Order, Distance, Closure and Convergence: Reconciling Competing Fundamental Topological Concepts

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H. Lai, W. Tholen: Quantale-valued topological spaces via closure and convergence, *Topology and Its Applications* (to appear) arXiv(math):1604.08813
H. Lai, W. Tholen: A note on the topologicity of quantale-valued topological spaces, *Logical Methods in Computer Science* (to appear) arXiv(cs):1612.09504

Convergence and Metric: Maurice Fréchet 1906

"Sur quelques points du calcul fonctionnel"
Rendiconti del Circolo Matematico di Palermo 22, 1–74

Axioms on the convergence of sequences (p 6):

- $(A) \to A$
- $(A_n) \to A \Longrightarrow (A_{n_i}) \to A$

Axioms on the distance between points (p 18):

- $(A, B) = (B, A) \geq 0$
- $(A, B) = 0 \iff A = B$
- For some function f with $\lim_{\varepsilon>0} f(\varepsilon) = 0$:
  $(A, B) \leq \varepsilon, \ (B, C) \leq \varepsilon \Longrightarrow (A, C) \leq f(\varepsilon)$

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"Grundzüge der Mengenlehre" (Veit & Comp., Leipzig), Ch. 7, p 211:

"Welchen der drei [...] Grundbegriffe Entfernung, Limes, Umgebung man zur Basis der Betrachtung wählen will, ist bis zu einem gewissen Grade Geschmacksache. [...] Danach scheint die Entfernungstheorie die speziellste, die Limestheorie die allgemeinste zu sein; auf der andern Seite bringt der Limesbegriff sofort eine Beziehung zum Abzählbaren (zu Elementfolgen) in die Theorie hinein, worauf die Umgebungstheorie verzichtet."

"Which of the three [...] fundamental notions, distance, limit, neighbourhood, one wants to choose as the basis of consideration is, to a certain degree, a matter of taste. [...] Accordingly, the distance theory seems to be the most special, the limit theory the most general; on the other side, the notion of limit brings immediately a connection to countability (to sequences of elements), which the neighbourhood theory foregoes."

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Haudorff's functional approach to ...

- Order:
  $M \times M \longrightarrow \{<, >, =\}, \quad M \times M \longrightarrow \{<, >, =, ||\}$
- Metric:
  $M \times M \longrightarrow \mathbb{R}$
- Convergence:
  $M^{\mathbb{N}} \longrightarrow\!\!\!\!\!\!+\ M \quad (\iff \ M^{\mathbb{N}} \times M \longrightarrow 2 = \{\text{true}, \text{false}\})$
- Topology (via neighbourhood systems):

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... and beyond, ...

“Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d. h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element $n = f(a, b)$ *einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen.”*

“Now there is no obstacle to generalizing this point of view, and we could think that there be defined an arbitrary function of pairs of a set, i.e., that, to every pair (a, b) of elements in a set M, there be assigned a certain element $n = f(a, b)$ in a second set N. In even further generalization we can consider a function of triples of elements, sequences of elements, complexes [= families] of elements, subsets of M, and the like.”

... but he also cautions:

"Eine ganz allgemein gehaltene Theorie dieser Art würde natürlich erhebliche Komplikationen bedingen und wenig positive Ausbeute liefern."

"If kept very general, a theory of this type would of course cause considerable complications and provide little positive outcome."

OUR GENERAL STRATEGY (to keep useless generality in check):
Put structure on Hausdorff's set N which can capture the syntax used in key axioms of the theory!

BUT WE MUST DECIDE EARLY ON:
Keep N fixed, or let N "float" (together with M)?
Guided by our principal examples, we'll keep it fixed, but we'll briefly return to this point at the end of this talk!

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Seeing a metric as "hom": F. William Lawvere 1973

Folklore since Eilenberg–Mac Lane in the 1940s:

Preordered sets	$\Leftrightarrow$	"thin" (small) categories
$\mathrm{true} \Rightarrow x \le x$		$1 \longrightarrow \mathrm{hom}(x,x)$
$x \le y \;\&\; y \le z \Rightarrow x \le z$		$\mathrm{hom}(x,y) \times \mathrm{hom}(y,z) \longrightarrow \mathrm{hom}(x,z)$

"Metric spaces, generalized logic, and closed categories",
Rendiconti del Seminario Matematico e Fisico di Milano 43, 135–166

$0 \ge d(x,x)$

$d(x,y) + d(y,z) \ge d(x,z)$

"homs" are just numbers!

