Topological Vistas from Considering Spaces as Small Categories

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Algebra, Topology and Their Interactions

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Overview

- Preliminaries on small categories
- From preordered sets to metric spaces to small quantale-enriched categories
- A step-by-step approach to the ultrafilter convergence axiomatization of a topology
- Small (monad,quantale)-enriched categories
- A fundamental adjunction
- Equationally defined topological properties for objects and morphisms
- Trading convergence relations for closure operations
- Comparison with the internal-category approach
- Problems, projects, references

Monoids vs (Pre)Orders

Monoids:	(Pre)Orders:
(A, i, m)	$A \subseteq X \times X$ (write $x \leq y$ for $(x, y) \in A$)
$i : 1 \longrightarrow A$ $m : A \times A \longrightarrow A$	$x \leq x$ $x \leq y \;\&\; y \leq z \Longrightarrow x \leq z$
subject to two unity axioms and the associativity axiom	subject to NO further conditions

Both types of structures are (extreme) examples of small categories:

$\mathrm{ob}A = 1 = \{*\}$	$\mathrm{ob}A = X$
$\mathrm{mor}A = A$	$\hom_A(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ \emptyset & \text{else} \end{cases}$

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Two presentations of the notion of (small) category

$$A \xrightarrow{(d,c)} X \times X \quad \Longleftrightarrow \quad X \times X \xrightarrow{\hom_A} \mathrm{Set}$$

$$A \cong \coprod_{x,y \in X} \hom_A(x,y) \quad \Longleftrightarrow \quad \hom_A(x,y) = (d,c)^{-1}(x,y)$$

$$X \xrightarrow{i} A \quad \Longleftrightarrow \quad 1 \xrightarrow{i_x} \hom_A(x,x)$$

$$A \times_X A \xrightarrow{m} A \quad \Longleftrightarrow \quad \hom_A(x,y) \times \hom_A(y,z) \xrightarrow{m_{x,y,z}} \hom_A(x,z)$$

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A (with X, d, c, i, m) category
internal to a category $\mathcal{C}$
with pullbacks,
rather than just $\mathcal{C} = \textbf{Set}$

A (with $\hom_A = A(-,-), i, m$) category
enriched in a category $(\mathcal{V}, \otimes, \mathrm{k})$ that is
(symmetric) monoidal (closed),
rather than just $\mathcal{V} = (\textbf{Set}, \times, 1)$

With the internal/enriched notions of functor we obtain the categories

$\mathsf{Cat}(\mathcal{C})$ $\qquad\qquad$ $\mathcal{V}$-**Cat** .

We may in particular talk about *monoid- or group-objects in* $\mathcal{C}$, as special category objects in $\mathcal{C}$, such as group objects in **Top**, a.k.a. topological groups: $\mathsf{Grp}(\textbf{Top}) = \textbf{TopGrp}$.

On the enriched side, we have in particular the one-object $\mathcal{V}$*-enriched categories*, i.e, $\mathcal{V}$*-monoids*. For instance, for $\mathcal{V} = (\textbf{AbGrp}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, we obtain (unital) rings: $\mathcal{V}$-**Mon** $=$ **Rng** .

A glimpse at Internal Category Theory vs Enriched Category Theory

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Lawvere 1973: From ordered sets to metric spaces

Ord $a : X \times X \to 2 = (\{\bot, \top\}, \Rightarrow, \&, \top)$ $\quad \top \Rightarrow a(x,x)$

$a(x, y \;\&\; a(y,z) \Rightarrow a(x,z)$

$f : (X, a) \to (Y, b)$ $\quad a(x,x') \Rightarrow b(fx, fx')$

Met $d : X \times X \to [0,\infty] = ([0,\infty], \geq, +, 0)$ $\quad 0 \geq d(x,x)$

$d(x,y) + d(y,z) \geq d(x,z)$

$f : (X, d) \to (Y, e)$ $\quad d(x,x') \geq e(fx, fx')$

Note: $(2, \&, \top) \in \mathrm{CMon}(\mathbf{Ord})$ Actually: $(2, \&, \top) \in \mathrm{CMon}(\mathbf{Sup})$

$([0,\infty], +, 0) \in \mathrm{CMon}(\mathbf{Ord})$ $\quad ([0,\infty], +, 0) \in \mathrm{CMon}(\mathbf{Sup})$

where **Sup** is the *monoidal* category of complete lattices and $\bigvee$-preserving maps, since the above structure maps *a* and *d* actually preserve $\bigvee$ in each variable.

Quantales – half of the syntax needed for Monoidal Topology

V unital and (for convenience) commutative *quantale*

= complete lattice with a commutative monoid structure, $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$, such that

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a commutative monoid in the cocomplete symmetric monoidal-closed category **Sup**

Some examples:

- $\mathsf{V} = 2$ with $u \otimes v = u \,\&\, v$, $\quad \mathsf{k} = \top$ (Boolean 2-chain)
- $\mathsf{V} = [0, \infty]$ with $u \otimes v = u + v$, $\quad \mathsf{k} = 0$ (Lawvere quantale)
- V any frame with $u \otimes v = u \wedge v$, $\quad \mathsf{k} = \top$ (a cartesian quantale)
- $\mathsf{V} = 2^M$, for any commutative monoid M (free quantale over M), with $A \otimes B = \{\alpha \cdot \beta \mid \alpha \in A, \ \beta \in B\}$, $\quad \mathsf{k} = \{\eta\}$, η with neutral in M

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One more quantale: {distance distribution functions}

$$[0,\infty] = ([0,\infty], \geq, +, 0) \;\cong\; ([0,1], \leq, \cdot, 1) = [0,1]$$

$$\Delta \ni \varphi : [0,\infty] \to [0,1] \qquad \varphi(\beta) = \sup_{\alpha<\beta} \varphi(\alpha)$$

$$(\varphi \otimes \psi)(\gamma) = \sup_{\alpha+\beta=\gamma} \varphi(\alpha)\cdot\psi(\beta) \qquad \kappa(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0. \end{cases}$$

The two interval quantales are fully embedded into the quantale Δ, via

$$[0,\infty] \xrightarrow{\;\sigma\;} \Delta \xleftarrow{\;\tau\;} [0,1]$$

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$$\varphi = \sup_{\alpha} \sigma(\alpha) \otimes \tau(\varphi(\alpha)) :$$ Δ as a coproduct in **Qnt** !

