

From Metric Spaces to Quantale-Enriched Categories

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Topology, Algebra, and Categories in Logic

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Learning goals for this lecture series

- Embrace (enriched) category theory as a guide for analytic inquiry
- Appreciate the quantalic structure of the real half-line as the key for studying metrics
- Get familiar with other important quantales and study the categories enriched in them
- Study the core of the theory: cocompleteness vs injectivity vs pseudo-algebraicity,
- in particular: Cauchy vs Lawvere
- Feel prepared to study monad-quantale-enriched categories (*Monoidal Topology*),
- normed/weighted categories, metrically enriched categories, metagories, *etc.*

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- 1 Metrics: from Frechét via Hausdorff to Lawvere
- 2 Quantales and the (small) categories enriched in them
- 3 Distributors and the presheaf monad
- 4 Weighted colimits, tensors, conical infima
- 5 Pseudo-algebras of the presheaf monad, injectivity
- 6 Cauchy- and Lawvere-completeness
- 7 A glance at normed/weighted categories

1.1 Fréchet 1906

A Fréchet metric $d : X \times X \rightarrow \mathbb{R}$ on a set X satisfies:

0-Self $0 = d(x, x)$

Sep $d(x, y) = 0 = d(y, x) \implies x = y$

Sym $d(x, y) = d(y, x)$

∇ -Inq $d(x, y) + d(y, z) \geq d(x, z)$

Necessarily then:

Pos $d(x, y) \geq 0$

Possible strengthenings:

Bdd $1 \geq d(x, y)$ (*bounded metric*)

Ult $\max\{d(x, y), d(y, z)\} \geq d(x, z)$ (*ultrametric*)

Met_{Fréchet} : morph's $f : X \rightarrow Y$ satisfy $d_X(x, x') \geq d_Y(fx, fx')$; write $X(x, x') \geq Y(fx, fx')$.

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1.2 Some shortcomings of $\mathbf{Met}_{\text{Fréchet}}$, Hausdorff's 1914 observations

- Finitely complete, but countable products (even of 2-point spaces) may not exist.
- Any two non-empty spaces fail to admit a coproduct.
- Neither its cartesian structure nor its natural monoidal structure are closed.
- The (non-symmetrized) Hausdorff distance

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$$

for $A, B \subseteq X$ will (when it exists in $[0, \infty)$) generally satisfy *only* (0-Self) and (Δ -Inq) of the Fréchet axioms, ...

... but this remains true even when the given distance function on X satisfies just these two conditions! Likewise for bounded metrics, ultrametrics, *etc.*

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1.3 Regrouping Fréchet's axioms à la Lawvere 1973

A Fréchet metric $d : X \times X \rightarrow [0, \infty]$ on a set X satisfies

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Finiteness: $\infty > d(x, y)$

$\emptyset \neq X(x, y)$

A map $f : X \rightarrow Y$ of metric spaces is non-expansive / short / 1-Lipschitz if

Contraction: $d_X(x, x') \geq d_Y(fx, fx')$

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The category **Met** is complete and cocomplete and symmetric monoidal-closed. But:



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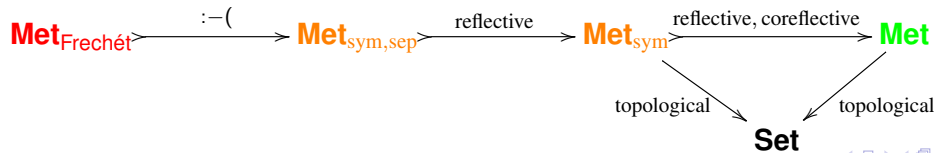
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$$X_{\text{rsym}}(x, x') = \inf_{x=x_0, \dots, x_n=x'} \sum_{j=1}^n \min\{X(x_{j-1}, x_j), X(x_j, x_{j-1})\}$$

- Separation: with $(x \simeq y : \iff X(x, y) = 0 = X(y, x))$, let

$$X / \simeq ([x], [y]) = X(x, y)$$

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2.1 Quantales

A (commutative) quantale $(\mathcal{V}, \leq, \otimes, k)$ is a commutative monoid in $(\mathbf{Sup}, \boxtimes, 2)$; that is:

- (\mathcal{V}, \leq) is a complete lattice;
- $(\mathcal{V}, \otimes, k)$ is a commutative monoid;
- $- \otimes v : \mathcal{V} \rightarrow \mathcal{V}$ preserves joins for all $v \in \mathcal{V}$.

Hence, as a monotone map, every $- \otimes v$ has a right adjoint; this means:

\mathcal{V} is a “thin” symmetric monoidal-closed category, with internal homs $[v, w]$ determined by

$$u \leq [v, w] \iff u \otimes v \leq w.$$

Some useful rules:

$$k \leq [u, u], [u, v] \otimes u \leq v, [k, v] = v, [u_1 \otimes u_2, v] = [u_1, [u_2, v]] = [u_2, [u_1, v]],$$

$$\left[\bigvee_{i \in I} u_i, v \right] = \bigwedge_{i \in I} [u_i, v], \quad \left[u, \bigwedge_{i \in I} v_i \right] = \bigwedge_{i \in I} [u, v_i].$$

2.1 Quantales

A (commutative) quantale $(\mathcal{V}, \leq, \otimes, \mathbf{k})$ is a commutative monoid in $(\mathbf{Sup}, \boxtimes, \mathbf{2})$; that is:

- (\mathcal{V}, \leq) is a complete lattice;
- $(\mathcal{V}, \otimes, \mathbf{k})$ is a commutative monoid;
- $- \otimes v : \mathcal{V} \rightarrow \mathcal{V}$ preserves joins for all $v \in \mathcal{V}$.

Hence, as a monotone map, every $- \otimes v$ has a right adjoint; this means:

\mathcal{V} is a “thin” symmetric monoidal-closed category, with internal homs $[v, w]$ determined by

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$\varphi : \mathcal{V} \rightarrow \mathcal{W}$ is a lax homomorphism if

$$\bigvee_{i \in I} \varphi u_i \leq \varphi(\bigvee_{i \in I} u_i), \quad \varphi u \otimes_{\mathcal{W}} \varphi v \leq \varphi(u \otimes_{\mathcal{V}} v), \quad k_{\mathcal{W}} \leq \varphi(k_{\mathcal{V}});$$

φ is a (strict) homomorphism if \leq may be replaced by $=$.

- 1 is the terminal quantale ($k = \perp$)
- $2 = (\{\perp, \top\}, \leq, \wedge, \top)$ is the initial quantale; more generally: $(\mathcal{P}S, \subseteq, \cap, S)$ (S any set)
- even more generally: any locale (frame) L is a “cartesian” quantale (L, \leq, \wedge, \top)
- the Lawvere quantale $[0, \infty]_+ \cong [0, 1]_{\times}$, that is: $([0, \infty], \geq, +, 0) \cong ([0, 1], \leq, \times, 1)$
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2.2 More examples of quantales

- the free quantale $(\mathcal{P}M, \subseteq, *, \{\eta\})$ over a commutative monoid $(M, *, \eta)$
- the quantale $(\mathcal{D}\mathcal{V}, \subseteq, \otimes_{\downarrow}, \downarrow k)$ of down(-closed) sets of a quantale $(\mathcal{V}, \otimes, k)$
- the quantale $\Delta_{\&} = (\Delta, \leq, \&, \kappa)$ of distance distribution functions, with

$$\Delta = \{\varphi: [0, \infty] \rightarrow [0, 1] \mid \forall \alpha \in [0, \infty] : \varphi(\alpha) = \sup_{\beta < \alpha} \varphi(\beta)\},$$

for any “t-norm” $\&$ on $[0, 1]$, i.e. any operation that makes $([0, 1], \leq, \&, 1)$ a quantale, extended to Δ by

$$(\varphi \& \psi)(\gamma) = \sup_{\alpha + \beta < \gamma} \varphi(\alpha) \& \psi(\beta);$$

the distance distribution function κ with $\kappa(0) = 0$ and $\kappa(\alpha) = 1$ for $\alpha > 0$ is $\&$ -neutral.

$$[0, \infty]_+ \xrightarrow{\sigma_+} \Delta_{\&} \xleftarrow{\tau_{\&}} [0, 1]_{\&}$$

is a coproduct in the category of quantales, since for any $\varphi \in \Delta$:

$$\varphi = \sup_{0 \leq \alpha \leq \infty} \sigma_+(\alpha) \& \tau_{\&}(\varphi(\alpha)) = \sup_{0 < \alpha < \infty} \sigma_+(\alpha) \& \tau_{\&}(\varphi(\alpha))$$

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2.3 Quantale-valued relations of sets

$$X \xrightarrow{r} Y \quad \iff \quad X \times Y \xrightarrow{r} \mathcal{V} \quad \iff \quad r = (r(x, y))_{x \in X, y \in Y}$$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

Set \longrightarrow \mathcal{V} -**Rel**, $(X \xrightarrow{f} Y) \longmapsto (X \xrightarrow{f_0} Y)$ $f_0(x, y) = k$ if $fx = y$, $= \perp$ else

\mathcal{V} -**Rel** is a 2-category with 2-cells given by the pointwise order of \mathcal{V} -relations.

\mathcal{V} -**Rel** has the involution $r^\circ(y, x) = r(x, y)$; put $f^\circ = (f_0)^\circ$; then $f_0 \dashv f^\circ$ (“maps are maps”)

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$\mathcal{V} \longrightarrow \mathcal{V}$ -**Rel**, $v \longmapsto (1 \xrightarrow{v} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \xrightarrow{r} Y \xleftarrow{h} Z$ $(h^\circ \cdot r \cdot g_0)(w, z) = r(gw, hz)$.

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\mathcal{V} -**Rel** has the involution $r^\circ(y, x) = r(x, y)$; put $f^\circ = (f_0)^\circ$; then $f_0 \dashv f^\circ$ (“maps are maps”)

\mathcal{V} -**Rel** is a quantaloid, *i.e.* a **Sup**-enriched category:

$$\left(\bigvee_{i \in I} s_i\right) \cdot r = \bigvee_{i \in I} (s_i \cdot r), \quad s \cdot \left(\bigvee_{i \in I} r_i\right) = \bigvee_{i \in I} (s \cdot r_i)$$

$\mathcal{V} \longrightarrow \mathcal{V}$ -**Rel**, $v \longmapsto (1 \xrightarrow{v} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \xrightarrow{r} Y \xleftarrow{h} Z$ $(h^\circ \cdot r \cdot g_0)(w, z) = r(gw, hz)$.

