Semi-Reflective Extensions of Dualities: A New Approach to the Fedorchuk Duality

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Standard: (dual) adjunctions give (dual) equivalences

$$T : \mathcal{A}^{(\mathrm{op})} \rightleftarrows \mathcal{B} : S$$

$$\varepsilon_A : A \longrightarrow STA \qquad \eta_B : B \longrightarrow TSB$$

$$T\varepsilon_A \cdot \eta_{TA} = 1_{TA} \qquad S\eta_B \cdot \varepsilon_{SB} = 1_{SB}$$

$$\mathrm{Fix}(\varepsilon) = \{A \,|\, \varepsilon_A \text{ iso}\} \qquad \mathrm{Fix}(\eta) = \{B \,|\, \eta_B \text{ iso}\}$$

$$T' : \mathrm{Fix}(\varepsilon)^{(\mathrm{op})} \rightleftarrows \mathrm{Fix}(\eta) : S'$$

Note: In what follows, we will suppress "op" throughout this talk.

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Conversely: How to extend a given duality naturally?

Given a dual equivalence

$$T : \mathcal{A} \rightleftarrows \mathcal{B} : S$$

with $\mathcal{B}$ a full subcategory of a category $\mathcal{C}$:
Find a natural description of a full extension category $\mathcal{D}$ of $\mathcal{A}$ and a dual equivalence

$$\tilde{T} : \mathcal{D} \rightleftarrows \mathcal{C} : \tilde{S}$$

extending the given one:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\tilde{T}} & \mathcal{C} \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle I} \\ \mathcal{A} & \xrightarrow{T} & \mathcal{B} \end{array} \qquad \begin{array}{ccc} \mathcal{D} & \xleftarrow{\tilde{S}} & \mathcal{C} \\ {\scriptstyle J}\uparrow & \cong & \uparrow{\scriptstyle I} \\ \mathcal{A} & \xleftarrow{S} & \mathcal{B} \end{array}$$

Challenge: describe such $\mathcal{D}$ and the extended duality naturally!

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Challenge: describe such $\mathcal{D}$ and the extended duality naturally!

$$\hom(-,2) : \textbf{Boole} \rightleftarrows \textbf{Set} : \hom(-,2) = \mathrm{P}$$

induces the ultrafilter monad

$$\mathrm{Ult}(X) = \textbf{Boole}(\mathrm{P}X, 2)$$

on **Set** whose Eilenberg-Moore category is **CHaus** (Manes 1967).
Get the (dual) comparison adjunction

$$\mathrm{Stone} : \textbf{Boole} \rightleftarrows \textbf{Set}^{\mathrm{Ult}} \cong \textbf{CHaus} : \mathrm{CO}$$

By inspection:

$$\mathrm{Fix}(\varepsilon) = \textbf{Boole} \rightleftarrows \textbf{Stone} = \textbf{ZDCHaus} = \mathrm{Fix}(\eta)$$

Voila! [ZD = zero-dimensional: the clopens form a base]

Warm-up: Stone, via [Porst-T 1991, Richard Garner]

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de Vries extends a restricted Stone: next talk, Dimov!

de Vries 1962:

$$\begin{array}{ccc} \textbf{deVries} & \xrightarrow{\simeq} & \textbf{CHaus} \\ \uparrow & & \uparrow \\ \textbf{CBoole} & \xrightarrow{\simeq} & \textbf{EDCHaus} \end{array}$$

Objects in **deVries** are Boolean algebras with structure $\surd$
and morphisms are maps that behave well w.r.t. the structure $\surd$

But: morphism composition in **deVries** does NOT proceed as in **Set**, which makes the category a bit cumbersome to deal with!

[extremally disconnected: closure of an open is open]

The Fedorchuk extension of the restricted Stone

Fedorchuk 1973:

$|\mathbf{Fedor}| = |\mathbf{deVries}|$, but take fewer morphisms to obtain a duality

$$\mathbf{Fedor} \rightleftarrows \mathbf{CHaus}_{\mathrm{qop}}$$

$[f : X \longrightarrow Y$ *quasi-open* $:\Longleftrightarrow \forall U \subseteq X$ open: $(\mathrm{int}\, f(U) = \emptyset \Longrightarrow U = \emptyset)]$

Dimov 2009:
Stone restricts to $\mathbf{Boole}_{\mathrm{sup}} \simeq \mathbf{Stone}_{\mathrm{qop}}$, and further:

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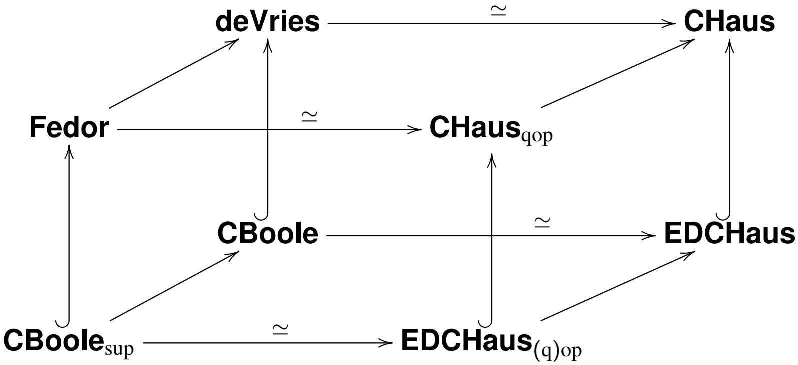
Viewing deVries and Fedorchuk as Stone extensions



Let's understand the top as a categorical extension of the bottom!
Front face: now! Back face: next talk!

de Vries representation of compact Hausdorff spaces

- a compact Hausdorff space X is determined by $(\mathrm{RC}(X), \ll)$, with ($\mathrm{RC}=$ regular closed) and ($F \ll G \Leftrightarrow F \subseteq \mathrm{int}\, G$)
- these pairs are algebraically described as *de Vries algebras* $(A, \ll)$: A complete Boolean algebra, axioms for the relation $\ll$
- (Bezhanishvili 2010) equivalently as $(A, p : \mathrm{Stone}(A) \longrightarrow X)$ where p is a projective cover of a compact Hausdorff space X *i.e.* $\mathrm{Stone}(A)$ is *the Gleason cover / the absolute* of X
- $p : Y \longrightarrow X$ projective cover: Y extremally disconnected and p is *irreducible*: $\forall F \subseteq X$ closed ($p(F) = Y \Longrightarrow F = X$); these maps are quasi-open!

Our strategy: Isolate the needed categorical properties of the class of irreducible maps in $\mathbf{CHaus}_{\mathrm{qop}}$ and build **Fedor** abstractly from them!

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A note of caution

While for a projective cover $p : \mathrm{Stone}(A) \longrightarrow X$, $\mathrm{Stone}(A)$, and therefore A, is determined by X in **CHaus** (up to isom.), there can be no functorial dependency of the domain on the codomain:

[Adámek, Herrlich, Rosický, T 2002]
In a category with projective covers (injective hulls) and a generator (cogenerator), the covering maps (injective embeddings) can never form a natural transformation, unless all objects of the category are projective (injective).

Applications: MacNeille compl., Gleason cover, algebraic closure, ...

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such that $\mathcal{B}$ is a full subactegory of $\mathcal{C}$, and a morphism class $\mathcal{P}$ in $\mathcal{C}$ such that

(P1) $\forall\ (p\colon B \longrightarrow C)\ \in \mathcal{P} :\ B \in |\mathcal{B}|$;

(P2) $\forall B \in |\mathcal{B}| : 1_B \in \mathcal{P}$;

(P3) $\mathcal{P} \cdot \mathrm{Iso}(\mathcal{B}) \subseteq \mathcal{P}$;

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(P5) for morphisms in $\mathcal{C}$, there is a functorial assignment

$$
\begin{CD}
B @. B' \\
@V{\mathcal{P}\ni p}VV @VV{p'\in\mathcal{P}}V \\
C @>>{v}> C'
\end{CD}
\qquad \mapsto \qquad
\begin{CD}
B @>{\hat{v}}>> B' \\
@V{p}VV @VV{p'}V \\
C @>>{v}> C'
\end{CD}
$$

$$
\begin{CD}
B @. B' @. B'' \\
@V{p}VV @VV{p'}V @VV{p''}V \\
C @>>{v}> C' @>>{w}> C''
\end{CD}
\qquad \mapsto \qquad
\begin{CD}
B @>{\widehat{w\circ v}}>{=\hat{w}\circ\hat{v}}> B'' \\
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\begin{CD}
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Characterization of (P1-5): $\mathcal{P}$ is a $(\mathcal{B},\mathcal{C})$-*covering class*

In the presence of $(\mathrm{P2})$, reformulate $(\mathrm{P4})$ as

$(\mathrm{P4}')$ $\forall C \in |\mathcal{C}|\ \exists\, (\pi_C : EC \longrightarrow C) \in \mathcal{P}$ *(with* $\pi_C = 1_C$ *when* $C \in |\mathcal{B}|$*)*.

PROPOSITION:

For $I : \mathcal{B} \hookrightarrow \mathcal{C}$ full and faithful, the following are equivalent:

- $\mathcal{B}$ admits a $(\mathcal{B},\mathcal{C})$-covering class;
- there are a functor $E : \mathcal{C} \longrightarrow \mathcal{B}$ and a natural transformation $\pi : IE \longrightarrow \mathrm{Id}_{\mathcal{C}}$, such that $\pi I : IEI \longrightarrow I$ is an isomorphism; E and π may actually be chosen to satisfy $EI = \mathrm{Id}_{\mathcal{B}}$ and $\pi I = 1_I$;
- I is *fully left semi-adjoint* (Medvedev 1974): there are $E : \mathcal{C} \longrightarrow \mathcal{B},\ \pi : IE \longrightarrow \mathrm{Id}_{\mathcal{C}},\ \sigma : \mathrm{Id}_{\mathcal{B}} \longrightarrow EI$ with $\pi I \cdot I\sigma = 1_I$ and σ iso.