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Quantale-enriched categories

$(\mathsf{V}, \otimes, \mathsf{k})$ (commutative) quantale

V-*relation*	$r : X \nrightarrow Y$	$r : X \times Y \to \mathsf{V}$	
	$s \cdot r : X \nrightarrow Z$	$s \cdot r(x,z) = \bigvee_y r(x,y) \otimes s(y,z)$	$(s : Y \nrightarrow Z)$
	$r^\circ : Y \nrightarrow X$	$r^\circ(y,x) = r(x,y)$	
V-*graph*	$f_\circ : X \nrightarrow Y$	$f_\circ(x,y) = \begin{cases} \mathsf{k} & \text{if } f(x) = y \\ \bot & \text{else} \end{cases}$	
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V-*category*	$(X, a : X \nrightarrow X)$	$\mathsf{k} \leq a(x,x)$	$1_X^\circ \leq a$
		$a(x,y) \otimes a(y,z) \leq a(x,z)$	$a \cdot a \leq a$
V-*functor*	$(X,a) \xrightarrow{f} (Y,b)$	$a(x,y) \leq b(fx, fy)$	$a \leq f^\circ \cdot b \cdot f_\circ$

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Some examples

$V = 1$	V-**Cat** = **Set**	objects = sets X
2	**Ord**	(pre)ordered sets $(X, \leq)$
$[0, \infty]$ $\cong [0, 1]$	**Met** $\cong$ **ProbOrd**	(generalized) metric spaces (X, d) probabilistic (pre)ordered sets $(X, p : X \times X \to [0, 1])$ (think $p(x, y)$ = probability of $x \leq y$, with $\leq$ a random order on X)
2^M	M-**Ord**	M-scaled (pre)ordered sets $(X, (\leq_\alpha)_{\alpha \in M})$ $x \leq_\eta x \quad (\eta \text{ neutral in } M); \qquad x \leq_\alpha y,\ y \leq_\beta z \Rightarrow x \leq_{\alpha \cdot \beta} z$
Δ	**ProbMet**	probabilistic (geneneralized) metric spaces $(X, (p_\alpha)_{\alpha \geq 0})$ (think $p_\alpha(x, y)$ = prob'ty of $d(x, y) < \alpha$, with d a random metric) $p_0(x, x) = 0, \quad (\alpha > 0 \Longrightarrow p_\alpha(x, x) = 1)$ $\alpha + \beta \leq \gamma \Longrightarrow p_\alpha(x, y) \cdot p_\beta(y, z) \leq p_\gamma(x, z)$

Another type of generalized preordered sets: **MultiOrd**

$\mathrm{L}X = \{\overline{x} = (x_1, ..., x_n) \mid n \geq 0, x_i \in X\} = \{$ lists of elements in $X\}$

A *multi-orderd set* X comes with a relation $\leq : \mathrm{L}X \nrightarrow X$ satisfying

(R) for all $x \in X$: $(x) \leq x$

(T) for all $\mathcal{X} = (\overline{x_1}, ..., \overline{x_m}) \in \mathrm{LL}X,\ \overline{y} = (y_1, ..., y_m) \in \mathrm{L}X,\ z \in X$:

$$\begin{array}{c} \overline{x_1} \leq y_1 \\ \vdots \\ \overline{x_m} \leq y_m \end{array}$$

$$\mathcal{X} \leq \overline{y} \quad \& \quad \overline{y} \leq z \quad \Longrightarrow \quad \Sigma\mathcal{X} = (x_{1,1}, x_{1,2}, ..., x_{m,1}, ..., x_{m,n_m}) \leq z$$

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Yet another type of generalized preordered sets: **Cls**

$\mathrm{P}X$ = powerset of X: Trade (finite) lists of elements of X for subsets of X

Then we consider a ("support") relation $\vdash$: $\mathrm{P}X \nrightarrow X$ satisfying

(R) for all $x \in X$: $\{x\} \vdash x$

(T) for all $\mathcal{A} \in \mathrm{PP}X, B \in \mathrm{P}X, z \in X$:

$$\underbrace{\forall\, y \in B\ \exists\, A \in \mathcal{A} :\ A \vdash y}_{\mathcal{A} \vdash B} \quad \& \quad B \vdash z \implies \bigcup \mathcal{A} \vdash z$$

Transcribing $A \vdash y$ as $y \in cA$, these conditions become equivalent to

(R') for all $A \in \mathrm{P}X$: $A \subseteq cA$

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Topological spaces as generalized (pre)ordered sets?

$\mathrm{U}X = \{$ ultrafilters on $X\}$: Consider a relation $\rightsquigarrow: \mathrm{U}X \mapsto X$ satisfying

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$$\underbrace{\forall \mathcal{A} \in \mathfrak{X}, B \in \mathfrak{y}\ \exists \mathfrak{x} \in \mathcal{A}, y \in B : \mathfrak{x} \rightsquigarrow y}_{\mathfrak{X} \rightsquigarrow \mathfrak{y}} \quad \& \quad \mathfrak{y} \rightsquigarrow z \quad \Longrightarrow \quad \underbrace{A \in \Sigma\mathfrak{X} \Longleftrightarrow \{\mathfrak{x} \mid A \in \mathfrak{x}\} \in \mathfrak{X}}_{\Sigma\mathfrak{X} \rightsquigarrow z}$$

Pictured as sequences:

$$\mathfrak{x}_1 = (x_{1,1}, \ldots) \quad \rightsquigarrow \quad y_1$$

$$\vdots \qquad\qquad \vdots$$

$$\mathfrak{x}_n = (x_{n,1}, \ldots) \quad \rightsquigarrow \quad y_n$$

$$\vdots \qquad\qquad \vdots$$

$$\mathfrak{X} = (\mathfrak{x}_1, \ldots..) \quad \rightsquigarrow \quad \mathfrak{y} \quad \rightsquigarrow z \quad \Rightarrow \quad \Sigma\mathfrak{X} = (x_{1,1}, .., x_{n,n}, \ldots) \quad \rightsquigarrow \quad z$$

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$$\mathfrak{X} = (\mathfrak{x}_1, \ldots..) \quad \rightsquigarrow \quad \mathfrak{y} \quad \rightsquigarrow z \quad \Rightarrow \quad \Sigma\mathfrak{X} = (x_{1,1}, .., x_{n,n}, \ldots) \quad \rightsquigarrow \quad z$$

Topological spaces as generalized (pre)ordered sets?