"Syntax providers":

$2 = ((\{\mathrm{true},\mathrm{false}\}, \Rightarrow), \&, \mathrm{true})$ $\qquad$ $\mathbf{Set} = ((\{\text{all sets}\}, \longrightarrow), \times, 1)$

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What's special about the extra structure?

Cartesian closedness, exponentiability, monoidal closedness:

$\dfrac{Z \times X \longrightarrow Y}{Z \longrightarrow Y^X}$	**Set**	$(\sum Z_i) \times X \cong \sum(Z_i \times X)$
$\dfrac{Z \times X \longrightarrow Y}{Z \longrightarrow \mathrm{C}(X, Y)}$	???**Top**???	$(\sum Z_i) \times X \cong \sum(Z_i \times X)$
$\dfrac{C \otimes A \longrightarrow B}{C \longrightarrow \hom(A, B)}$	**AbGrp**	$(\bigoplus C_i) \otimes A \cong \bigoplus(C_i \otimes A)$
$\dfrac{z + x \geq y}{z \geq \max\{y\text{-}x, 0\}}$	$\overleftarrow{[0, \infty]}$	$(\inf z_i) + x = \inf(z_i + x)$

Quantales

V unital (but not necessarily commutative) *quantale*
= complete lattice with a monoid structure $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$ s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i, \quad (\bigvee_{i \in I} v_i) \otimes u = \bigvee_{i \in I} (v_i \otimes u)$$

- $\mathsf{V} = 2$ with $u \otimes v = u \wedge v, \quad \mathsf{k} = \mathrm{true}$
- $\mathsf{V} = \overleftarrow{[0, \infty]}$ with $u \otimes v = u + v, \quad \mathsf{k} = 0$
- V any frame with $u \otimes v = u \wedge v, \quad \mathsf{k} = \top$
- $\mathsf{V} = 2^M$ (M monoid) with $A \otimes B = \{\alpha\beta \mid \alpha \in A, \beta \in B\}, \ \mathsf{k} = \{\eta_M\}$

"Internal hom" operations:

$$v \le w \swarrow u \iff v \otimes u \le w \iff u \le v \searrow w$$

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Quantale-enriched categories

$(\mathsf{V}, \otimes, \mathsf{k})$ V-**Cat**:

V-category : $(X, a : X \times X \longrightarrow \mathsf{V})$: $\mathsf{k} \leq a(x, x)$
$a(y, z) \otimes a(x, y) \leq a(x, z)$

V-functor : $(X, a) \xrightarrow{f} (Y, b)$: $a(x, y) \leq b(fx, fy)$

$\mathsf{V} = 2$ V-**Cat** = **Ord** = {(pre)ordered sets}

$\mathsf{V} = \overleftarrow{[0, \infty]}$ V-**Cat** = **Met** = {(gen'd) metric spaces}

$\mathsf{V} = 2^M$ (for a monoid M) V-**Cat** = M-**Ord** = {"M-ordered sets"} :

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One more quantale: {distance distribution functions}

$$\overleftarrow{[0,\infty]} \cong \overrightarrow{[0,1]} = (([0,1],\leq),\cdot,1)$$

$$\Delta \ni \varphi : [0,\infty] \longrightarrow [0,1] \qquad \varphi(\beta) = \sup_{\alpha<\beta} \varphi(\alpha)$$

$$(\varphi \otimes \psi)(\gamma) = \sup_{\alpha+\beta=\gamma} \varphi(\alpha)\psi(\beta) \qquad \kappa(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0. \end{cases}$$

$$\sigma : \overleftarrow{[0,\infty]} \hookrightarrow \Delta \hookleftarrow \overrightarrow{[0,1]} : \tau$$

$\varphi = \sup_{\gamma} \sigma(\gamma) \otimes \tau(\varphi(\gamma))$: Δ as a coproduct in **CQuant** !