Then: I left adjoint $\iff$ $IE\pi = \pi IE$ $\iff$ $E\pi \cdot \sigma E = 1_E$ $\iff$

$(\mathrm{P5}^*)$ In $(\mathrm{P5})$, the morphism $\hat{v}$ is uniquely determined by p, v, p'.

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Building the category $\mathcal{D}$ from the class $\mathcal{P}$ ...

- objects (A, p): $A \in |\mathcal{A}|$ and $p : TA \longrightarrow C$ in $\mathcal{P}$;
- morphisms $(\varphi, f) : (A, p) \longrightarrow (A', p')$: $\varphi : A \longrightarrow A'$ in $\mathcal{A}$ and $f : C' \longrightarrow C$ in $\mathcal{C}$, with $T\varphi = \hat{f}$:

$$\begin{array}{ccc} TA & \xleftarrow{T\varphi = \hat{f}} & TA' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ C & \xleftarrow[f]{} & C' \end{array}$$

- composition = horizontal pasting of diagrams: $(\varphi', f') \cdot (\varphi, f) = (\varphi' \cdot \varphi, f \cdot f')$
- identity morphisms: $1_{(A, p : TA \to C)} = (1_A, 1_C)$.

...containing $\mathcal{A}$ as a full semi-reflective subcategory ...

$$J : \mathcal{A} \hookrightarrow \mathcal{D}$$
$$(\varphi : A \to A') \longmapsto ((\varphi, T\varphi) : (A, 1_{TA}) \to (A', 1_{TA'}))$$

$$\mathcal{A} \longleftarrow \mathcal{D} : F$$
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$$\rho_{(A,p)} : (A, p) \longrightarrow JF(A, p)$$

$$\begin{array}{ccc} TA & \xleftarrow{\hat{p} = T\iota_p} & TA \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle 1_{TA}} \\ C & \xleftarrow[p]{} & TA \end{array}$$

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$$\begin{array}{ccc} TA & \xleftarrow{\hat{p} = T\iota_p} & TA \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle 1_{TA}} \\ C & \xleftarrow[p]{} & TA \end{array}$$

... and allowing for the extension of the given duality

$$\tilde{T} : \mathcal{D} \longrightarrow \mathcal{C}$$

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The augmented Extension Theorem

If (P4) may be strengthened to

(P4*) $\forall C \in |\mathcal{C}|\ \exists\ (\pi_C : EC \longrightarrow C) \in \mathcal{P}$ *rigid*: $(\forall \alpha$ iso: $\pi \cdot \alpha = \pi \Rightarrow \alpha = 1)$,

then there are

- a dual equivalence $\tilde{T}\colon \mathcal{D} \longleftrightarrow \mathcal{C} :\tilde{S}$, with natural isomorphisms $\tilde{\eta} : \mathrm{Id}_{\mathcal{C}} \longrightarrow \tilde{T}\tilde{S}$ and $\tilde{\varepsilon} : \mathrm{Id}_{\mathcal{D}} \longrightarrow \tilde{S}\tilde{T}$ satisfying the triangular identities
- and natural isomorphisms $\beta : TF \longrightarrow E\tilde{T}$ and $\gamma : JS \longrightarrow \tilde{S}I$

satisfying the following identities:

(1) $\tilde{T}J = IT$ and $F\tilde{S} = SE$;

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$\mathcal{P} = \{$irreducible maps with domain in $\mathcal{B}\}$. Must confirm (P1–5)!

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$|\mathbf{Fedor}| = |\mathbf{deVries}| \ni (A, \ll)$: A compl. Boolean alg., relation $\ll$ s.th.

- $a \ll b \Longrightarrow a \leq b$.
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- $a \leq b \ll c \leq d \Longrightarrow a \ll d$
- $a \ll c,\ b \ll c \Longrightarrow a \vee b \ll c$
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Morphisms in **Fedor**: Boolean homs preserving sups and $\ll$
Role model: $\mathrm{RC}(X)$ with $(F \ll G \iff F \subseteq \mathrm{int}G)$
Equivalent axiomatization as *normal contact algebras* $(A, \asymp)$

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Wanted: (covariant) equivalence

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Key step:
For a complete Boolean algebra A, define a bijective correspondence

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$$u \approx v :\iff \forall a \in u,\ b \in v : a \asymp b$$

Finally consider

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