$\mathrm{U}X = \{$ ultrafilters on $X\}$: Consider a relation $\rightsquigarrow: \mathrm{U}X \nrightarrow X$ satisfying

(R) for all $x \in X$: $\dot{x} \rightsquigarrow x$

(T) for all $\mathfrak{X} \in \mathrm{UU}X, \mathfrak{y} \in \mathrm{U}X, z \in X$:

$$\underbrace{\forall \mathcal{A} \in \mathfrak{X}, B \in \mathfrak{y}\ \exists \mathfrak{x} \in \mathcal{A}, y \in B : \mathfrak{x} \rightsquigarrow y}_{\mathfrak{X} \rightsquigarrow \mathfrak{y}} \quad \& \quad \mathfrak{y} \rightsquigarrow z \implies \underbrace{A \in \Sigma\mathfrak{X} \iff \{\mathfrak{x} \mid A \in \mathfrak{x}\} \in \mathfrak{X}}_{\Sigma\mathfrak{X} \rightsquigarrow z}$$

Pictured as sequences:

$$\begin{array}{lcl} \mathfrak{x}_1 = (x_{1,1}, \ldots) & \rightsquigarrow & y_1 \\ \vdots & & \vdots \\ \mathfrak{x}_n = (x_{n,1}, \ldots) & \rightsquigarrow & y_n \\ \vdots & & \vdots \\ \mathfrak{X} = (\mathfrak{x}_1, \ldots..) & \rightsquigarrow & \mathfrak{y} \quad \rightsquigarrow z \quad \Rightarrow \quad \Sigma\mathfrak{X} = (x_{1,1}, .., x_{n,n}, \ldots) \quad \rightsquigarrow \quad z \end{array}$$

Yes: Manes 1967 → Barr 1970 → Wyler 1995

$f : X \to Y$ continuous : $\mathfrak{x} \rightsquigarrow x \Longrightarrow f[\mathfrak{x}] \rightsquigarrow fx$ where $(B \in f[\mathfrak{x}] \Longleftrightarrow f^{-1}B \in \mathfrak{x})$

Retrieving the topology from the two ultrafilter convergence axioms: define

$$x \in cA \quad \Longleftrightarrow \quad \exists\, \mathfrak{x} \in \mathrm{U}A \subseteq \mathrm{U}X : \mathfrak{x} \rightsquigarrow x$$

Then:

(R') $A \subseteq cA$

(T') $B \subseteq cA \Longrightarrow cB \subseteq cA$

(FA) $c\emptyset \subseteq \emptyset$ and $c(A \cup B) \subseteq cA \cup cB$

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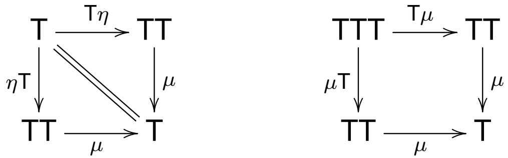
Monads–the other half needed for the syntax of Monoidal Topology

$(\mathrm{L}, (-), \Sigma)$, $(\mathrm{P}, \{-\}, \bigcup)$, $(\mathrm{U}, \dot{-}, \Sigma)$ are examples of *monads* (T, η, μ) on **Set**:

A monad (on **Set**) is a monoid in the monoidal category $([\mathbf{Set}, \mathbf{Set}], \circ, \mathrm{Id}_{\mathbf{Set}})$, that is:

$\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ functor with natural transform's $\eta : \mathrm{Id}_{\mathbf{Set}} \to \mathsf{T}$ and $\mu : \mathsf{TT} \to \mathsf{T}$ satisfying



Every adjunction $F \dashv G : \mathcal{A} \longrightarrow \mathbf{Set}$ with unit $\eta : \mathrm{Id}_{\mathbf{Set}} \to GF$, counit $\varepsilon : FG \to \mathrm{Id}_{\mathcal{A}}$ induces $(GF, \eta, G\varepsilon F)$.

Every monad T is induced by a "largest" adjunction: $\mathcal{A} = \mathbf{Set}^{\mathsf{T}}$ Eilenberg-Moore cat. of T.

$\mathbf{Set}^{\mathrm{L}} = \mathbf{Mon}$ $\quad$ $\mathbf{Set}^{\mathrm{P}} = \mathbf{Sup}$ $\quad$ $\mathbf{Set}^{\mathrm{U}} = \mathbf{CompHaus}$

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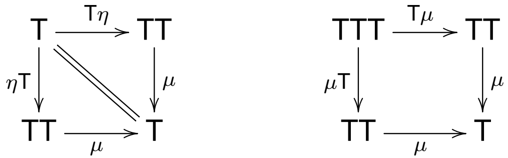

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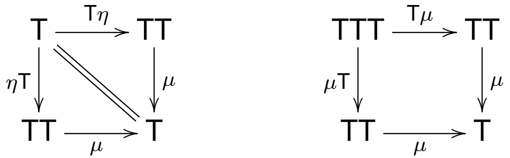

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Plus:the quantale V and the monad T must be linked

The **Ord**-enriched category V-**Rel**: same objects as **Set**; hom-sets ordered pointwise:

$$(r : X \nrightarrow Y) \le (r' : X \nrightarrow Y) \iff \forall x \in X, y \in Y : r(x,y) \le r'(x,y)$$

$$\mathbf{Set} \xrightarrow{(-)_\circ} \text{V-}\mathbf{Rel} \xleftarrow{(-)^{\mathrm{op}}} \mathbf{Set}^{\mathrm{op}}$$

T comes with a *lax extension* $\hat{\mathrm{T}}$ from maps to V-relations $(r : X \nrightarrow Y) \longmapsto (\hat{\mathrm{T}}r : \mathrm{T}X \nrightarrow \mathrm{T}Y)$:

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(T, V)-Cat $(X, a : \mathsf{T}X {\to\!\!\!\!\!\mapsto} X)$

$\mathsf{k} \leq a(\eta_X(x), x) \quad (x, z \in X)$

$\hat{\mathsf{T}}a(\mathfrak{X}, \mathfrak{y}) \otimes a(\mathfrak{y}, z) \leq a(\mu_X(\mathfrak{X}), z) \quad (\mathfrak{X} \in \mathsf{TT}X, \mathfrak{y} \in \mathsf{T}X)$

$(X, a) \xrightarrow{f} (Y, b) \quad a(\mathfrak{x}, x) \leq b(\mathsf{T}f(\mathfrak{x}), fx) \quad (\mathfrak{x} \in \mathsf{T}X)$