2.3 Quantale-valued relations of sets

$$X \xrightarrow{r} Y \quad \iff \quad X \times Y \xrightarrow{r} \mathcal{V} \quad \iff \quad r = (r(x, y))_{x \in X, y \in Y}$$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

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2.4 Extensions and liftings of \mathcal{V} -relations

Consider $X \xrightarrow{r} Y$, $Y \xrightarrow{s} Z$, $X \xrightarrow{t} Z$. Obtain:

$$\mathcal{V}\text{-Rel}(Y, Z) \begin{array}{c} \xrightarrow{-\cdot r} \\ \perp \\ \xleftarrow{[r, -]} \end{array} \mathcal{V}\text{-Rel}(X, Z)$$

$$\mathcal{V}\text{-Rel}(X, Y) \begin{array}{c} \xrightarrow{s \cdot -} \\ \perp \\ \xleftarrow{]s, -[} \end{array} \mathcal{V}\text{-Rel}(X, Z)$$

$$s \leq [r, t] \iff s \cdot r \leq t \iff r \leq]s, t[$$

“extension of t along r ”

$$\begin{array}{ccc} & & Z \\ & \nearrow t & \uparrow [r, t] \\ X & \xrightarrow{r} & Y \end{array}$$

$$\begin{array}{ccc} Y & \xrightarrow{s} & Z \\ \uparrow]s, t[& \leq & \nearrow t \\ X & & \end{array}$$

“lifting of t along s ”

$$[r, t](y, z) = \bigwedge_{x \in X} [r(x, y), t(x, z)]$$

$$]s, t[(x, y) = \bigwedge_{z \in Z} [s(y, z), t(x, z)]$$

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2.5 Small categories and functors enriched in \mathcal{V}

$(X, a) \in \mathcal{V}\text{-Cat}$ $\iff a$ is a monoid in the monoidal category $(\mathcal{V}\text{-Rel}(X, X), \leq, \cdot, 1_X^\circ)$

$$\iff 1_X^\circ \leq a, \quad a \cdot a \leq a$$

$$\iff k \leq a(x, x), \quad a(x, y) \otimes a(y, z) \leq a(x, z)$$

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$f : (X, a) \rightarrow (Y, b)$ $\iff a \leq f^\circ \cdot b \cdot f_\circ \iff f_\circ \cdot a \leq b \cdot f_\circ \iff a \cdot f^\circ \leq f^\circ \cdot b$

Some prominent objects in $\mathcal{V}\text{-Cat}$:

$$\emptyset, \quad 1 = (\{*\}, \top), \quad E = (\{*\}, k), \quad \mathcal{V} = (\mathcal{V}, [-, -])$$

Lax homomorphisms of quantales facilitate change-of-base functors:

$\varphi : \mathcal{V} \rightarrow \mathcal{W}$ lax homomorphism $\implies B_\varphi : \mathcal{V}\text{-Cat} \rightarrow \mathcal{W}\text{-Cat}, \quad (X, a) \mapsto (X, \varphi a)$

$p : \mathcal{V} \rightarrow 2$ with $(p(v) = \top \iff k \leq v) \implies B_p : \mathcal{V}\text{-Cat} \rightarrow \mathbf{Ord}$ with $(x \leq y \iff k \leq X(x, y))$

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$$\begin{aligned}(X, a) \in \mathcal{V}\text{-Cat} &\iff a \text{ is a monoid in the monoidal category } (\mathcal{V}\text{-Rel}(X, X), \leq, \cdot, 1_X^\circ) \\ &\iff 1_X^\circ \leq a, \quad a \cdot a \leq a \\ &\iff k \leq a(x, x), \quad a(x, y) \otimes a(y, z) \leq a(x, z) \\ X \in \mathcal{V}\text{-Cat} &\iff k \leq X(x, x), \quad X(x, y) \otimes X(y, z) \leq X(x, z) \\ f : X \rightarrow Y \text{ in } \mathcal{V}\text{-Cat} &\iff X(x, x') \leq Y(fx, fx') \\ f : (X, a) \rightarrow (Y, b) &\iff a \leq f^\circ \cdot b \cdot f_\circ \iff f_\circ \cdot a \leq b \cdot f_\circ \iff a \cdot f^\circ \leq f^\circ \cdot b\end{aligned}$$

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2.6 Some examples of (categories of) \mathcal{V} -categories

1-Cat = Set

2-Cat = Ord: preordered sets and monotone maps

$[0, \infty]_+$ -**Cat** = **Met** \cong $[0, 1]_{\times}$ -**Cat** = **ProbOrd**: probabilistic (pre)ordered sets

$[0, \infty]_{\max}$ -**Cat** = **UMet** \cong $[0, 1]_{\min}$ -**Cat**: (Lawvere) ultrametric spaces

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Δ_{\times} -**Cat** = **ProbMet** probabilistic (Lawvere) metric spaces $(X, p : X \times X \rightarrow \Delta)$,
with $p(x, y)(\alpha)$ to be interpreted as probability of “ $d(x, y) < \alpha$ for a random metric on X ”

$(2 \longrightarrow [0, \infty]_+ \longrightarrow \Delta_{\times}) \quad \implies \quad (\mathbf{Ord} \longrightarrow \mathbf{Met} \longrightarrow \mathbf{ProbMet})$

$\mathcal{P}(M, *, \eta)$ -**Cat** $\ni (X, (\leq_{\alpha})_{\alpha \in M})$ with $x \leq_{\eta} x$, $(x \leq_{\alpha} y \ \& \ y \leq_{\beta} z \implies x \leq_{\alpha * \beta} z)$

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2.7 \mathcal{V} -Cat as a concrete category over Set

$\mathcal{V}\text{-Cat}_X = (\mathbf{O} : \mathcal{V}\text{-Cat} \rightarrow \mathbf{Set})^{-1} X$ is a complete lattice, with \bigwedge as in $\mathcal{V}\text{-Rel}(X, X)$, $\perp = \mathbf{1}_X^\circ$

Every $r \in \mathcal{V}\text{-Rel}(X, X)$ has a $\mathcal{V}\text{-Cat}_X$ -hull $\bar{r} \geq r$: $\bar{r} = \bigvee_{n \geq 0} r^n$.

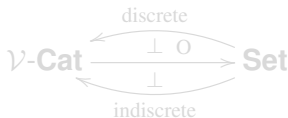
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2.7 \mathcal{V} -Cat as a concrete category over Set

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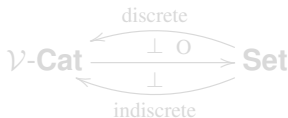
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2.8 \mathcal{V} -Cat as a closed category and a 2-category

For $X, Y \in \mathcal{V}\text{-Cat}$, consider $(\text{ev}_x : \mathcal{V}\text{-Cat}(X, Y) \longrightarrow Y)_{x \in X}$ and put the initial structure on

$$\begin{aligned}[X, Y] := \mathcal{V}\text{-Cat}(X, Y): \quad [X, Y](f, g) &= \bigwedge_{x \in X} Y(fx, gx) \\ &= \bigwedge_{x, x' \in X} [X(x, x'), Y(fx, gx')]\end{aligned}$$

The induced (pre)order on $[X, Y]$ is

$$f \leq g \iff \forall x \in X : k \leq Y(fx, gx) \iff \forall x \in X : fx \leq gx.$$

With its 2-cells given by \leq , $\mathcal{V}\text{-Cat}$ is thus a 2-category.

Adjunction in $\mathcal{V}\text{-Cat}$:

$$(X \xrightarrow{f} Y) \dashv (Y \xleftarrow{g} X) \iff X(x, gy) = Y(fx, y) \iff g^\circ \cdot a = b \cdot f_\circ$$

Note: RHS forces f, g to be \mathcal{V} -functors and gives $f \dashv g$ in **Ord**, i.e. $fg \leq 1_Y$ and $1_X \leq gf$, but $f \dashv g$ in **Ord** secures $f \dashv g$ in $\mathcal{V}\text{-Cat}$ only when f, g are actually \mathcal{V} -functors.

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2.9 \mathcal{V} -Cat as a symmetric monoidal-closed category, Yoneda

$$X \otimes Y((x, y), (x', y')) = X(x, x') \otimes Y(y, y'), \quad E(*, *) = k$$

Enriched Universal Property: $[Z \otimes X, Y] \cong [Z, [X, Y]]$

$$\begin{array}{ccc}
 [X, Y] \otimes X & \xrightarrow{\epsilon} & Y \\
 \uparrow f^\# \otimes 1_X & \nearrow f & \\
 Z \otimes X & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 [X, \mathcal{V}] \otimes X & \xrightarrow{\epsilon} & \mathcal{V} \\
 \uparrow \mathbf{y}_{X^{\text{op}}} \otimes 1_X & \nearrow X(-, -) & \\
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Yoneda \mathcal{V} -functor:

$$\mathbf{y}_X : X \longrightarrow \mathcal{P}_{\mathcal{V}}X = [X^{\text{op}}, \mathcal{V}], y \longmapsto X(-, y), \quad \mathbf{y}_X^\# : X \longrightarrow \mathcal{P}_{\mathcal{V}}^\#X = [X, \mathcal{V}]^{\text{op}}, x \longmapsto X(x, -)$$

Yoneda Lemma:

$$\mathcal{P}_{\mathcal{V}}X(\mathbf{y}_X y, \sigma) = \sigma y, \quad \mathcal{P}_{\mathcal{V}}^\#X(\tau, \mathbf{y}_X^\# x) = \tau x$$

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3.1 Distributors (profunctors, (bi)modules)

Slogan: function/relation = functor/distributor

For $\mathcal{V} = [0, \infty]_+$, think of them as “compatible one-way metrics” between two spaces.

Generally:

$$\begin{aligned}(X, a) \xrightarrow{\rho} (Y, b) &\iff b \cdot \rho \cdot a \leq \rho \\ &\iff X(x', x) \otimes \rho(x, y) \otimes Y(y, y') \leq \rho(x', y') \\ &\iff X^{\text{op}}(x, x') \otimes Y(y, y') \leq [\rho(x, y), \rho(x', y')] \\ &\iff \rho : X^{\text{op}} \otimes Y \rightarrow \mathcal{V} \text{ is a } \mathcal{V}\text{-functor}\end{aligned}$$

\mathcal{V} -distributors are closed under \mathcal{V} -relational composition and under \wedge, \vee formed in $\mathcal{V}\text{-Rel}$.