Met $\hookrightarrow$ **ProbMet** $\hookleftarrow$ **ProbOrd** ($\cong$ **Met**)

[Menger 1942], [Schweizer-Sklar 1983]

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Hausdorff's vision, I: Replace points by subsets

Recall: $a : X \times X \longrightarrow 2$ (R) true $\Rightarrow a(x,x)$

(T) $a(x,y)\ \&\ a(y,z) \Rightarrow a(x,z)$

$f : (X,a) \longrightarrow (Y,b)$ (C) $a(x,y) \Rightarrow b(fx,fy)$

Now: $a : \mathsf{P}X \times X \longrightarrow 2$ (R) true $\Rightarrow a(\{x\},x)$

Need $\hat{\mathsf{P}}a$: $\mathsf{PP}X \times \mathsf{P}X \longrightarrow 2$ (T) $\hat{\mathsf{P}}a(\mathcal{A},B)\ \&\ a(B,z) \Rightarrow a(\bigcup\mathcal{A},z)$

Choose: $\mathbb{P} = (\mathsf{P},\bigcup,\{\text{-}\})$ $\hat{\mathsf{P}}a(\mathcal{A},B) :\iff \forall y \in B\, \exists A \in \mathcal{A} : a(A,y)$

Rewrite: $c : \mathsf{P}X \longrightarrow 2^X$ (R) $\{x\} \subseteq c\{x\}$

(T) $B \subseteq \bigcup_{A\in\mathcal{A}} cA\ \&\ z \in cB \Rightarrow z \in c(\bigcup\mathcal{A})$

Simplify: (R*) $A \subseteq cA$

(T*) $B \subseteq cA\ \&\ z \in cB \Rightarrow z \in cA$

Equiv'ly: $B \subseteq cA \Rightarrow cB \subseteq cA$

c extensive and idempotent!

Also: $f : X \longrightarrow Y$ (C*) $f(c_X A) \subseteq c_Y(fA)$

Hence: $(\mathbb{P},2)$-**Cat** $=$ **Cls** closure spaces & continuous maps

[Seal 2005]

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Cls $\leftrightarrow$ Top, V-Cls $\leftrightarrow$ V-Top

Top $c : PX \longrightarrow 2^X$

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c fin'ly additive: (A) $c(A \cup B) = cA \cup cB$

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[Seal 2009]

V-**Cls** $c : PX \longrightarrow V^X$

$= (\mathbb{P}, V)$-**Cat**

(R) $\forall x \in A : k \leq (cA)(x)$

(T) $\left(\bigwedge_{y \in B} (cA)(y)\right) \otimes (cB)(x) \leq (cA)(x)$

[Lai-T 2016]

V-**Top** c fin'ly additive: (A) $c(A \cup B)(x) = (cA)(x) \vee (cB)(x)$

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(Not to be confused with other V-concepts in topology!)

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$\Delta_{\&}$-**Top**, with & denoting any left-continuous t-norm on [0,1] – all "generated" by only three such norms!

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Properties: Is V-**Top** topological over **Set**?

V-**Cls** certainly is:

Initial structure c on X w.r.t. $(f_i : X \longrightarrow (Y_i, c_i),\ i \in I)$ (I any size)
(= largest c making all f_i continuous):

$$(cA)(x) = \bigwedge_{i \in I} c_i(fA)(fx)$$

But: $(\forall i \in I : c_i \text{ finitely additive}) \not\Rightarrow c$ finitely additive

Fix:

$$(c^+A)(x) = \bigwedge_{A = M_1 \cup \ldots \cup M_n} (cM_1)(x) \vee \ldots \vee (cM_n)(x)$$

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[Dikranjan-T 1995]: $\overline{c^+}$ is finitely additive; hence:

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If $v = p$ coprime ($p > \bot$, $p \leq u \vee v \Rightarrow p \leq u$ or $p \leq v$):

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Topologicity of V-**Top** over **Set**

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Without invoking Choice:

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i.e. $\bigvee : \mathrm{Down}\mathsf{V} \longrightarrow \mathsf{V}$ preserves all infima

" $\Longleftarrow$ " needs Choice, and then (ccd $\Longleftrightarrow$ cd)

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Now:	$a : \mathsf{U}X \times X \longrightarrow 2$	(R) true $\Rightarrow a(\dot{x},x)$
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	$\mathbb{U} = (\mathsf{U},\Sigma,(\dot{\text{-}}))$	$\overline{\mathsf{U}}a(\mathfrak{X},\mathfrak{y}) :\iff \forall\mathcal{A}\in\mathfrak{X}, B\in\mathfrak{y}$
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Again:		(R) $\dot{x}\to x$
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[Barr 1970]: $(\mathbb{U},2)$-**Cat** $\cong$ **Top**

[Seal 2005]: Works also for filters!

