State University

Georgian Academy of Sciences (since 1975)

McGill, York, Milan, Chicago, Bielefeld, Sydney

Hungarian Academy of Sciences, Trieste, Genova, Wales (at Bangor)

Tours, Louvain-la-Neuve, Littoral (at Calais), Coimbra

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University of Cape Town (since 2004)

Major areas of work

Categorical Galois Theory

Descent Theory

Categories for Algebra

Categories for Topology

Categorical Galois Theory

- Galois Theory in categories with inclusions (Proc. Junior Sci. 1974)
- The fundamental theorem of Galois Theory (USSR Sbornik 1989)
- Pure Galois Theory in Categories (J. Algebra 1990)

- Galois Theories (Cambridge 2001, with F. Borceux)

- Categorical Galois Theory: Revision and some recent developments (Potsdam 2001)
- Descent and Galois Theory (Haute Bodeux 2007)

$A \xrightarrow{\alpha} B$ surjective

$(A, \alpha) \in (\mathbf{Grp} \downarrow B)$ *central extension*

$\iff \ker \alpha \subseteq \text{centre}\,(A)$

(A, α) *trivial* central extension

$\iff (A, \alpha) \cong (K \times B, K \times B \to B)$ with K Abelian

Central extensions – Categorically

$(A, \alpha) \in (\mathbf{Grp} \downarrow B)$ central extension

$\iff$ $\exists\, p : E \to B$ surjective such that

$p^*(A, \alpha)$ trivial:

$$\begin{array}{ccc} E \times_B A & \longrightarrow & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

$\iff$: (A, α) split over (E, p)

Separable extensions – Classically

$A \overset{\alpha}{\leftarrow} B$ in **CR**, B field

Example

$$f \in B[x],\ \deg f \geqslant 1,\ B_f = B[x]/(f) \leftarrow B$$

Facts

$$f = g \cdot h,\ (g,h) = 1 \Longrightarrow B_f \cong B_g \times B_h$$

$$B_{(x-b)^n} \cong B_{x^n}$$

$$f \text{ separable } \iff f = a \cdot \prod_{i=1}^{n} (x - b_i),\ b_i \neq b_j \text{ for } i \neq j$$

$$\iff B_f \cong B \times \ldots \times B$$

$$\iff B_f \text{ is a } \textit{trivial } B\text{-algebra}$$

Separable extensions – Classically (continued)

If $f \in B[x]$ does not split: $\exists\, E \supseteq B$ such that $f \in E[x]$ splits,

$$E_f \cong E \otimes_B B_f$$

f separable $\iff$ $E \otimes_B B_f$ trivial E-algebra

$$\begin{tikzcd}
E \otimes_B B_f & B_f \arrow[l] \\
E \arrow[u, "\text{trivial}"] & B \arrow[l, hook'] \arrow[u]
\end{tikzcd}$$

A separable B-algebra

$\iff$ $\dim_B A < \infty$, $\forall a \in A$: a separable over B

$\iff$ $\exists$ field extension $E \hookleftarrow B$: $E \otimes_B A$ trivial E-algebra

$\iff$: A is split over B

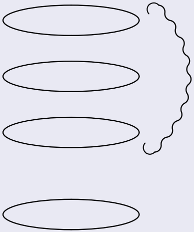
$$\begin{tikzcd} E \otimes_B A & A \arrow[l] \\ E \arrow[u] & B \arrow[l] \arrow[u, "\alpha"'] \end{tikzcd}$$

Covering spaces – Classically

$A \xrightarrow{\alpha} B$ local homeomorphism $\iff$ (A, α) étale space over B

Very trivial example

$A_i \subseteq A$ open, $A_i \xrightarrow{\cong} B$



$$\begin{array}{c} A = \bigcup\limits_{i \in I} A_i \quad \text{(disjoint)} \\ \Big\downarrow \alpha \\ B \end{array}$$

Trivial example

$$B = \bigcup_{\lambda \in \Lambda} B_\lambda \text{ (disjoint)}$$

$B_\lambda \subseteq B$ open, $\alpha^{-1}(B_\lambda) \to B_\lambda$ very trivial

(A, α) covering space over B

$\iff$ $\forall\, b \in B\ \exists$ open $V \ni b$ in B: $\alpha^{-1}(V) \to V$ very trivial

$\iff$ $\exists\, E \xrightarrow{p} B$ surjective, étale:

$p^{\star}(A, \alpha)$ trivial

$$\begin{tikzcd} E \times_B A \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$\iff$: (A, α) split over (E, p)

$$\mathbf{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbf{X}, \quad \mathbf{C} \text{ with pullbacks, } B \in \mathbf{C}$$

$$(\mathbf{C} \downarrow B) \underset{H^B}{\overset{I^B}{\rightleftarrows}} (\mathbf{X} \downarrow IB)$$

$$(A, \alpha) \longmapsto (IA, I\alpha)$$

$$\begin{tikzcd} B \times_{HIB} HX \arrow[r] \arrow[d, "\pi_1"'] & HX \arrow[d, "H\varphi"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$

$$(B \times_{HIB} HX, \pi_1) \longleftarrow\!\shortmid (X, \varphi)$$

Split objects

$$(A,\alpha) \text{ \textit{trivial} } :\Longleftrightarrow \quad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\alpha"'] & HIA \arrow[d, "HI\alpha"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd} \quad \text{pullback}$$

(A,α) *split over* (E,p) $:\Longleftrightarrow$ $p^*(A,\alpha)$ trivial

Example 1

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{AbGrp} \arrow[l, shift left] \end{tikzcd}$$

α, p surjective, E free

Split objects, continued

Example 2

$$(\mathbf{CR}^{op} \downarrow k)_{\text{fin}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{FinSet}$$

$$A \longmapsto \{\text{minimal non-zero idempotents}\}$$

$$\underbrace{k \times \ldots \times k}_{X \text{ times}} \longleftarrow\!\!\!| \; X$$

$E \xleftarrow{p} B$ fields

Split objects, continued

Example 3

$$\begin{tikzcd} \mathbf{LCTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, shift left] \end{tikzcd}$$

$$B \longmapsto \pi_0 B$$

(discrete) $X \longleftarrow\!\!\shortmid X$

$p : E \to B$ surjective, étale

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, shift left, "H"] \end{tikzcd} \qquad \mathcal{F} \subseteq \operatorname{mor}\mathbf{C},\ \Phi \subseteq \mathbf{X}\text{: “\textit{fibrations}”}$$

Hypothesis

- pullbacks of fibrations exist and are fibrations
- isomorphisms are fibrations, closed under composition
- I and H preserve fibrations
- ("*Admissibility*") $\phi : X \to IB$ fibration $\Rightarrow$ $(I(B \times_{HIB} HX) \to IHX \to X)$ isomorphism

Theorem

$p^* : \mathcal{F}(B) \to \mathcal{F}(E)$ monadic $\Rightarrow \mathrm{Spl}(E, p) \simeq \mathbf{X}^{\mathrm{Gal}(E,p)} \bigcap \Phi$

$$\begin{tikzcd}
\mathrm{Spl}(E,p) \arrow[r] \arrow[d, hook] & \mathrm{TrivCov}(E) \arrow[d] \arrow[r, phantom, "\simeq"] & \Phi(IE) \arrow[d, "H^E"] \\
\Phi(B) \arrow[r] \arrow[ur, phantom, "\text{pullback}"] & \Phi(E) \arrow[r, equal] \arrow[ur, phantom, "\text{(admissible)}"] & \Phi(E)
\end{tikzcd}$$

$$\mathrm{Gal}(E,p) = I(\mathrm{Eq}(p)) = (\, I(E \times_B E \times_B E) \rightrightarrows\!\!\rightarrow I(E \times_B E) \underset{Ic}{\overset{Id}{\rightrightarrows}} I(E))$$

$$\mathbf{X}^{\mathrm{Gal}(E,p)} \ni (A_0, \pi, \xi) \qquad
\begin{tikzcd}
I(E \times_B E) \times_{(Id,\pi)} A_0 \arrow[r, "\xi"] \arrow[d] & A_0 \arrow[d, "\pi"] \\
I(E \times_B E) \arrow[r, "Ic"'] & IE
\end{tikzcd}$$

First proof generalizing Magid's Theorem: 1984. In full generality: 1991

Descent Theory

$p : E \to B$ *effective* (for) *descent*

$\iff$ $p_! \dashv p^* : \mathcal{F}(B) \to \mathcal{F}(E)$ monadic

$\iff$ rebuild $\mathcal{F}(B)$ from $\mathcal{F}(E)$ as

$\{(C, \gamma; \xi) : \ \xi : E \times_B C \to C, \ 2 \text{ equations}\}$

$$\begin{tikzcd}
E \times_B C \arrow[r] \arrow[d] & C \arrow[d, "\gamma"] \\
E \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