$\mathcal{V}\text{-Dist}$: objects are \mathcal{V} -categories $X = (X, a)$; identity distributor on X : $1_X^* = (X \xrightarrow{a} X)$

$\mathcal{V}\text{-Dist}$ is **Sup**-enriched (a quantaloid) AND also ($\mathcal{V}\text{-Cat}$)-enriched:

$$\mathcal{V}\text{-Dist}(X, Y) = [X^{\text{op}} \otimes Y, \mathcal{V}], \quad \mathcal{V}\text{-Dist}(X, Y) \otimes \mathcal{V}\text{-Dist}(Y, Z) \longrightarrow \mathcal{V}\text{-Dist}(X, Z)$$

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3.2 \mathcal{V} -functors vs. \mathcal{V} -distributors; extensions, liftings, tensor products

$$(X, a) \xrightarrow{f} (Y, b) \implies X \xrightarrow{f_* = b \cdot f} Y, \quad Y \xrightarrow{f^* = f \circ a} X, \quad f_* \dashv f^* \text{ in } \mathcal{V}\text{-Dist}$$

$$(-)_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{co}} \quad (-)^* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{op}}$$

$$f \dashv g \iff f_* = g^* \iff g^* \dashv f^* \iff g_* \dashv f_*$$

\mathcal{V} -distributors are closed under the formation of extensions and liftings in $\mathcal{V}\text{-Rel}$:

“extension of τ along ρ ”

$$\begin{array}{ccc} & & Z \\ & \nearrow \tau & \uparrow [\rho, \tau] \\ X & \xrightarrow{\rho} & Y \end{array}$$

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ \uparrow [\sigma, \tau] & \searrow \tau & \uparrow \\ X & & \end{array}$$

“lifting of τ along σ ”

$$\mathcal{V}\text{-Dist}(Y, Z)(\sigma, [\rho, \tau]) = \mathcal{V}\text{-Dist}(X, Z)(\sigma \cdot \rho, \tau) = \mathcal{V}\text{-Dist}(X, Y)(\rho,]\sigma, \tau[)$$

$\mathcal{V}\text{-Dist}$ is symmetric monoidal: $\rho \otimes \varphi : X \otimes S \rightarrow Y \otimes T$

$$\rho \otimes \varphi ((x, s), (y, t)) = \rho(x, y) \otimes \varphi(s, t)$$

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$$(X, a) \xrightarrow{f} (Y, b) \implies X \xrightarrow{f_* = b \cdot f \circ} Y, \quad Y \xrightarrow{f^* = f \circ \cdot b} X, \quad f_* \dashv f^* \text{ in } \mathcal{V}\text{-Dist}$$

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$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ \uparrow [\sigma, \tau] & \searrow \tau & \uparrow \\ X & & \end{array}$$

“lifting of τ along σ ”

$$\mathcal{V}\text{-Dist}(Y, Z)(\sigma, [\rho, \tau]) = \mathcal{V}\text{-Dist}(X, Z)(\sigma \cdot \rho, \tau) = \mathcal{V}\text{-Dist}(X, Y)(\rho,]\sigma, \tau[)$$

$\mathcal{V}\text{-Dist}$ is symmetric monoidal: $\rho \otimes \varphi : X \otimes S \rightarrow Y \otimes T$

$$\rho \otimes \varphi ((x, s), (y, t)) = \rho(x, y) \otimes \varphi(s, t)$$

3.2 \mathcal{V} -functors vs. \mathcal{V} -distributors; extensions, liftings, tensor products

$$(X, a) \xrightarrow{f} (Y, b) \implies X \xrightarrow{f_* = b \cdot f} Y, \quad Y \xrightarrow{f^* = f \circ a} X, \quad f_* \dashv f^* \text{ in } \mathcal{V}\text{-Dist}$$

$$(-)_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{co}} \quad (-)^* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{op}}$$

$$f \dashv g \iff f_* = g^* \iff g^* \dashv f^* \iff g_* \dashv f_*$$

\mathcal{V} -distributors are closed under the formation of extensions and liftings in $\mathcal{V}\text{-Rel}$:

“extension of τ along ρ ”

$$\begin{array}{ccc} & & Z \\ & \nearrow \tau & \uparrow [\rho, \tau] \\ X & \xrightarrow{\rho} & Y \end{array}$$

“lifting of τ along σ ”

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ \uparrow [\sigma, \tau] & \leq & \nearrow \tau \\ X & & \end{array}$$

$$\mathcal{V}\text{-Dist}(Y, Z)(\sigma, [\rho, \tau]) = \mathcal{V}\text{-Dist}(X, Z)(\sigma \cdot \rho, \tau) = \mathcal{V}\text{-Dist}(X, Y)(\rho,]\sigma, \tau[)$$

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3.2 \mathcal{V} -functors vs. \mathcal{V} -distributors; extensions, liftings, tensor products

$$(X, a) \xrightarrow{f} (Y, b) \implies X \xrightarrow{f_* = b \cdot f \circ} Y, \quad Y \xrightarrow{f^* = f \circ \cdot b} X, \quad f_* \dashv f^* \text{ in } \mathcal{V}\text{-Dist}$$

$$(-)_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{co}} \quad (-)^* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Dist}^{\text{op}}$$

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“extension of τ along ρ ”

$$\begin{array}{ccc} & & Z \\ & \nearrow \tau & \uparrow [\rho, \tau] \\ X & \xrightarrow{\rho} & Y \end{array}$$

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3.3 \mathcal{V} -distributors vs. \mathcal{V} -presheaves

$$\mathcal{V}\text{-Dist}(X, Y) = [X^{\text{op}} \otimes Y, \mathcal{V}] \cong [Y, \mathcal{P}_{\mathcal{V}}X] \cong [X^{\text{op}}, [Y, \mathcal{V}]] \cong [X, \mathcal{P}_{\mathcal{V}}^{\sharp}Y]^{\text{op}}$$

$$\mathcal{P}_{\mathcal{V}}X \cong \mathcal{V}\text{-Dist}(X, E) \quad \mathcal{P}_{\mathcal{V}}^{\sharp}Y \cong (\mathcal{V}\text{-Dist}(E, Y))^{\text{op}}$$

The Fundamental Presheaf Adjunction: $\mathcal{V}\text{-Dist}^{\text{op}} \begin{array}{c} \xleftarrow{(-)^*} \\ \perp \\ \xrightarrow{\mathcal{P}_{\mathcal{V}} \cong \mathcal{V}\text{-Dist}(-, E)} \end{array} \mathcal{V}\text{-Cat}$

For $X \xrightarrow{\rho} Y$: $\mathcal{P}_{\mathcal{V}}\rho : \mathcal{P}_{\mathcal{V}}Y \rightarrow \mathcal{P}_{\mathcal{V}}X$, $(Y \xrightarrow{\sigma} E) \mapsto (X \xrightarrow{\sigma \circ \rho} E)$

$$(\mathcal{P}_{\mathcal{V}}\rho)(\sigma)(x) = \bigvee_{y \in Y} \rho(x, y) \otimes \sigma(y)$$

adjunction units: $\mathbf{y}_X : X \rightarrow \mathcal{P}_{\mathcal{V}}X$, adjunction counits: $(\mathbf{y}_X)_* : X \xrightarrow{\circ} \mathcal{P}_{\mathcal{V}}X$

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3.4 The \mathcal{V} -presheaf monad $(\mathcal{P}, \mathbf{s}, \mathbf{y})$ and its discretization

$$\mathcal{P} : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Cat}, \quad (f : X \rightarrow Y) \longmapsto (\mathcal{P}_\mathcal{V} f^* : \mathcal{P}_\mathcal{V} X \rightarrow \mathcal{P}_\mathcal{V} Y, \sigma \mapsto \sigma \cdot f^*)$$

$$(\mathcal{P}f)(\sigma)(y) = \bigvee_{x \in X} Y(y, fx) \otimes \sigma x$$

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$(\mathcal{P}, \mathbf{s}, \mathbf{y})$ is a 2-monad, with \mathcal{P} locally fully faithful: $(f \leq g \iff \mathcal{P}f \leq \mathcal{P}g)$

$(\mathcal{P}, \mathbf{s}, \mathbf{y})$ is lax idempotent (Kock–Zöberlein): $\mathcal{P}\mathbf{y}_X \leq \mathbf{y}_{\mathcal{P}X}$

$$\begin{array}{ccccc} & & (-)^* & & \text{d(iscrete)} \\ & & \longleftarrow & & \longleftarrow \\ \mathcal{V}\text{-Dist}^{\text{op}} & & \perp & & \mathcal{V}\text{-Cat} & & \text{Set} \\ & & \longrightarrow & & \longrightarrow & & \\ & & \mathcal{P}_\mathcal{V} \cong \mathcal{V}\text{-Dist}(-, E) & & \text{O} & & \end{array}$$

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3.5 Distributors are Kleisli morphisms

$$X \xrightarrow{\circlearrowleft \varphi} Y \iff Y \xrightarrow{\varphi^\#} \mathcal{P}X$$

$$X \xrightarrow{\circlearrowleft 1_X^*} X \iff X \xrightarrow{\mathbf{y}_X} \mathcal{P}X$$

$$(X \xrightarrow{\circlearrowleft \varphi} Y \xrightarrow{\circlearrowleft \psi} Z)^\# = (Z \xrightarrow{\psi^\#} \mathcal{P}Y \xrightarrow{\mathcal{P}\varphi^\#} \mathcal{P}\mathcal{P}X \xrightarrow{\mathbf{s}_X} \mathcal{P}X)$$

$$(\mathcal{V}\text{-Dist})^{\text{op}} \cong \text{Kl}(\mathcal{P})$$

$$(\mathcal{V}\text{-Rel})^{\text{op}} \cong \text{Kl}(\mathcal{P}_d)$$

Q: What “is” $\text{EM}(\mathcal{P})$?

3.5 Distributors are Kleisli morphisms

$$X \xrightarrow{\circlearrowleft \varphi} Y \iff Y \xrightarrow{\varphi^\#} \mathcal{P}X$$

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4.1 Order completeness vs. conical (co)completeness

Consider $X \in \mathcal{V}\text{-Cat}$ with its induced order ($x \leq y \iff k \leq X(x, y)$). Then:

$$y \simeq \bigwedge_{i \in I} x_i \iff \forall z (k \leq X(z, y) \iff \forall i \in I : k \leq X(z, x_i))$$

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$$X \text{ conically complete} \iff \mathbf{y}_X y = \bigwedge_{i \in I} \mathbf{y}_X x_i \text{ in } \mathcal{P}_{\mathcal{V}} X = [X^{\text{op}}, \mathcal{V}]$$

$$\iff X \text{ has all conical infima}$$

$$\iff X \text{ has all infima and } \mathbf{y}_X \text{ preserves them}$$

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In $X \in \mathbf{2-Cat} = \mathbf{Ord}$, every inf/sup is conical; hence:
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In $X \in [0, \infty]\text{-Cat} = \mathbf{Met}$, not even binary infs/sups have to be conical.