Equivalently: $\eta_X^\circ \leq a$

$a \circ a \leq a$ (Kleisli convolution)

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Kleisli convolution for $r : \mathsf{T}X {\to\!\!\!\!\!\mapsto} Y,\ s : \mathsf{T}Y {\to\!\!\!\!\!\mapsto} Z$: $\quad s \circ r := s \cdot \hat{\mathsf{T}}r \cdot \mathsf{m}_X^\circ : \mathsf{T}X {\to\!\!\!\!\!\mapsto} Z$

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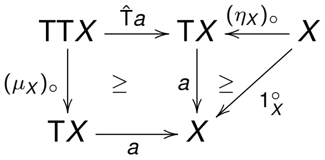

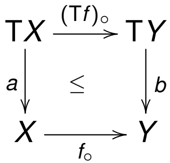

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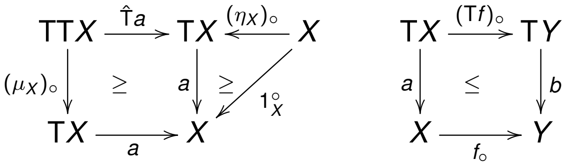

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Examples for $(\mathbb{T}, \mathsf{V})$-**Cat**

$$(\mathrm{Id}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cat}$$

$\mathbb{T}$	$(\mathbb{T}, 2)$-**Cat**	$(\mathbb{T}, [0, \infty])$-**Cat**	$(\mathbb{T}, \Delta)$-**Cat**
Id	**Ord**	**Met**	**ProbMet**
L	**MultiOrd**	?	.
P	**Cls**	??	..
U	**Top**	???	...

Let's clarify ?, ?? , ??? and leave ., .., ... to the audience (sorry!)

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$\mathbb{P}$	**Cls**	??	..
$\mathbb{U}$	**Top**	???	...

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Trading 2 for $[0, \infty]$

T	Reflexivity	Transitivity = ∇-Inequality	$(\mathrm{T}, [0,\infty])$-**Cat**
$\mathbb{L}$	$0 \geq d((x), x)$	$\underbrace{d(\mathcal{X}, \overline{y})}_{} + d(\overline{y}, z) \geq d(\Sigma\mathcal{X}, z)$ $\hat{\mathbb{L}}d(\mathcal{X}, \overline{y}) = d(\overline{x_1}, y_1) + \ldots + d(\overline{x}_n, y_n)$	**MultiMet**
$\mathbb{P}$	$0 \geq d(\{x\}, x)$	$\underbrace{d(\mathcal{A}, B)}_{} + d(B, z) \geq d(\bigcup A, z)$ $\hat{\mathbb{P}}d(\mathcal{A}, B) = \sup_{y\in B} \inf_{A\in\mathcal{A}} d(A, y)$	**PrApp**
$\Leftrightarrow$	$0 \geq d(A, x)$ if $x \in A$	$\sup_{y\in B} d(A, y) + d(B, z) \geq d(A, z)$	Hausdorff distance
$\mathbb{U}$	$0 \geq d(\dot{x}, x)$	$\underbrace{d(\mathfrak{X}, \mathfrak{y})}_{} + d(\mathfrak{y}, z) \geq d(\Sigma\mathfrak{X}, z)$ $\hat{\mathbb{U}}d(\mathfrak{X}, \overline{y}) = \sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathfrak{y}} \inf_{\mathfrak{x}\in\mathcal{A}, y\in B} d(\mathfrak{x}, y)$	$\cong$ **App** Clem.-Hofm. 2003
$\Leftrightarrow$	Pre-approach + finite additivity	$d(\emptyset, x) \geq \infty$ $d(A \cup B, x) \geq \min\{d(A, x), d(B, x)\}$	Lowen 1989

Some properties and results

- (T,V)-**Cat** $\longrightarrow$ **Set** is topological and, hence, has both adjoints; the initial structure a on X of a family $(f_i : X \to (Y_i, b_i))_{\in I}$ is given by $a(\mathfrak{x}, x) = \bigwedge_{i \in I} b(\hat{\mathsf{T}} f_i(\mathfrak{x}), f_i x)$;
- (T,V)-**Cat** is complete and cocomplete, with limits (colimits) of diagrams formed by initial (final) lifting of the limits (colimits) of the underlying **Set**-diagrams ;
- the formation of (T,V)-**Cat** is functorial in T (contra-) and V (co-variantly); example:

Theorem (Lowen-Vroegrijk 2008, Hofmann 2014)

For every **Set**-*monad* T, *laxly extended to* V-**Rel**, *one can find a monad* Π *(encoding both,* T *and* V*) , laxly extended to* **Rel** $= 2$-**Rel**, *such that* (T,V)-**Cat** $\cong (\Pi, 2)$-**Cat**.

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$$\begin{array}{ccc}
\mathbf{Top} \cong (\mathrm{U}, 2)\text{-}\mathbf{Cat} & \underset{\longleftarrow \;\bot}{\overset{\longleftarrow \;\bot}{\longrightarrow}} & (\mathrm{U}, [0,\infty])\text{-}\mathbf{Cat} \cong \mathbf{App} \\
\uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\
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For every **Set***-monad* $\mathbb{T}$*, laxly extended to* V-**Rel***, one can find a monad* Π *(encoding both,* $\mathbb{T}$ *and* V*) , laxly extended to* **Rel** $= 2$-**Rel***, such that* $(\mathbb{T},\mathsf{V})$-**Cat** $\cong (\Pi, 2)$-**Cat**.

A fundamental adjunction

The **Set**-monad $\mathbb{T}$ with its lax extension $\hat{\mathrm{T}}$ to V-**Rel** may be considered as a (KZ-)monad on V-**Cat** (T 2009): $\mathrm{T}(X, a_0 : X \nrightarrow X) = (\mathrm{T}X, \hat{\mathrm{T}}a_0 : \mathrm{T}X \nrightarrow \mathrm{T}X)$

If the Kleisli convolution is associative, then (Clementino-Hofmann 2009):

$$(X, a_0, \xi : \mathrm{T}X \to X) \longmapsto (X, a_0 \cdot \xi_\circ : \mathrm{T}X \nrightarrow X)$$

$$(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\rightleftarrows}} (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \qquad (\top)$$

$$(\mathrm{T}X, \underbrace{\hat{\mathrm{T}}a \cdot \mu_X^\circ}_{=: \hat{a} : \mathrm{T}X \nrightarrow \mathrm{T}X}, \mu_X) \longleftarrow (X, a : \mathrm{T}X \nrightarrow X)$$

In particular (Hofmann 2007): If the V-category $(\mathsf{V}, \mathrm{hom})$ has a *good* $\mathbb{T}$-structure ξ, then K makes V a $(\mathbb{T}, \mathsf{V})$-category, enables dualization, Yoneda embedding, ...