Equivalent presentation of ξ:

$$\begin{tikzcd}
 & & C \arrow[dr, "p\cdot\gamma"] & \\
E \times_B C \arrow[r, "\bar{\xi}"] \arrow[urr, bend left, "\xi"] \arrow[drr, bend right, "\gamma\cdot\pi_2"'] & E \times_B C \arrow[ur, "\pi_2"] \arrow[dr, "\pi_1"] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}$$

Descent Theory (continued)

$$\mathcal{F}(B) = (\mathbf{Top} \downarrow B)$$

$(x, y) \in E \times_B E$

$$\begin{tikzcd}
\gamma^{-1}x \arrow[r, "\xi_{x,y}"] \arrow[d, "j_{y,x}"'] & \gamma^{-1}y \arrow[d, "j_{x,y}"] & c \arrow[d, mapsto] \\
E \times_B C \arrow[r, "\bar{\xi}"] & E \times_B C & (x, c)
\end{tikzcd}$$

$\xi_{x,x} = \mathrm{id}$, $\xi_{x,z} = \xi_{y,z} \cdot \xi_{x,y}$ $(p(x) = p(y) = p(z))$, *Glueing Condition*

Example

$$E = \sum_{i \in I} U_i \xrightarrow{\ p\ } B = \bigcup_{i \in I} U_i \quad (U_i \subseteq B \text{ open})$$

$\xi_{i,j} : \gamma_i^{-1}(U_i \bigcap U_j) \longrightarrow \gamma_j^{-1}(U_i \bigcap U_j)$ satisfying the *Cocycle Condition*

$$
\begin{tikzcd}
\begin{array}{c}\text{Categorical Algebra}\\ \text{(commutators)}\end{array} \arrow[r, leftrightarrow] & \begin{array}{c}\text{Internal Category}\\ \text{Theory}\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Topology}\\ \text{(crossed modules)}\end{array} \\
\text{Galois Theory} \arrow[r, leftrightarrow] & \textit{Descent Theory} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Sheaf \&}\\ \text{Cohomology Theory}\end{array} \\
 & \text{Monad Theory} &
\end{tikzcd}
$$

Descent Theory (continued)

Two of George's "simple" observations:

- descent $\neq$ effective descent, even in algebra:

$$\{A \in \mathbf{AbGrp} \mid n^2x = 0 \Rightarrow nx = 0\}, \quad p : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

- $\mathbf{C} \hookrightarrow \mathbf{D}$ closed under pullbacks,

  $p : E \to B$ in $\mathbf{C}$, effective descent in $\mathbf{D}$. Then:

  p effective descent in $\mathbf{C} \Longleftrightarrow \forall\,(A, \alpha) \in (\mathbf{D} \downarrow B)$:

$$p^*(A, \alpha) \in (\mathbf{C} \downarrow E) \Rightarrow (A, \alpha) \in (\mathbf{C} \downarrow B)$$

  $\rightsquigarrow$ Reiterman-T characterization of effective descent morphisms in **Top**

  $\rightsquigarrow$ Clementino-Hofmann characterization of effective descent morphisms in **Top**

Descent Theory (continued)

PreOrd $\cong$ **Alexandroff**, **FinPreOrd** $\cong$ **FinTop**

universal quotient:
(=descent)

$$\begin{tikzcd} x \arrow[r, dotted] \arrow[d, dash] & y \arrow[d, dash] & E \arrow[d, "p"] \\ u \arrow[r] & v & B \end{tikzcd}$$

$\uparrow\ \neq$

effective descent:

$$\begin{tikzcd} x \arrow[r, dotted] \arrow[d, dash] & y \arrow[r, dotted] \arrow[d, dash] & z \arrow[d, dash] \\ u \arrow[r] & v \arrow[r] & w \end{tikzcd}$$

$\uparrow\ \neq$

triquotient:

$$\begin{tikzcd} x_n \arrow[r, dotted] \arrow[d, dash] & x_{n-1} \arrow[r, dotted] \arrow[d, dash] & \dots \arrow[r, dotted] & x_1 \arrow[r, dotted] \arrow[d, dash] & x_0 \arrow[d, dash] \\ u_n \arrow[r] & u_{n-1} \arrow[r] & \dots \arrow[r] & u_1 \arrow[r] & u_0 \end{tikzcd}$$