The order of $X \in \mathbf{Met}_{\text{sym,sep}}$ is discrete, so that X is order-complete only when $|X| = 1$.

$\mathcal{V} \in \mathcal{V}\text{-Cat}$ is always conically (co)complete, and so is $\mathcal{P}_{\mathcal{V}}X$, for all $X \in \mathcal{V}\text{-Cat}$.

For \mathcal{V} non-integral ($k < \top$), one finds $X \in \mathcal{V}\text{-Cat}$ order-compl., but not conically (co)compl.

There is a subspace of $[0, \infty] \in [0, \infty]_+\text{-Cat}$ is conically complete category but fails to be conically cocomplete (Clementino).

We need:

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4.3 Tensored and cotensored \mathcal{V} -categories

Recall:

X conically complete $\iff X$ order complete and $(\forall x \in X : X(x, -) : X \rightarrow \mathcal{V}$ pres. infs)

Definition:

X tensorable : $\iff \forall x \in X : X(x, -) : X \rightarrow \mathcal{V}$ has a left adjoint $- \odot x : \mathcal{V} \rightarrow X$

$$X(u \odot x, y) = [u, X(x, y)] \quad (*)$$

X cotensorable : $\iff \forall y \in X : X(-, y) : X^{\text{op}} \rightarrow \mathcal{V}$ has a left adjoint $- \pitchfork y : \mathcal{V} \rightarrow X$

$$X(x, u \pitchfork y) = [u, X(x, y)]$$

Note:

Necessarily $u \odot x = \bigwedge \{y \in X \mid u \leq X(x, y)\}$,

but the the existence of these infima does not guarantee $(*)$!

Trivially: X (co)tensorable $\implies (X$ conically (co)complete $\iff X$ order-complete).

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4.4: Tensored \mathcal{V} -categories: examples and remarks

$\emptyset \neq X \in \mathbf{2-Cat} = \mathbf{Ord}$ is tensoried $\iff X$ has a least element.

$\mathcal{V} \in \mathcal{V}\text{-Cat}$ is tensoried and cotensoried, with $u \odot x = u \otimes x$ and $u \pitchfork x = [u, x]$.
More generally, $\mathcal{P}_{\mathcal{V}}X$ is (co-)tensoried, for every \mathcal{V} -category X .

A full \mathcal{V} -subcategory of $\mathcal{V} \in \mathcal{V}\text{-Cat}$ may fail to be tensoried or cotensoried.

$[0, \infty]_{\text{csym}}$ (= $[0, \infty]$ with the Euclidean metric) fails to be tensoried or cotensoried in \mathbf{Met} .

Products of (co)tensoried \mathcal{V} -categories are (co)tensoried.

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4.5.1 Presenting tensored \mathcal{V} -categories via the action of \mathcal{V} : prelims

Rules for the action of \mathcal{V} on a tensored \mathcal{V} -category X :

- (1) $k \odot x \simeq x$
- (2) $(u \otimes v) \odot x \simeq u \odot (v \odot x)$
- (3) $(\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I} (u_i \odot x)$ (with the RHS \bigvee existing in X , as part of the condition)
- (4⁻) $x \leq y \implies u \odot x \leq u \odot y$

Conversely:

Let X be just a preordered set equipped with a map $\odot : \mathcal{V} \times X \rightarrow X$ satisfying (1) – (4⁻). Then, for every $x \in X$, the map $- \odot x : \mathcal{V} \rightarrow X$ has a right adjoint $X(x, -)$, defined by

$$X(x, y) = \bigvee \{u \mid u \odot x \leq y\},$$

making X a \mathcal{V} -category, whose underlying preorder is the given one and, by the given rules and adjunction, satisfies

$$X(u \odot x, y) = [u, X(x, y)],$$

making X a tensored \mathcal{V} -category.

4.5.1 Presenting tensored \mathcal{V} -categories via the action of \mathcal{V} : prelims

Rules for the action of \mathcal{V} on a tensored \mathcal{V} -category X :

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4.5.2 Presenting tensored \mathcal{V} -categories via the action of \mathcal{V} : theorem

Theorem (Martinelli 2021)

There is a 2-equivalence

$$\mathcal{V}\text{-Cat}_{\text{tensor}} \simeq \mathbf{Ord}_{\frac{1}{2}\text{cocts}}^{\mathcal{V}}$$

$\mathcal{V}\text{-Cat}_{\text{tensor}}$:

small tensored \mathcal{V} -categories, with tensor-preserving \mathcal{V} -functors

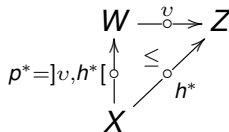
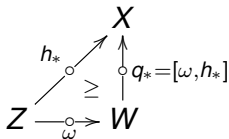
$\mathbf{Ord}_{\frac{1}{2}\text{cocts}}^{\mathcal{V}}$:

preordered sets on which \mathcal{V} acts, satisfying conditions (1), (2), (3), (4⁻), with monotone and pseudo-equivariant maps.

4.6 Weighted colimits and limits: definitions

Given a “diagram” $Z \xrightarrow{h} X$ in X and “weights” $Z \xrightarrow{\omega} W$ and $W \xrightarrow{v} Z$. Then:

$$(W \xrightarrow{q} X) \simeq \operatorname{colim}^{\omega} h : \iff q_* = [\omega, h_*], \quad (W \xrightarrow{p} X) \simeq \operatorname{lim}^v h : \iff p^* =]v, h^*[$$



$$X(qt, x) = \bigwedge_{z \in Z} [\omega(z, t), X(hz, x)]$$

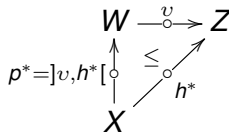
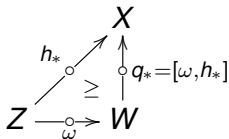
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for all $x \in X, t \in W$.

4.7 Tensors and conical sups as weighted colimits, and conversely

Let $(x \in X \iff x : E = (\{*\}, k) \rightarrow X)$ and $(f = (x_i)_{i \in I} \text{ in } X \iff f : I_d \cong \coprod_{i \in I} E \rightarrow X)$, let $\nabla : \coprod_{i \in I} E \rightarrow E$ be the “codiagonal”. Then:

$$u \odot x \simeq \operatorname{colim}^u x, \quad \bigvee_{i \in I}^{\nabla} x_i \simeq \operatorname{colim}^{\nabla^*} f, \quad u \pitchfork x \simeq \operatorname{lim}^u x, \quad \bigwedge_{i \in I}^{\nabla} x_i \simeq \operatorname{lim}^{\nabla^*} f.$$

Theorem

Let $Z \xrightarrow{h} X$ be a diagram in the tensored \mathcal{V} -category X with weight $Z \xrightarrow{\omega} W$. Then

$$(\operatorname{colim}^{\omega} h)(t) \simeq \bigvee_{z \in Z}^{\nabla} \omega(z, t) \odot h(z)$$

for all $t \in W$, with the colimit on the left existing precisely when the conical supremum on the right exists in X for all $t \in W$.

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4.8 Hiding the diagram in the weight

Corollary

- 1 X cocomplete $\iff X$ is tensored and conically cocomplete;
- 2 X complete $\iff X$ is cotensored and conically complete.
- 3 X complete and cocomplete $\iff X$ tensored, cotensored and order-complete.

Given a diagram $Z \xrightarrow{h} X$ in X and weights $Z \xrightarrow{\omega} W$ and $W \xrightarrow{v} Z$. Then

$$\operatorname{colim}^{\omega} h \simeq \operatorname{colim}^{\omega \cdot h^*} 1_X \quad \text{and} \quad \operatorname{lim}^v h \simeq \operatorname{lim}^{h_* \cdot v} 1_X,$$

with the (co)limit on either side of \simeq existing when the (co)limit on the other side exists.
In particular:

$$u \odot x \cong \operatorname{colim}^u x \cong \operatorname{colim}^{u \cdot x^*} 1_X = \operatorname{colim}^{u \cdot \mathbf{y}_X x} 1_X \quad \text{and} \quad \bigvee_{i \in I}^{\nabla} x_i \simeq \operatorname{colim}^{\omega} 1_X, \quad \text{with } \omega = \bigvee_{i \in I} \mathbf{y}_X x_i.$$

Hence: It suffices to let $Z = X$, $h = 1_X$ and $W = E$; presheaves on X suffice as weights!

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4.9.1 Preservation of (co)limits: definition

Definition

Let $h : Z \rightarrow X$, $f : X \rightarrow Y$ be \mathcal{V} -functors and $Z \xrightarrow{\omega} W \xrightarrow{v} Z$ be \mathcal{V} -distributors.

- 1 If $q \simeq \operatorname{colim}^{\omega} h$ exists in X , one says that $f : X \rightarrow Y$ preserves the colimit if the colimit $\operatorname{colim}^{\omega}(f \cdot h)$ exists in Y and is given by $f \cdot q$; equivalently, if one has the implication

$$q_* = [\omega, h_*] \implies (f \cdot q)_* = [\omega, (f \cdot h)_*].$$

- 2 Dually, if $p \simeq \operatorname{lim}^v h$ exists in X , one says that $f : X \rightarrow Y$ preserves the limit if the limit $\operatorname{lim}^v(f \cdot h)$ exists in Y and is given by $f \cdot p$; equivalently, if one has the implication

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4.9.2 Preservation of (co)limits: criteria, examples

Let $f : X \rightarrow Y$, $g : Y \rightarrow X$, $h : Z \rightarrow [X, Y]$ be \mathcal{V} -functors, $x \in X$, $Z \xrightarrow{\omega} W$.

- 1 If X is tensored: f is cocontinuous $\iff f$ preserves tensors and conical suprema.
- 2 If X is cotensored: f is continuous $\iff f$ preserves cotensors and conical infima.
- 3 $X(x, -) : X \rightarrow \mathcal{V}$ is continuous, $X(-, x) : X \rightarrow \mathcal{V}^{\text{op}}$ is cocontinuous.
- 4 $\text{colim}^{\omega}(h : Z \rightarrow [X, Y])$ exists if $\text{colim}^{\omega} \text{ev}_x h$ exists in Y for all x , and it is then preserved by every $\text{ev}_x : [X, Y] \rightarrow Y$.
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- 6 If $f \dashv g$, then g is continuous and f is cocontinuous.