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A fundamental adjunction

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In particular (Hofmann 2007): If the V-category $(\mathrm{V}, \mathrm{hom})$ has a *good* $\mathbb{T}$-structure ξ, then K makes V a $(\mathbb{T}, \mathrm{V})$-category, enables dualization, Yoneda embedding, ...

$$(X, a) \longmapsto (X, a \cdot (\eta_X)_\circ)$$

$$(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\rightleftarrows}} (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \underset{A^\circ}{\overset{A_\circ}{\rightleftarrows}} \mathsf{V}\text{-}\mathbf{Cat} \qquad (\top,\ \top)$$

$$(X, \eta_X^\circ \cdot \hat{T} a_0) \longleftarrow\!\!\shortmid (X, a_0)$$

$\mathbb{T} = \mathbb{U},\ \mathsf{V} = 2$:

$$\mathbf{OrdCompHaus} \rightleftarrows \mathbf{Top} \rightleftarrows \mathbf{Ord} \qquad (\top,\ \top)$$

K topologizes $(X, \leq, \xi)$ by $(\mathfrak{x} \rightsquigarrow y \iff \xi(\mathfrak{x}) \leq y)$; $A_\circ$ puts (dual of) specialization order

M orders UX by $(\mathfrak{x} \leq \mathfrak{y} \iff \forall A \in \mathfrak{x}$ closed in $X : A \in \mathfrak{y})$; A° puts Alexandroff topology

$M \dashv K$ is a factor of the Eilenberg-Moore adjunction

$$(X,a) \longmapsto (X, a\cdot(\eta_X)_\circ)$$

$$(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\underset{\top}{\rightleftarrows}}} (\mathbb{T},\mathsf{V})\text{-}\mathbf{Cat} \underset{A^\circ}{\overset{A_\circ}{\underset{\top}{\rightleftarrows}}} \mathsf{V}\text{-}\mathbf{Cat}$$

$$(X, \eta_X^\circ \cdot \hat{\mathbb{T}}a_0) \longleftarrow\!\dashv (X, a_0)$$

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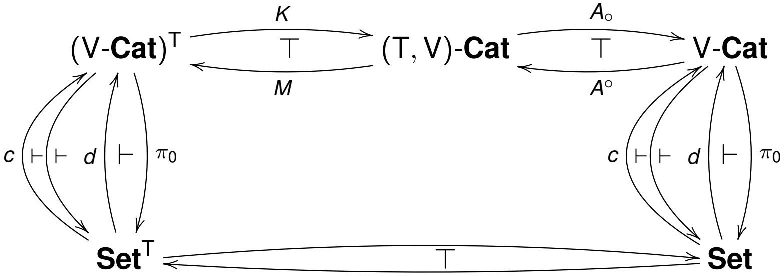
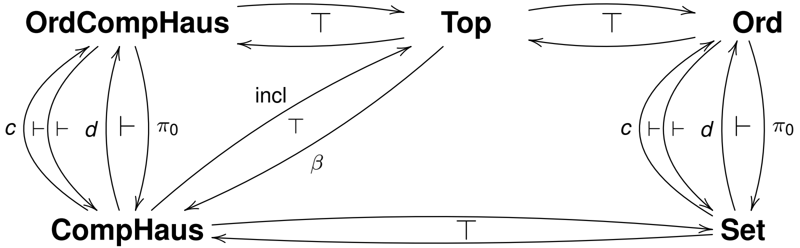
The greater picture (when $\mathbb{T}$ is flat and V integral)



The greater picture (when $\mathrm{T} = \mathrm{U}$ and $\mathsf{V} = 2$)



Note:
Up to now, we are able to justify the name "π_0" only when $X \in$ **Top** is normal;
that is: when X is normal, βX is homeomorphic to the space of connected components w.r.t. the order that is imposed on the space $\mathrm{U}X$ by the functor M.
(An elaborate) Exercise:
Trade 2 for $[0, \infty]$!

PART 2

Replacing inequalities by equalities: T_1-sep'tion, core compactness

Recall the two defining inequalities for a (T,V)-category $(X, a : \mathsf{T}X \nrightarrow X)$:

(R) $1_X \leq a \cdot (\eta_X)_\circ$ **T_1** : $1_X \geq a \cdot (\eta_X)_\circ$

$\mathsf{T} = \mathbb{U}, \mathsf{V} = 2$: $(\dot{x} \rightsquigarrow y \Rightarrow x = y)$

(T) $a \cdot \hat{\mathsf{T}}a \leq a \cdot (\mathsf{m}_X)_\circ$ **core compact:** $a \cdot \hat{\mathsf{T}}a \geq a \cdot (\mu_X)_\circ$

Pisani 1999: $\mathsf{T} = \mathbb{U}, \mathsf{V} = 2$: $\Sigma\mathfrak{X} \rightsquigarrow z \Rightarrow \exists \mathfrak{y}\, (\mathfrak{X} \rightsquigarrow \mathfrak{y} \rightsquigarrow z)$

$\iff \forall x \in B \subseteq X$ open $\exists A \subseteq X$ open $(x \in A \ll B)$

$\iff X$ exponentiable in **Top**

Note:

If we express (R) and (T) equivalently as $\eta_X^\circ \leq a$ and $a \circ a \leq a$ resp., and "strictify" these inequalities, *different* properties will emerge: discrete and no condition at all!