Semi-Abelian Categories

Mac Lane, Duality for groups, Bull. AMS 1950

"Abelian bicategory" $\rightsquigarrow$ "exact category" (Buchsbaum 1955)
$=$ abelian category

$$\begin{tikzcd}
\mathbf{AbGrp} \arrow[r, leftrightarrow] \arrow[d, hook] & \text{abelian category} \arrow[d, hook] \\
\mathbf{Grp} \arrow[r, leftrightarrow] & ?
\end{tikzcd}$$

Old-style generalizations in the realm of pointed/additive categories

Semi-Abelian Categories (continued)

New style approaches 1970 - 1998

- "Barr-exact": finite limits
  pullback stable regular epi-mono factorizations
  equivalence relations are effective

  Tierney's "equation": Barr-exact $+$ additive $=$ Abelian

- "Malcev": from varieties to categories
  (Carboni, Kelly, Lambek, Pedicchio, ...)

Barr-exact $+$ Malcev $\rightsquigarrow$ Commutator theory (Janelidze, Pedicchio ...)

- "Bourn-protomodular" (Como 1990)

$$\mathrm{Pt}\mathbf{C} = (1 \downarrow \mathbf{C}), \quad \mathrm{Pt}(B) = \mathrm{Pt}(\mathbf{C} \downarrow B) \qquad \begin{tikzcd} E \arrow[d, shift right, "p"'] \\ B \arrow[u, shift right, "s"'] \end{tikzcd} \qquad p \cdot s = 1$$

$h : C \to B$: $\quad h^* : \mathrm{Pt}(B) \to \mathrm{Pt}(C)$ reflects isomorphisms

If $C = 0$: $\quad \ker_B : \mathrm{Pt}(B) \to \mathbf{C}$ reflects isomorphisms

$\iff$ Split Short-Five lemma:

$$\begin{tikzcd} \bullet \arrow[r, tail] \arrow[d, "u"'] & \bullet \arrow[r, two heads] \arrow[d, "w"'] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, tail] & \bullet \arrow[r, two heads] & \bullet \end{tikzcd}$$

u, v iso $\Rightarrow$ w iso

- Mac Lane (1950): "ABC extension equivalence theorem"

Semi-Abelian Categories (continued)

$$\begin{aligned}
\text{Semi-Abelian} \quad &= \quad \text{Barr-exact} \;+\; \text{Bourn-protomodular} \\
&\qquad + \;\text{finite coproducts} \;+\; 0 \cong 1 \\[1em]
&= \quad \text{Barr-exact} \;+\; \text{semi-additive} \\[1em]
\text{Semi-additive} \quad &= \quad \forall B: \; \ker_B : \mathrm{Pt}(B) \to \mathbf{C} \text{ monadic} \\
&\qquad + \;\text{finite coproducts} \;+\; 0 \cong 1 \\[1em]
\text{Abelian} \quad &= \quad \text{Semi-Abelian} \;+\; (\text{Semi-Abelian})^{\mathrm{op}}
\end{aligned}$$

Semi-Abelian Categories (continued)

Examples:

- varieties of Ω-groups, $\{$crossed modules$\}$
- $\mathcal{T}$-$\mathbf{Alg}(\mathbf{Set})$ semi-Abelian $\Rightarrow$ $\mathcal{T}$-$\mathbf{Alg}(\mathbf{C})$ Semi-Abelian
  (finite coproducts granted)
- $(\mathbf{Set}_*)^{\mathrm{op}}$

Pointed naturally-Malcev $\subsetneqq$ protomodular $\subsetneqq$ Malcev

Semi-Abelian Categories (continued)

"Old-style" axioms:

- F. Hofmann (1960):

$$\begin{tikzcd} F \arrow[r, two heads, "q", "\text{normal}"'] \arrow[d, tail, "w"'] & C \arrow[d, tail, "v"] \\ E \arrow[r, two heads, "p"', "\text{normal}"] & B \end{tikzcd} \quad v \text{ normal}, \qquad \ker p \leqslant w \Rightarrow w \text{ normal}$$

  $\rightsquigarrow$ protomodular

- p, q, w normal $\Rightarrow$ v normal

  $\rightsquigarrow$ equivalence relations are effective

Semi-Abelian Categories (continued)

Cleaned-up version of "new $=$ old" (Hartl, Loiseau - 2011)

$$\left.\begin{array}{l} \bullet\ \text{pointed, finitely complete, finitely cocomplete} \\[1em] \bullet\ p : E \to B \text{ split epi with sections } s : B \to E \\ \quad \Rightarrow \ker p + B \to E \text{ normal epi} \\[1em] \bullet\ \text{normal epis pullback stable} \end{array}\right\} \text{"homological"}$$

- image of normal mono by normal epi is normal