Theorem (Adjoint Functor Theorem)

- 1 Y complete: $g : Y \rightarrow X$ has a left adjoint \mathcal{V} -functor $\iff g$ is continuous.
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4.10 Completeness Theorem

Theorem

For every \mathcal{V} -category X , the following statements are equivalent:

- (i) X is cocomplete;*
- (ii) for every presheaf ω on X , the colimit of 1_X weighted by ω exists in X ;*
- (iii) $\mathbf{y}_X : X \rightarrow [X^{\text{op}}, \mathcal{V}]$ has a left adjoint \mathcal{V} -functor;*
- (iv) X is tensored, cotensored and order-complete;*
- (v) X is complete;*
- (vi) for every copresheaf v on X , the limit of 1_X weighted by v exists in X ;*
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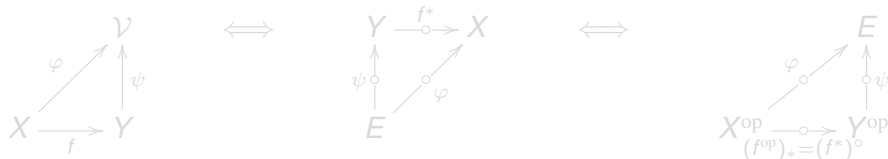
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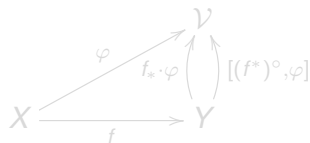
5.1 \mathcal{V} is injective in $\mathcal{V}\text{-Cat}$

$f : X \rightarrow Y$ fully faithful $\iff f^* \cdot f_* = 1_X^* \iff X(x, x') = Y(fx, fx')$ for all $x, x' \in X$



Given f fully faithful and φ , there is a least and a largest extension, ψ_- and ψ^- :

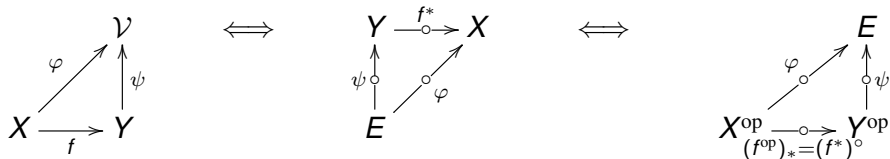
$$\psi_- = f_* \cdot \varphi \quad \text{and} \quad \psi^- = [(f^*)^{\circ}, \varphi] = [(f^{\text{op}})_*, \varphi]$$



$$\psi_- y = \bigvee_{x \in X} Y(fx, y) \otimes \varphi x \quad \text{and} \quad \psi^- y = \bigwedge_{x \in X} [Y(y, fx), \varphi x]$$

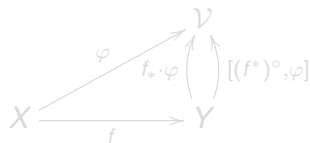
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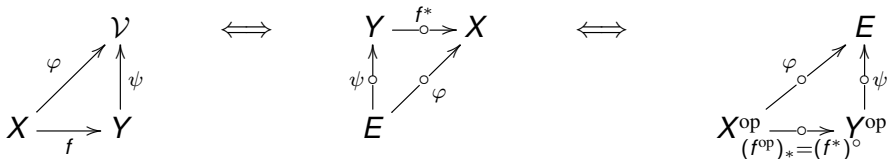
$$\psi_- = f_* \cdot \varphi \quad \text{and} \quad \psi^- = [(f^*)^{\circ}, \varphi] = [(f^{\text{op}})_*, \varphi]$$



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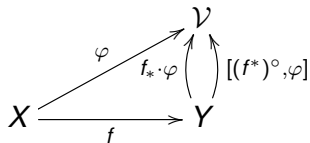
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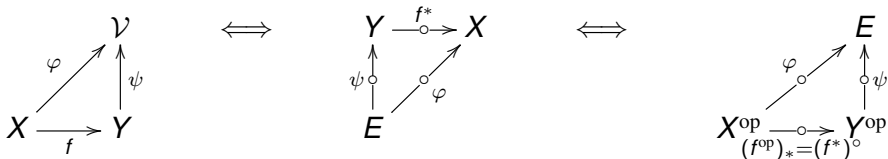
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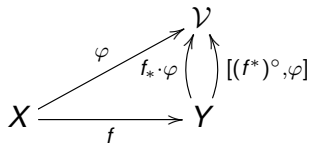
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$$f \neq g : X \rightarrow Y, Y \text{ separated} \implies \exists h : Y \rightarrow \mathcal{V} : hf \neq hg$$

$$\kappa_Y : Y \longrightarrow \mathcal{V}^{[Y, \mathcal{V}]} = \prod_{h \in [Y, \mathcal{V}]} \mathcal{V}, \quad y \longmapsto (hy)_{h \in [Y, \mathcal{V}]}$$

$$\pi_Y : \mathcal{V}^{[Y, \mathcal{V}]} \longrightarrow \mathcal{V}^Y, \quad (v_h)_{h \in [Y, \mathcal{V}]} \longmapsto (v_{\mathbf{y} \#_Y z})_{z \in Y}$$

Theorem

\mathcal{V} is a regular cogenerator of the category $\mathcal{V}\text{-Cat}_{\text{sep}}$, and it is injective with respect to fully faithful \mathcal{V} -functors. Every separated \mathcal{V} -category Y embeds fully into the Y -fold power \mathcal{V}^Y of \mathcal{V} , which is injective again.

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5.3.1 Colimit and limit completion of a \mathcal{V} -category

Every \mathcal{V} -presheaf ω on $X \in \mathcal{V}\text{-Cat}$ is a colimit of \mathbf{y}_X in $\mathcal{P}_{\mathcal{V}}X$ weighted by ω : $\omega \simeq \text{colim}^{\omega} \mathbf{y}_X$.

$$\begin{array}{ccc}
 & & \mathcal{P}_{\mathcal{V}}X \\
 & \nearrow (\mathbf{y}_X)_* & \uparrow \omega_* \\
 X & \xrightarrow{\omega} & E
 \end{array}$$

$$\begin{array}{ccc}
 E & \xrightarrow{v} & X \\
 \uparrow v^* & \searrow (\mathbf{y}_X^\#)_* & \nearrow \\
 \mathcal{P}_{\mathcal{V}}^\# X & &
 \end{array}$$

Dually, every \mathcal{V} -copresheaf v on X is a limit of representables in $\mathcal{P}_{\mathcal{V}}^\# X$; that is: $v \simeq \text{lim}^v \mathbf{y}_X^\#$.

Wanted for $f : X \rightarrow Y$, Y cocomplete/complete:

$$\begin{array}{ccc}
 X & \xrightarrow{\mathbf{y}_X} & \mathcal{P}_{\mathcal{V}}X \\
 & \searrow f & \downarrow \tilde{f} \text{ cocontinuous} \\
 & & Y
 \end{array}$$

$$\begin{array}{ccc}
 X & \xrightarrow{\mathbf{y}_X^\#} & \mathcal{P}_{\mathcal{V}}^\# X \\
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5.3.2 Proof of the Colimit Completion Theorem

Uniqueness:

$$\tilde{f}(\omega) \simeq \tilde{f}(\operatorname{colim}^{\omega} \mathbf{y}_X) \simeq \operatorname{colim}^{\omega} (\tilde{f} \mathbf{y}_X) \simeq \operatorname{colim}^{\omega} f$$

Existence:

$$\begin{aligned} \tilde{f}(\omega) &= \operatorname{colim}^{\omega} f \simeq \operatorname{colim}^{\omega \cdot f^*} 1_Y \\ \tilde{f} &\simeq (\operatorname{colim}^{(-)} 1_Y)(\mathcal{P}_V f^*) \end{aligned}$$

$$\tilde{f} \mathbf{y}_X \simeq (\operatorname{colim}^{(-)} 1_Y)(\mathcal{P}_V f^*) \mathbf{y}_X \simeq (\operatorname{colim}^{(-)} 1_Y) \mathbf{y}_Y f \simeq f.$$

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5.4 Cocomplete \mathcal{V} -categories as pseudo-algebras and as injectives

Theorem

The following properties for a \mathcal{V} -category X are equivalent:

- (i) X is (co)complete;
- (ii) X carries the structure of a pseudo-algebra with respect to the presheaf monad on $\mathcal{V}\text{-Cat}$;
- (iii) The Yoneda \mathcal{V} -functor \mathbf{y}_X has a pseudo-retraction; that is: there is a \mathcal{V} -functor $h : \mathcal{P}_{\mathcal{V}}X \rightarrow X$ with $h\mathbf{y}_X \simeq 1_X$;
- (iv) X is pseudo-injective in $\mathcal{V}\text{-Cat}$ with respect to fully faithful functors.

5.5.1 Cocomplete \mathcal{V} -categories via cocontinuous action

Let X be a (co)complete preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying

- (1) $k \odot x \simeq x$
- (2) $(u \otimes v) \odot x \simeq u \odot (v \odot x)$
- (3) $(\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I} (u_i \odot x)$
- (4) $u \odot (\bigvee_{i \in I} x_i) \simeq \bigvee_{i \in I} (u \odot x_i)$

Condition (4) (= sup-preservation of every $u \odot - : X \longrightarrow X$) makes the (existing) sups in X conical colimits:

$$X(\bigvee_{i \in I} x_i, y) = \bigwedge_{i \in I} X(x_i, y).$$

Combine this with two fundamental enriched colimit formulae we have already seen:

$$(\operatorname{colim}^{\omega} h)(w) \simeq \bigvee_z \omega(z, w) \odot h(z) \quad (h : Z \rightarrow X, \omega : Z^{\operatorname{op}} \otimes W \rightarrow \mathcal{V})$$

$$X(\operatorname{colim}^{\omega} 1_X, x) \simeq [X^{\operatorname{op}}, \mathcal{V}](\omega, \mathbf{y}_X x) \quad (\omega : X^{\operatorname{op}} \cong X^{\operatorname{op}} \otimes E \rightarrow \mathcal{V}), \text{ saying } \operatorname{colim}^{(-)} \dashv \mathbf{y}_X,$$

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5.5.2 Cocomplete \mathcal{V} -categories via cocontinuous action: Theorem

Theorem (Folklore 19??)