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Replacing inequalities by equalities: proper maps, open maps

Recall (two forms of) the defining inequality for a (T, V)-functor $f : (X, a) \to (Y, b)$:

$f_\circ \cdot a \le b \cdot (\mathsf{T}f)_\circ$ **proper:** $f_\circ \cdot a \ge b \cdot (\mathsf{T}f)_\circ$ $\quad \bigvee_{x \in f^{-1}y} a(\mathfrak{x}, x) \ge b(\mathsf{T}f(\mathfrak{x}), y))$

Manes 1974: $\mathsf{T} = \mathrm{U}, \mathsf{V} = 2:$

$$\begin{array}{ccc} \mathfrak{x} & \cdots\cdots\!> & x \\ \Big| & & \vdots \\ \mathrm{Tf}(\mathfrak{x}) & \longrightarrow & y \end{array}$$

$a \cdot (\mathsf{T}f)^\circ \le f^\circ \cdot b$ **open**: $a \cdot (\mathsf{T}f)^\circ \ge f^\circ \cdot b$ $\quad \bigvee_{\mathfrak{x} \in (\mathsf{T}f)^{-1}\mathfrak{y}} a(\mathfrak{x}, x) \ge b(\mathfrak{y}, f(x))$

Möbus 1981: $\mathsf{T} = \mathrm{U}, \mathsf{V} = 2:$

$$\begin{array}{ccc} \mathfrak{x} & \cdots\cdots\!> & x \\ \vdots & & \Big| \\ \mathfrak{y} & \longrightarrow & f(x) \end{array}$$

Some stability properties for proper and open maps

- Isomorphisms are proper/open
- Proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\Longrightarrow$ f proper/open
- $g \cdot f$ proper/open, f surjective $\Longrightarrow$ g proper/open
- f proper/open $\Longrightarrow$ every pullback of f is proper/open

Theorem (Tychonoff-Frolík-Bourbaki Theorem)

Let V be completely distributive. Then:

$$f_i : X_i \to Y_i \quad \text{proper} \;\; (i \in I) \Longrightarrow \prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \quad \text{proper}$$

Note that, by contrast (*not* by categorical dualization!), one has:

$$f_i : X_i \to Y_i \quad \text{open} \;\; (i \in I) \Longrightarrow \coprod_{i \in I} f_i : \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i \quad \text{open}$$

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The adjunction of structures: Hausdorff separation and compactness

NOTE: Graphs of maps are Lawverian maps in V-**Rel**; that is: $f_\circ \cdot f^\circ \leq 1^\circ$ and $1 \leq f^\circ \cdot f_\circ$.
Conversely, under some light assumptions on V (excluding 2^M, but none of the other quantales mentioned), one has: if $r \dashv s$ in V-**Rel**, then $r = f_\circ, s = f^\circ$, for a unique map f.

(X, a) **Hausdorff:** $a \cdot a^\circ \leq 1_X$ $\quad \bot < a(\mathfrak{z}, x) \otimes a(\mathfrak{z}, y) \Longrightarrow x = y \quad (x, y \in X, \mathfrak{z} \in TX)$

(X, a) **compact:** $1_{TX} \leq a^\circ \cdot a$ $\quad \mathrm{k} \leq \bigvee_{x \in X} a(\mathfrak{z}, x) \quad (\mathfrak{z} \in TX)$

Note: $(X, a) \to (1, \top)$ proper $\Longleftrightarrow$ (X, a) compact

Theorem (Manes, Lawvere, Clementino-Hofmann, T)

$$\mathbf{Set}^{\mathbb{T}} = (\mathrm{T}, \mathrm{V})\text{-}\mathbf{Cat}_{\mathrm{Comp}} \cap (T, V)\text{-}\mathbf{Cat}_{\mathrm{Haus}}$$

Proof:

$$(a \cdot a^\circ \leq 1_X \text{ and } 1_{TX} \leq a^\circ \cdot a) \Longleftrightarrow a \dashv a^\circ \Longleftrightarrow a \text{ is a map}$$

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Compact + Hausdorff = Eilenberg-Moore

$\mathbb{T}$	V	$(\mathbb{T}, \mathsf{V})$-**Cat**$_{\text{Comp}}$	$(\mathbb{T}, \mathsf{V})$-**Cat**$_{\text{Haus}}$
Id	2	**Ord**	**Set** $\cong$ {discretely ordered sets}
Id	$[0, \infty]$	**Met**	**Set** $\cong$ {discrete (gen'ed) metric spaces}
U	2	**Comp**	**Haus**
U	$[0, \infty]$	**App**$_{\text{0-Comp}}$	{approach spaces whose induced pseudotopology is Hausdorff}

Theorem (Tychonoff)

Let the quantale V be completely distributive. Then, if all $(\mathbb{T}, \mathsf{V})$*-categories* $X_i = (X_i, a_i)$ $(i \in I)$ *are compact, then also* $(X, a) = \prod_{i \in I} X_i$ *is compact.*

Proof (Schubert 2005): For all $\mathfrak{z} \in TX$:

$$\bigvee_{x \in X} a(\mathfrak{z}, x) = \bigvee_{x \in X} \bigwedge_{i \in I} a_i(Tp_i(\mathfrak{z}), p_i(x)) = \bigwedge_{i \in I} \bigvee_{x_i \in X_i} a_i(Tp_i(\mathfrak{z}), x_i) \geq k$$

Compact + Hausdorff = Eilenberg-Moore

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Kuratowski-Mrówka Theorem

Under mild hypotheses on $\mathbb{T}$ and V:

Theorem (Clementino-T 2007)

$f : (X, a) \to (Y, b)$ proper $\iff$ • f has compact fibres
• $\mathsf{T}f : (X, \hat{a}) \to (Y, \hat{b})$ proper

(in **Top, App, ...**) $\iff$ • f has compact fibres   Corollary
• f is closed

$\iff$ f is *stably* closed

X compact $\iff \forall Z : X \times Z \to Z$ closed (equ'ly: proper)

$(X \xrightarrow{f} Y)$ proper $\iff \forall (Z \to Y) : (X \times_Y Z \to Z)$ closed (proper)

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Normality and extremal disconnectedness

Reminder:

$X \in \mathbf{Top}$ normal $\iff$ disjoint closed sets have disjoint nbhds in X

X extremally disconnected $\iff$ closures of open sets are open in X

How do these properties fare in our setting? Recall:

$(\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \xrightarrow{M} \mathsf{V}\text{-}\mathbf{Cat}^{\mathbb{T}} \to \mathsf{V}\text{-}\mathbf{Cat}, \quad (X, a) \mapsto (TX, \hat{a}, \mu_X) \mapsto (TX, \hat{a}),$

$$\text{with } \hat{a} = (TX \xrightarrow{\mu_X^{\circ}} TTX \xrightarrow{\hat{T}a} TX)$$

For $\mathbb{T} = \mathbb{U}$, $\mathsf{V} = 2$ and $X \in \mathbf{Top}$, the functor provides UX with the order

$$\mathfrak{x} \le \mathfrak{y} \iff \forall A \subseteq X \text{ closed} : (A \in \mathfrak{x} \Rightarrow A \in \mathfrak{y})$$

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$$\text{with } \hat{a} = (\mathrm{T}X \xrightarrow{\mu_X^\circ} \mathrm{TT}X \xrightarrow{\hat{T}a} \mathrm{T}X)$$

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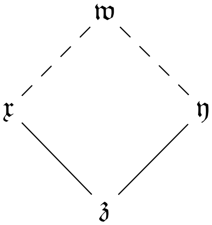
Normality and extremal disconnectedness are dual to each other!