There are 2-equivalences

$$\mathcal{V}\text{-Cat}^{\mathcal{P}\simeq} \simeq \mathcal{V}\text{-Cat}_{\text{colim}} \simeq (\mathbf{Ord}_{\text{sup}})^{\mathcal{V}}$$

$\mathcal{V}\text{-Cat}_{\text{colim}}$:

(co)complete \mathcal{V} -categories, with cocontinuous \mathcal{V} -functors

$\mathbf{Ord}_{\text{sup}}^{\mathcal{V}}$:

(co)complete preordered sets on which \mathcal{V} acts satisfying conditions (1), (2), (3), (4), with sup-preserving and pseudo-equivariant maps

Corollary

There are 2-equivalences

$$(\mathcal{V}\text{-Cat}_{\text{sep}})^{\mathcal{P}} \simeq \mathcal{V}\text{-Cat}_{\text{sep, colim}} \simeq \mathbf{Sup}^{\mathcal{V}}$$

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5.6.1 Presenting conically cocomplete \mathcal{V} -categories algebraically?

Consider moving from the presheaf-monad \mathcal{P} on $\mathcal{V}\text{-Cat}$:

$$\mathcal{P} : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Cat}, \quad X \longmapsto [X^{\text{op}}, \mathcal{V}], \quad \mathcal{P}X(\sigma, \tau) = \bigwedge_{z \in X} [\sigma z, \tau z]$$

to the Hausdorff submonad \mathcal{H} via

$$j_X : \mathcal{H}X = \{A \mid A \subseteq X\} \longrightarrow \mathcal{P}X, \quad A \longmapsto (z \mapsto X(z, A) = \bigvee_{x \in A} X(z, x)).$$

where $\mathcal{H}X$ carries the initial (= cartesian) structure inherited from $\mathcal{P}X$ via j_X :

$$\mathcal{H}X(A, B) = \bigwedge_{z \in X} [\bigvee_{x \in A} X(z, x), \bigvee_{y \in B} X(z, y)] = \dots = \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y).$$

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5.6.2 Algebraic presentation of conically cocomplete \mathcal{V} -categories

Theorem (Akhvlediani-Clementino-T 2009, Stubbe 2009)

*Just like \mathcal{P} , also \mathcal{H} becomes a lax-idempotent monad of the 2-category $\mathcal{V}\text{-Cat}$, lifting the power-set monad of **Set**, and making $j : \mathcal{H} \rightarrow \mathcal{P}$ a monad morphism, which induces the forgetful functor*

$$(\mathcal{V}\text{-Cat})^{\mathcal{P}\simeq} \simeq \mathcal{V}\text{-Cat}_{\text{colim}} \longrightarrow \mathcal{V}\text{-Cat}_{\text{consup}} \simeq (\mathcal{V}\text{-Cat})^{\mathcal{H}\simeq},$$

$\mathcal{V}\text{-Cat}_{\text{colim}}$:

(co)complete (= all weighted (co)limts exist) \mathcal{V} -categories, with cocontinuous \mathcal{V} -functors;

$\mathcal{V}\text{-Cat}_{\text{consup}}$:

conically cocomplete (= sups exist, Yoneda preserves) \mathcal{V} -cats, with sup-preserving \mathcal{V} -funs

5.7 $\mathcal{V}\text{-Cat}_{\text{sep, colim}}$ as a quantification of **Sup**?

- Monadicity:



- Self-duality:

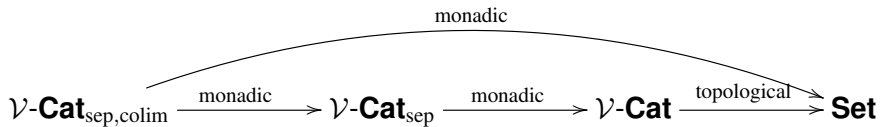
$$\mathcal{V}\text{-Cat}_{\text{sep, colim}} \xrightarrow{\cong} (\mathcal{V}\text{-Cat}_{\text{sep, colim}})^{\text{op}}$$

$$\begin{array}{ccc}
 X & \xleftrightarrow{\quad} & X^{\text{op}} \\
 \downarrow f \dashv f_* & & \uparrow (f_*)^{\text{op}} \\
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- Symmetric monoidal-closed?

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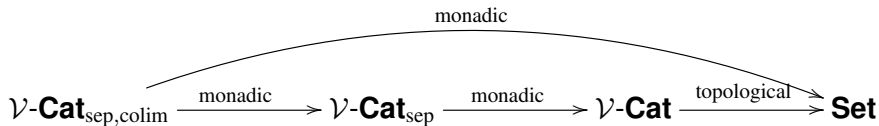
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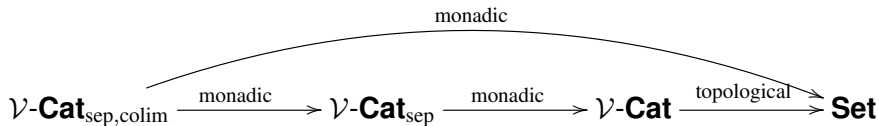
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- Symmetric monoidal-closed?

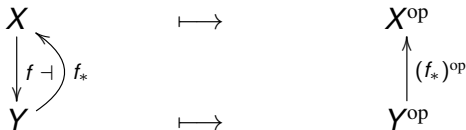
5.7 $\mathcal{V}\text{-Cat}_{\text{sep, colim}}$ as a quantification of **Sup**?

- **Monadicity:**



- **Self-duality:**

$$\mathcal{V}\text{-Cat}_{\text{sep, colim}} \xrightarrow{\cong} (\mathcal{V}\text{-Cat}_{\text{sep, colim}})^{\text{op}}$$



- **Symmetric monoidal-closed?**

5.8 \mathcal{V} -Cat_{sep,colim} is symmetric monoidal closed

Having an equational presentation of separated cocomplete \mathcal{V} -categories, we construct the tensor product classifying “bimorphisms” in a standard manner:

Given objects X, Y , form the free object $\mathcal{P}_d(X \times Y)$ (with the \mathcal{V} -powerset monad of **Set**) and then put

$$X \boxtimes Y = \mathcal{P}_d(X \times Y) / \sim$$

with the least congruence relation \sim making the Yoneda map

$\mathbf{y} : X \times Y \longrightarrow \mathcal{P}_d(X \times Y) / \sim$ a bimorphism; so, \sim is generated by:

$$\begin{aligned} \mathbf{y}(u \odot x, y) &\sim u \odot \mathbf{y}(x, y) \sim \mathbf{y}(x, u \odot y), \\ \mathbf{y}\left(\bigvee_{i \in I} x_i, y\right) &\sim \bigvee_{i \in I} \mathbf{y}(x_i, y), & \mathbf{y}\left(x, \bigvee_{i \in I} y_i\right) &\sim \bigvee_{i \in I} \mathbf{y}(x, y_i) \end{aligned}$$

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6.1 Cauchy sequences

$s = (x_n)_{n \in \mathbb{N}}$ sequence in $X \in \mathcal{V}\text{-Cat}$, $x \in X$

$$\text{Cauchy}(s) := \bigvee_{N \in \mathbb{N}} \bigwedge_{m, n \geq N} X(x_m, x_n)$$

$$s \text{ is Cauchy} : \iff k \leq \text{Cauchy}(s)$$

$$\lambda_s(x) := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} X(x_n, x) \quad (\text{"left-convergence value of } s \rightsquigarrow x\text{"})$$

$$\rho_s(x) := \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} X(x, x_n) \quad (\text{"right-convergence value of } s \rightsquigarrow x\text{"})$$

Facts:

$$E \xrightarrow{\lambda_s} X, \quad X \xrightarrow{\rho_s} E, \quad \text{with } \lambda_s \cdot \rho_s \leq 1_X^*$$

$$s \text{ Cauchy} \iff 1_E^* \leq \rho_s \cdot \lambda_s \iff \lambda_s \dashv \rho_s$$

Definitions:

$$s \rightsquigarrow x : \iff k \leq \bigvee_{N \in \mathbb{N}} (\bigwedge_{m \geq N} X(x_m, x) \otimes \bigwedge_{n \geq N} X(x, x_n)) \iff k \leq \lambda_s(x) \otimes \rho_s(x)$$

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$$\begin{aligned} X \text{ Lawvere-complete} &: \iff \forall \varphi \dashv \psi : X \multimap W \exists f : W \longrightarrow X : \varphi = f_*, \psi = f^* \\ &\iff \forall \varphi \dashv \psi : X \multimap E \exists x \in X : \varphi = x_*, \psi = x^* \\ &\implies X \text{ Cauchy-complete} \end{aligned}$$

Conversely?

Auxiliary conditions on \mathcal{V} :

\mathcal{V} integral ($k = \top$) and $\exists (\varepsilon_n)_{n \in \mathbb{N}}$ in \mathcal{V} : 1. $\varepsilon_n \leq \varepsilon_{n+1}$, 2. $\varepsilon_n \ll k$, 3. $\bigvee_{n \in \mathbb{N}} \varepsilon_n = k$

Then: $\forall \varphi \dashv \psi \exists s$ Cauchy in X : $\varphi = \lambda_s$, $\psi = \rho_s$

Theorem (Hofmann-Reis 2018)

If \mathcal{V} satisfies the auxiliary conditions: X Lawvere-complete \iff X Cauchy-complete

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6.3 Cocompletion wrt a given class Φ of weights: conditions on Φ

W1 $f^* \in \Phi$, for every \mathcal{V} -functor f ;

W2 $f^* \cdot \psi, \psi \cdot g^*, \psi \cdot h_* \in \Phi$, for all $\psi \in \Phi$ and \mathcal{V} -functors f, g, h with $h_* \in \Phi$, provided that the composites are defined;

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Φ *cocompletion class* : \iff (W1-3) hold; Φ *monadic cocompl. class* : \iff (W1-4) hold.

Largest cocompletion class: all \mathcal{V} -distributors; trivially, it is monadic.

Least cocompletion class: $\{f^* \mid f \text{ } \mathcal{V}\text{-functor}\}$; it may obviously fail to be monadic.

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$X \in \mathcal{V}\text{-Cat}$ is Φ -*cocomplete* : \iff all colimits of diagrams in X with weights in Φ exist.

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Least cocompletion class: $\{f^* \mid f \text{ } \mathcal{V}\text{-functor}\}$; it may obviously fail to be monadic.

Lawvere cocompletion class: $\{\psi \mid \psi \text{ right adjoint}\}$; it fails to be monadic already for $\mathcal{V} = 2$.

$X \in \mathcal{V}\text{-Cat}$ is Φ -*cocomplete* : \iff all colimits of diagrams in X with weights in Φ exist.

$f : X \rightarrow Y$ is Φ -*cocontinuous* : \iff f preserves Φ -weighted colimits of X .

6.4 Cocompletion wrt a given class Φ of weights: pseudo- Φ -injectivity

For a cocompletion class Φ call

$f : X \rightarrow Y$ Φ -dense $\iff f_* \in \Phi$;

X pseudo- Φ -injective $\iff X$ pseudo-injective wrt fully faithful Φ -dense \mathcal{V} -functors;

Put

$$X \xrightarrow{\mathbf{y}_X^\Phi} \Phi X := \{\psi \in \mathcal{P}X \mid \psi \in \Phi\} \xrightarrow{\text{inc}_X^\Phi} \mathcal{P}X$$

\mathbf{y}_X

Check:

- f has a right adjoint $\implies f$ Φ -dense;
- f and $g : Y \rightarrow Z$ Φ -dense $\implies g \cdot f$ Φ -dense;
- $g \cdot f$ Φ -dense and $f_* \cdot f^* = 1_Y^* \implies g$ Φ -dense;
- $g \cdot f$ Φ -dense and g fully faithful $\implies f$ Φ -dense;
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Theorem (Clementino-Hofmann 2009)

Let Φ be a cocompletion class.

- The following properties for a \mathcal{V} -category X are equivalent:
 - (i) X is Φ -cocomplete, i.e. X has all colimits with weights in Φ ;
 - (ii) X carries the structure of a pseudo-algebra with respect to the Φ -presheaf monad $(\Phi, \mathbf{s}^\Phi, \mathbf{y}^\Phi)$ on $\mathcal{V}\text{-Cat}$;
 - (iii) the Yoneda \mathcal{V} -functor \mathbf{y}_X^Φ has a pseudo-retraction; that is: there is a \mathcal{V} -functor $h : \mathcal{P}^\Phi X \rightarrow X$ with $h\mathbf{y}_X^\Phi \simeq 1_X$;
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- $\Phi \dashv (\mathcal{V}\text{-Cat}_{\text{sep}, \Phi\text{-colim}} \longrightarrow \mathcal{V}\text{-Cat})$.
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6.6 Cauchy completion of a \mathcal{V} -category à la Lawvere

Let \mathcal{V} satisfy $k = \top$ and $\exists (\varepsilon_n)_{n \in \mathbb{N}}$ in \mathcal{V} : 1. $\varepsilon_n \leq \varepsilon_{n+1}$, 2. $\varepsilon_n \ll k$, 3. $\bigvee_{n \in \mathbb{N}} \varepsilon_n = k$;

Consider $\Phi := \{\psi \mid \psi \text{ right adjoint } \mathcal{V}\text{-distributor}\}$, and let $X \in \mathcal{V}\text{-Cat}$. Then

$$\Phi X = \{\psi \in \mathcal{P}_{\mathcal{V}} X \mid \psi \text{ right adjoint}\} = \{\rho_s \mid s = (x_n)_n \text{ Cauchy sequence in } X\}$$

with $\rho_s(x) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} X(x, x_n)$ ($x \in X$), and

- 1 (trivially) ($s \sim s' \iff \rho_s = \rho_{s'}$) is an equivalence relation on the set of all Cauchy sequences in X , with projection $s \mapsto \rho_s$;
- 2 ΦX is Cauchy complete;
- 3 the restricted Yoneda \mathcal{V} -functor $X \rightarrow \Phi X$, $y \mapsto \rho_{(y)_n}$, is a reflection of X into the full subcategory of Cauchy complete \mathcal{V} -categories.

7.1 The category $\mathbf{Set} // \mathcal{V} = \mathcal{V}\text{-wSet}$ of \mathcal{V} -weighted or -normed sets

- Defining $\mathbf{Set} // \mathcal{V}$:

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi} & B \\
 & \searrow & \swarrow \\
 & \mathcal{V} &
 \end{array}
 \begin{array}{c}
 \leq \\
 |-\!|_A \quad |-\!|_B
 \end{array}
 \iff \forall a \in A: |a|_A \leq |\varphi a|_B$$

- $\mathbf{Set} // \mathcal{V}$ is topological over \mathbf{Set} :

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi_i} & B_i \\
 & \searrow & \swarrow \\
 & \mathcal{V} &
 \end{array}
 \begin{array}{c}
 \leq \\
 |-\!| \quad |-\!|
 \end{array}
 \text{ initial } \iff |a| = \bigwedge_{i \in I} |\varphi_i a|$$

- $\mathbf{Set} // \mathcal{V}$ is symmetric monoidal-closed:

$$A \otimes B = (A \times B, |(a, b)| = |a| \otimes |b|), \quad E = (1 = \{*\}, |*| = k)$$

$$[A, B] = (\mathbf{Set}(A, B), |\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|])$$

7.2 The category $\mathbf{Cat} // \mathcal{V} = \mathcal{V}\text{-wCat}$ of (small) \mathcal{V} -weighted categories

Objects of $\mathbf{Cat} // \mathcal{V}$ are (small) categories \mathbb{X} enriched in $\mathbf{Set} // \mathcal{V}$; this means (neglecting \forall):

$$\mathbb{X}(x, y) \otimes \mathbb{X}(y, z) \longrightarrow \mathbb{X}(x, z) \quad \text{and} \quad E \longrightarrow \mathbb{X}(x, x) \quad \text{live in } \mathbf{Set} // \mathcal{V}$$

$$\iff |f| \otimes |g| = |(f, g)| \leq |g \cdot f| \quad \text{and} \quad k \leq |1_x|$$

$$\iff |-| : \mathbb{X} \longrightarrow (\mathcal{V}, \otimes, k) \text{ is a lax functor}$$

For a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ to be enriched in $\mathbf{Set} // \mathcal{V}$ means (without universal quantifiers):

$$\mathbb{X}(x, y) \longrightarrow \mathbb{Y}(Fx, Fy) \quad \text{lives in } \mathbf{Set} // \mathcal{V}$$

$$\iff |f| \leq |Ff|$$

$$\iff \begin{array}{ccc} \mathbb{X} & \xrightarrow{F} & \mathbb{Y} \\ & \searrow & \swarrow \\ & \mathcal{V} & \end{array} \begin{array}{c} \\ \leq \\ \\ \end{array} \begin{array}{c} \\ | \cdot | \\ \\ \end{array}$$

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7.3 The adjunction $s \dashv i$, monoidal-closed structure, preserved by i, s

\mathcal{V} -Cat

$$X, \quad X(x, y) \otimes X(y, z) \leq X(x, z)$$

$$k \leq X(x, x)$$

$$s\mathbb{X} = \text{ob}\mathbb{X}, \quad s\mathbb{X}(x, y) = \bigvee \{|f| \mid f : x \rightarrow y\}$$

$$X \otimes Y = X \times Y \text{ (as a set)}$$

$$(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$$

$$[X, Y] = \mathcal{V}\text{-Cat}(X, Y) \text{ (as a set)}$$

$$[X, Y](f, g) = \bigwedge_{x \in X} Y(fx, gx)$$

Cat// \mathcal{V}

$$\xrightarrow{i}$$

$$iX, \quad \text{ob}(iX) = X$$

$$x \xrightarrow{(x, y)} y, \quad |(x, y)| = X(x, y)$$

$$\xleftarrow{s}$$

$$\mathbb{X}, \quad |f| \otimes |g| \leq |g \cdot f|$$

$$k \leq |1_x|$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$|(f, g)| = |f| \otimes |g|$$

$$[\mathbb{X}, \mathbb{Y}] = (\mathbf{Cat//}\mathcal{V})(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$|F \xrightarrow{\alpha} G| = \bigwedge_{x \in \text{ob}\mathbb{X}} |\alpha_x|$$

7.4.1 Example: $(\mathcal{V}, \leq, \otimes, \mathbf{k}) = (2, \perp < \top, \wedge, \top)$

2-Cat = Ord

$$X, \quad x \leq y \wedge y \leq z \implies x \leq z$$

$$\top \implies x \leq x$$

$$\mathbf{sX} = \text{obX}, \quad x \leq y \iff \exists(f : x \rightarrow y) \in \mathcal{S}$$

$$X \otimes Y = X \times Y$$

$$(x, y) \leq (x', y') \iff x \leq x' \wedge y \leq y'$$

$$[X, Y] = \mathbf{Ord}(X, Y)$$

$$f \leq g \iff \forall x \in X : fx \leq gx$$

Cat//2 = sCat

$$\xrightarrow{i} \quad iX, \quad \text{ob}(iX) = X$$

$$(x \xrightarrow{(x,y)} y) \in \mathcal{S} \iff x \leq y$$

$$\xleftarrow{s} \quad \mathbb{X}, \mathcal{S}, \quad f, g \in \mathcal{S} \implies g \cdot f \in \mathcal{S}$$

$$\top \implies 1_x \in \mathcal{S}$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$\mathcal{S}_{\mathbb{X} \otimes \mathbb{Y}} = \mathcal{S}_{\mathbb{X}} \times \mathcal{S}_{\mathbb{Y}}$$

$$[\mathbb{X}, \mathbb{Y}] = \mathbf{sCat}(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$\alpha \in \mathcal{S}_{[\mathbb{X}, \mathbb{Y}]} \iff \forall x \in \text{obX} : \alpha_x \in \mathcal{S}_{\mathbb{Y}}$$

7.4.2 Example: $(\mathcal{V}, \leq, \otimes, \mathbf{k}) = ([0, \infty], \geq, +, 0)$

$[0, \infty]$ -**Cat** = **Met**

$$X, \quad d(x, y) + d(y, z) \geq d(x, z)$$

$$0 \geq d(x, x)$$

$$s\mathbb{X} = \text{ob}\mathbb{X}, \quad d(x, y) = \inf_{f: x \rightarrow y} |f|$$

$$X \otimes Y = X \times Y$$

$$d((x, y), (x', y')) = d(x, x') + d(y, y')$$

$$[X, Y] = \mathbf{Met}(X, Y)$$

$$d(f, g) = \sup_{x \in X} d(fx, gx)$$

Cat// $[0, \infty]$ = **wCat**

$$iX, \quad \text{ob}(iX) = X$$

$$x \xrightarrow{(x,y)} y, \quad |(x, y)| = d(x, y)$$

$$\mathbb{X}, \quad |f| + |g| \geq |g \cdot f|$$

$$0 \geq |1_x|$$

$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$|(f, g)| = |f| + |g|$$

$$[\mathbb{X}, \mathbb{Y}] = \mathbf{wCat}(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$|F \xrightarrow{\alpha} G| = \sup_{x \in \text{ob}\mathbb{X}} |\alpha_x|$$

7.5.1 Some elementary examples of weighted categories, I

We saw:

\mathcal{V} -categories (and their functors) are \mathcal{V} -weighted categories (and their functors); in fact, they are precisely the \mathcal{V} weighted categories with indiscrete underlying category.

Question: May **Set** be “naturally” $[0, \infty]$ -weighted?