$X \in \mathbf{Top}$ normal



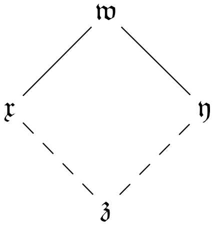
$(X, a) \in (\mathsf{T}, \mathsf{V})$-**Cat** **normal**

$$\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}$$

$$\Leftrightarrow$$

$(\mathsf{T}X, \hat{a})$ normal in V-**Cat**

X extremally disconnected



(X, a) **extremally disconnected**

$$\hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ$$

$$\Leftrightarrow$$

$(\mathsf{T}X, \hat{a}^\circ)$ normal in V-**Cat**

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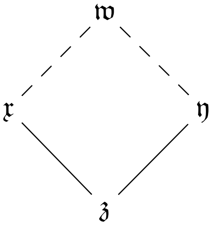
$X \in$ **Top** normal



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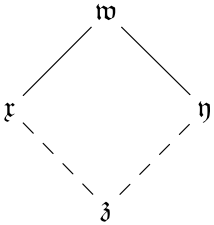

$(X, a) \in (\mathsf{T}, \mathsf{V})$-**Cat** **normal**

$$\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}$$

$$\Leftrightarrow$$

$(\mathsf{T}X, \hat{a})$ normal in V-**Cat**

(X, a) **extremally disconnected**

$$\hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ$$

$$\Leftrightarrow$$

$(\mathsf{T}X, \hat{a}^\circ)$ normal in V-**Cat**

Monoidal topology without convergence relations?

Cls	$c : PX \to 2^X$	(R)	$A \subseteq cA$
		(T)	$B \subseteq cA \Rightarrow cB \subseteq cA$
Top	c finitely additive:	(FA)	$c(A \cup B) \subseteq cA \cup cB$
			$c(\emptyset) \subseteq \emptyset$
		(C)	$f(c_X A) \subseteq c_Y(fA)$

[Seal 2009]

V-Cls	$c : PX \to V^X$	(R)	$\forall x \in A : k \leq (cA)(x)$
$= (\mathbb{P}, V)$-**Cat**		(T)	$(\bigwedge_{y \in B} (cA)(y)) \otimes (cB)(x) \leq (cA)(x)$

[Lai-T 2016]

V-Top	c finitely additive:	(FA)	$c(A \cup B)(x) \leq (cA)(x) \vee (cB)(x)$
			$(c\emptyset)(x) \leq \bot$
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$[0, \infty]$-**Top** $=:$ **App** [Lowen 1989] Δ-**Top** $=:$ **ProbApp** [Jäger 2015]

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How to reconcile closure and ultrafilter convergence?

For V completely distributive:

P and U interact via the V-relation $\varepsilon_X : \mathsf{P}X \nrightarrow \mathsf{U}X$ via $\varepsilon_X(A, \mathfrak{x}) = \begin{cases} \mathsf{k} & \text{if } A \in \mathfrak{x} \\ \bot & \text{else} \end{cases}$

$$A_\varepsilon : (\mathrm{U}, \mathsf{V})\text{-}\mathbf{Cat} \to (\mathrm{P}, \mathsf{V})\text{-}\mathbf{Cat}, \quad (X, a) \mapsto (X, c_a = a \circ \varepsilon_X)$$

$$(c_a A)(y) = \bigvee_{\mathfrak{x} \ni A} a(\mathfrak{x}, y)$$

A_ε has a right adjoint $(X, c) \mapsto (X, a_c)$, with

$$a_c(\mathfrak{x}, y) = \bigwedge_{A \in \mathfrak{x}} (cA)(y)$$

$(\mathbb{U}, \mathsf{V})$-**Cat** $\cong$ V-**Top**

Theorem (Lai-T 2016)

Let V *be completely distributive. Then:*

$$A_\varepsilon : (\mathbb{U}, \mathsf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{P}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cls}$$

is a full coreflective embedding; its image is V-**Top** $\cong (\mathbb{U}, \mathsf{V})$-**Cat**.

Corollary (Clementino-Hofmann 2003)

$$\mathbf{App} \cong (\mathbb{U}, [0, \infty])\text{-}\mathbf{Cat}$$

$$\mathbf{ProbApp} \cong (\Delta, [0, \infty]\text{-}\mathbf{Cat}$$

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Burroni 1971: How to internalize multicategories ...

$\mathcal{C}$ a category with pullbacks, $\mathsf{T} = (\mathsf{T}, \eta, \mu)$ any monad on $\mathcal{C}$. Define the category

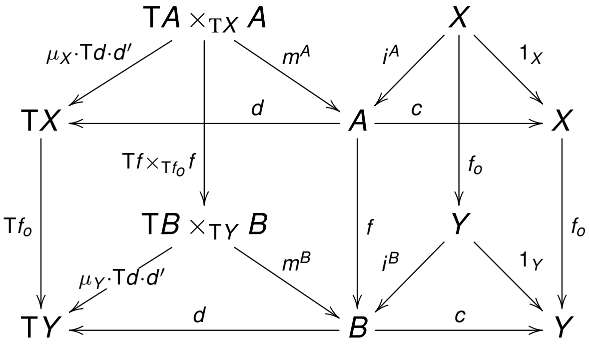
Cat(T)

Objects are (small) T-categories which are monoids in a bicategory of T-spans in $\mathcal{C}$; explicitly, they have an "object of objects" X and an "object of arrows" A, plus

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow^{\eta_X} & \downarrow{\scriptstyle i} & \searrow^{1_X} & \\
\mathsf{T}X & \xleftarrow{\;d\;} & A & \xrightarrow{\;c\;} & X \\
{\scriptstyle \mu_X \cdot \mathsf{T}d}\uparrow & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle c} \\
\mathsf{T}A & \xleftarrow{\;d'\;} & \mathsf{T}A \times_{\mathsf{T}X} A & \xrightarrow{\;c'\;} & A
\end{array}$$

subject to (somewhat cumbersome) unity and associativity laws.