Goal 1: Let $|f|$ measure the degree to which a map $f : X \rightarrow Y$ fails to be surjective.

Simply put $|f| := \#(Y \setminus f(X)) \in \mathbb{N} \cup \{\infty\} \subseteq [0, \infty]$.

Then: $0 \geq |\text{id}_X|$, and with $g : Y \rightarrow Z$ we have $|f| + |g| \geq |g \cdot f|$

since (assuming Choice and $Y \cap Z = \emptyset$) there is an injective map

$$Z \setminus (g(f(X))) \longrightarrow (Y \setminus f(X)) + (Z \setminus g(Y)).$$

Note: f surjective $\iff |f| = 0$.

7.5.1 Some elementary examples of weighted categories, I

We saw:

\mathcal{V} -categories (and their functors) are \mathcal{V} -weighted categories (and their functors); in fact, they are precisely the \mathcal{V} weighted categories with indiscrete underlying category.

Question: May **Set** be “naturally” $[0, \infty]$ -weighted?

Goal 1: Let $|f|$ measure the degree to which a map $f : X \rightarrow Y$ fails to be surjective.

Simply put $|f| := \#(Y \setminus f(X)) \in \mathbb{N} \cup \{\infty\} \subseteq [0, \infty]$.

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7.5.2 Some elementary examples of weighted categories, II

Question: May something similar be done for injectivity? That is:

Goal 2: Let $|f|$ measure the degree to which a map $f : X \rightarrow Y$ fails to be injective.

First consider $\#f := \sup_{y \in Y} \#f^{-1}y$; then, with $g : Y \rightarrow Z$, we have:

$$\#g \cdot \#f = \left(\sup_{z \in Z} \#g^{-1}z \right) \cdot \left(\sup_{y \in Y} \#f^{-1}y \right) \geq \sup_{z \in Z} \# \left(\bigcup_{y \in g^{-1}z} f^{-1}y \right) = \#(g \cdot f), \quad 1 \geq \#\text{id}_X$$

Not what we wanted! But $([1, \infty], \geq, \cdot, 1) \xrightarrow{\log} ([0, \infty], \geq, +, 0)$ comes to the rescue:

Put $|f| := \max\{0, \log \#f\}$; then: $|g| + |f| \geq |g \cdot f|$. $0 \geq |\text{id}_X|$.

Note: f injective $\iff |f| = 0$.

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7.5.3 A more interesting example of a (large) weighted category: **Lip**

ob **Lip** = ob **Met**, **Lip**(X, Y) = **Set**(X, Y); why call this category **Lip**??

Recall: $f : X \rightarrow Y$ is $K(\geq 0)$ -Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K d(x, x')$

In particular: $f : X \rightarrow Y$ is a morphism in **Met** $\iff f$ is 1-Lipschitz

Question: How far is an arbitrary map f away from being 1-Lipschitz?

Answer: Find the least Lipschitz constant $K \geq 1$ for f (admitting $K = \infty$)

That is: $\text{Lip}(f) = \max\left\{1, \sup_{x \neq x'} \frac{d(fx, fx')}{d(x, x')}\right\}$ (assuming temporarily that X be separated)

Then: $\text{Lip}(g) \cdot \text{Lip}(f) \geq \text{Lip}(g \cdot f)$, $1 \geq \text{Lip}(\text{id}_X)$

No problem:

$$([1, \infty], \geq, \cdot, 1) \xrightarrow{\log} ([0, \infty], \geq, +, 0), \quad |f| = \max\left\{0, \sup_{x, x'} (\log d(fx, fx') - \log d(x, x'))\right\}$$

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7.6 On the axiomatics for weighted/normed categories

The category \mathbb{X} is \mathcal{V} -weighted by $|-| : \mathbb{X} \rightarrow \mathcal{V}$ if

$$\mathbf{k} \leq |1_x|$$

$$\begin{aligned} |g| \otimes |f| \leq |g \cdot f| &\iff |f| \leq \bigwedge_g [|g|, |g \cdot f|] &&\iff |f| = \bigwedge_g [|g|, |g \cdot f|] \\ &\iff |g| \leq \bigwedge_f [|f|, |g \cdot f|] &&\iff |g| = \bigwedge_f [|f|, |g \cdot f|] \end{aligned}$$

The \mathcal{V} -weighted category \mathbb{X} is *right/left cancellable* if

$$|f| \otimes |g \cdot f| \leq |g| \iff |f| \leq \bigwedge_g [|g \cdot f|, |g|] =: |f|^R \quad (\text{right cancellable})$$

$$|g| \otimes |g \cdot f| \leq |f| \iff |g| \leq \bigwedge_f [|g \cdot f|, |f|] =: |g|^L \quad (\text{left cancellable; Kubiś: "norm"})$$

Facts (Insall-Luckhardt for $\mathcal{V} = [0, \infty]$): \mathbb{X} weighted by $|-| \implies \mathbb{X}$ weighted by $|-|^R$ and $|-|^L$,
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7.7 The underlying ordinary category \mathbb{X}_0 of a \mathcal{V} -weighted category \mathbb{X}

Note:

An isomorphism f in \mathbb{X} may not satisfy $k \leq |f|$, and even when it does, we may not have $k \leq |f^{-1}|$ (unless the weight is left/right cancellable). Still, in many of the examples with $\mathcal{V} = [0, \infty]$ considered in the literature, morphisms f , and especially isomorphisms, of norm 0 play an important role. They are called “modulators” by Insall-Luckhardt.

Question:

What is the “enriched significance” of considering morphisms f with $k \leq |f|$?

Answer:

These are precisely the morphisms of the underlying ordinary category \mathbb{X}_0 of the $(\mathbf{Set} // \mathcal{V})$ -enriched category \mathbb{X} .

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7.8.1 \mathcal{V} -weighted cats vs. \mathcal{V} -metrically enriched cats: syntax prep

Recall: groups $(X, -, 0)$ in subtractive notation:

$$x - 0 = x, \quad x - x = 0, \quad (x - y) - (z - y) = x - z$$

Write \mathcal{V} -**Met** for \mathcal{V} -**Cat**_{sym}: “ \mathcal{V} -metric spaces” = \mathcal{V} -categories X with $X(x, y) = X(y, x)$

Form the category \mathcal{V} -**MetGrp** of “ \mathcal{V} -metric groups”:

objects are \mathcal{V} -metric spaces X with a group structure that makes distances invariant under translations:

$$X(x, y) = X(x - z, y - z);$$

morphisms are \mathcal{V} -contractive homomorphisms.

\mathcal{V} -**MetGrp** inherits its symmetric monoidal structure from \mathcal{V} -**Cat** and the cartesian cat **Grp**.

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7.8.2 \mathcal{V} -metric groups as \mathcal{V} -weighted groups

The category $\mathbf{Grp} // \mathcal{V}$ has as

objects: \mathcal{V} -weighted sets $(X, |-|)$ with a group structure such that

$$k \leq |0|, \quad |x| \otimes |y| \leq |x - y|;$$

morphisms live in both, $\mathbf{Set} // \mathcal{V}$ and \mathbf{Grp} .

Obtain:

$$\mathbf{Grp} // \mathcal{V} \xleftrightarrow{\cong} \mathcal{V}\text{-MetGrp}$$

$$X \longmapsto X(x, y) = |x - y|$$

$$|x| = X(x, 0) \longleftarrow X$$

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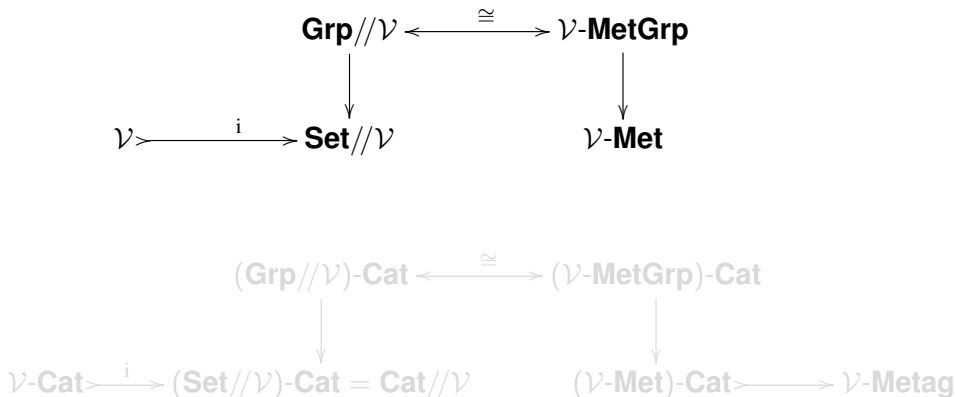
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7.9 \mathcal{V} -weighted cats vs \mathcal{V} -metrically enriched cats vs \mathcal{V} -metagories



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$$\begin{array}{ccc}
 \mathbf{Grp} // \mathcal{V} & \xleftarrow{\cong} & \mathcal{V}\text{-MetGrp} \\
 \downarrow & & \downarrow \\
 \mathcal{V} \xrightarrow{i} \mathbf{Set} // \mathcal{V} & & \mathcal{V}\text{-Met}
 \end{array}$$

$$\begin{array}{ccc}
 (\mathbf{Grp} // \mathcal{V})\text{-Cat} & \xleftarrow{\cong} & (\mathcal{V}\text{-MetGrp})\text{-Cat} \\
 \downarrow & & \downarrow \\
 \mathcal{V}\text{-Cat} \xrightarrow{i} (\mathbf{Set} // \mathcal{V})\text{-Cat} = \mathbf{Cat} // \mathcal{V} & & (\mathcal{V}\text{-Met})\text{-Cat} \xrightarrow{\quad} \mathcal{V}\text{-Metag}
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7.10.1 Principal references, I

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Rendiconti del Seminario Matematico e Fisico di Milano 43:135–166, 1973.

Republished in *Reprints in Theory and Applications of Categories* 1, 2002.

(This paper not only introduces metric spaces as small categories enriched in the extended real half-line, considered as a symmetric monoidal-closed category under addition, but it is also the birthplace of *normed categories*, introduced as categories enriched in a certain symmetric monoidal category of *normed sets*.)

D. Hofmann, G.J. Seal, W.T. (eds.): *Monoidal Topology—A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

(This book studies the category $(\mathbb{T}, \mathcal{V})\text{-Cat}$, for a **Set**-monad \mathbb{T} which is assumed to interact with the \mathcal{V} -presheaf monad $\mathcal{P}_{\mathcal{V}}$ via a lax distributive law; for \mathbb{T} the identity monad on **Set**, one obtains the category $\mathcal{V}\text{-Cat}$ as considered in these lectures.)

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