Morphisms are T-functors which are *more* than morphisms of such monoids; rather:



One may, however, set up a double category of T-spans in $\mathcal{C}$ such that T-functors are *precisely* homomorphisms of monoids.

... and multifunctors

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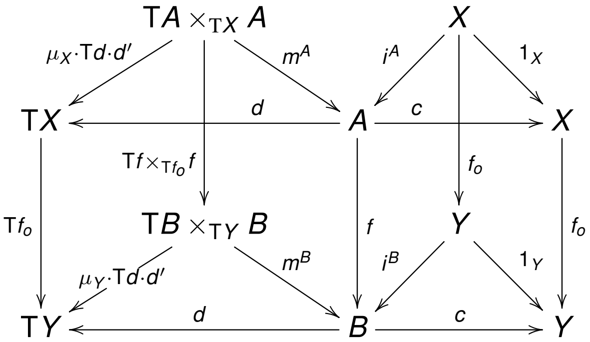

One may, however, set up a double category of T-spans in $\mathcal{C}$ such that T-functors are *precisely* homomorphisms of monoids.

Making things easier: "Spaces" as simplified T-categories

Definition

A T-category (X, A, d, c, i, m) is a T-*order* in $\mathcal{C}$ if (d, c) is a monic pair in $\mathcal{C}$.

In that case, i and m are uniquely determined by (X, A, d, c), and their existence becomes a property of (X, A, d, c): *reflexivity* and *transitivity*. The unity and associativity laws now come for free! For a T-functor $(f_o, f) : (X, A) \longrightarrow (Y, B)$, the arrow part f is determined by its object part f_o, and its existence becomes a property of f_o: *monotonicity*.

Some properties of the full subcategory Ord(T) of Cat(T):

- $\mathrm{Ord}(\mathbb{T}) \longrightarrow \mathcal{C}$ is *topological* (= fibration + cofibration + fibres are large-complete), provided that $\mathcal{C}$ is complete and wellpowered.
- If $\mathcal{C}$ is also cocomplete, so is Ord(T).
- Every Eilenberg-Moore T-algebra $(X, a : TX \to X)$ gives the T-order $(X, TX, 1_{TX}, a)$; in fact:

  T-*algebras are precisely those* T-*categories with domain map an identity.*

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Aspirational inclusions: Algebra ⊂ Topology ⊂ Category Theory

$$\begin{array}{ccccccc}
\mathcal{C}^{\mathsf{T}} = \mathsf{EM}(\mathsf{T}) & \rightarrowtail & \mathsf{Ord}(\mathsf{T}) & \rightarrowtail & \mathsf{Cat}(\mathsf{T}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C} & \rightarrowtail & \mathsf{Ord}(\mathcal{C}) & \rightarrowtail & \mathsf{Cat}(\mathcal{C}) & \longrightarrow & \mathcal{C}
\end{array}$$

Role models: $\mathsf{T} = \mathrm{L}$ (list monad) and $\mathsf{T} = \mathrm{U}$ (ultrafilter monad) on **Set**

$$\begin{array}{ccccccc}
\mathbf{Mon} & \rightarrowtail & \mathbf{MultiOrd} & \rightarrowtail & \mathbf{MultiCat} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbf{Set} & \rightarrowtail & \mathbf{Ord} & \rightarrowtail & \mathbf{Cat} & \longrightarrow & \mathbf{Set} \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{CompHaus} & \rightarrowtail & \mathbf{Top} & \rightarrowtail & \mathbf{UltraCat}_{\mathrm{internal}} & &
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More properties, a glimpse at the “topological potential” of Ord(T)

- Ord(T) $\rightarrowtail$ Cat(T) is reflective, provided that $\mathcal{C}$ is finitely complete, has a stable (strong epi, mono)-factorization system, and that T preserves strong epimorphisms (which is no restriction on $\mathbb{T}$ in case $\mathcal{C}$ = **Set**, under Choice).
- EM(T) $\rightarrowtail$ Ord(T) is reflective, under the additional provision that $\mathcal{C}$ is complete and weakly cowellpowered (by Freyd’s GAFT): Stone-Čech if T = $\mathbb{U}$.

In this case, define a T-order $X = (X, A, d, c)$ to be

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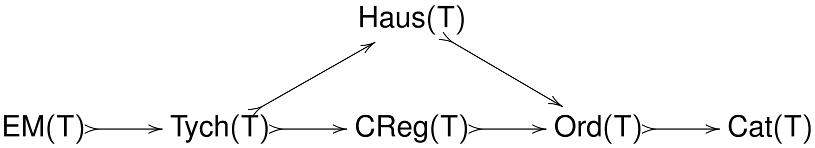

An example of category theory embracing topology

Let $f : X = (X, A) \longrightarrow Y = (Y, B)$ be in Tych(T). Then:

$$
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\quad \Rightarrow \quad
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Consequently:

Comprehensive factorization of f means (antiperfect, perfect)-factorization of f, a.k.a. the fibrewise Stone-Čech compactification of f.

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$\mathcal{C} =$ **Set**, T laxly extended to **Rel** á la Barr. Then:

$$\mathrm{Ord}(\mathsf{T}) \cong (\mathsf{T}, 2)\text{-}\mathbf{Cat}$$

In particular: $\mathrm{Ord}(\mathrm{U}) \cong$ **Top**

But what about arbitray quantales V, rather than 2?
For example: Is V-**Cat** of the form $\mathrm{Ord}(\mathsf{T})$, for some T? ...

Recall:
While for every T, laxly extended to V-**Rel**, one can find a monad Π laxly extendable to **Rel** (encoding both, T and V) such that (Lowen-Vroegrijk 2008, Hofmann 2014)

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Comparison with $(\mathbb{T}, \mathsf{V})$-Cat

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To-Do list

- Pursue monoidal topology (enriched) in the (internal) context ...
- ... and conversely!
- To which extent are (T, V)-categories covered by an internal setting?
- Apply the emerging theory in particular in topological algebra.
- Apply (T, V)-category theory to “probabilistic” quantales or monads.
- Dualization, Yoneda, (monoidal) closedness, 2-categorical structure, ...

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