Hausdorff's vision, II: Replace points by (ultra)filters

Recall:	$a : X \times X \longrightarrow 2$	(R) $\mathrm{true} \Rightarrow a(x,x)$
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$(\mathbb{U}, \mathsf{V})$-**Cat** $\cong$ V-**Top** ?

V completely distributive

For $a : \mathsf{U}X \times X \longrightarrow \mathsf{V}$, define $\overline{\mathsf{U}}a : \mathsf{UU}X \times \mathsf{U}X \longrightarrow \mathsf{V}$:

$$\overline{\mathsf{U}}a(\mathfrak{X}, \mathfrak{y}) := \bigwedge_{\mathcal{A}\in\mathfrak{X}, B\in\mathfrak{y}} \bigvee_{\mathfrak{x}\in\mathcal{A}, y\in B} a(\mathfrak{x}, y)$$

$\mathbb{P}$ and $\mathbb{U}$ interact via the V-relation $\varepsilon_X : \mathsf{P}X \times \mathsf{U}X \longrightarrow \mathsf{V}$:

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$$(X, a) \mapsto (X, c_a = a \circ \varepsilon_X)$$

$$(c_a A)(y) = \bigvee_{\mathfrak{x}\ni A} a(\mathfrak{x}, y)$$

A_ε has a right adjoint $R : (X, c) \mapsto (X, a_c)$, $\quad a_c(\mathfrak{x}, y) = \bigwedge_{A\in\mathfrak{x}} (cA)(y)$

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Theorem: Yes !

[Lai-T 2016] V completely distributive. Then:

- $A_\varepsilon : (\mathbb{U}, \mathsf{V})$-**Cat** $\hookrightarrow (\mathbb{P}, \mathsf{V})$-**Cat** $=$ V-**Cls** coreflective embedding
- Its image is V-**Top** $\cong (\mathbb{U}, \mathsf{V})$-**Cat**.

Topologicity of V-**Top** revisited:
$(\mathbb{U}, \mathsf{V})$-**Cat** is (almost) trivially topological over **Set** !

Corollaries:

[Clementino-Hofmann 2003]

App $\cong (\mathbb{U}, \overleftarrow{[0,\infty]})$-**Cat**

$0 \geq a(\dot{x}, x)$

$(\sup_{\mathcal{A}\in\mathfrak{X}, B\in\mathfrak{y}} \inf_{\mathfrak{x}\in\mathcal{A}, y\in B} a(\mathfrak{x}, y)) + a(\mathfrak{y}, z) \geq a(\Sigma\mathfrak{X}, z)$

[Jäger 2016],[Lai-T 2016]

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Change-of-base functors

V with $\mathsf{k} = \top > \bot$, compl. distributive. Then $2 \hookrightarrow \mathsf{V}$ has both adjoints:

$$2 \rightleftarrows \mathsf{V}$$

Expect transfer of this double adjunction to the categories of models:

$$\mathbf{Ord} = 2\text{-}\mathbf{Cat} \rightleftarrows \mathsf{V}\text{-}\mathbf{Cat}$$

Moreover,

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Justification

[Lai-T 2016] V, W completely distributive and integral ($\mathsf{k} = \top$), $\varphi : \mathsf{V} \longrightarrow \mathsf{W}$ monotone, $\psi : \mathsf{W} \longrightarrow \mathsf{V}$ lax homomorphism of quantales. Equivalent are:

- $\varphi \dashv \psi : \mathsf{W} \longrightarrow \mathsf{V}$
- $\overline{B_\varphi} \dashv B_\psi$: W-**Cat** $\longrightarrow$ V-**Cat**, $(X, b) \mapsto (X, \psi b)$
- $\overline{B_\varphi} \dashv B_\psi$: $(\mathbb{U}, \mathsf{W})$-**Cat** $\longrightarrow$ $(\mathbb{U}, \mathsf{V})$-**Cat**

[Here $\overline{B_\varphi}$ assigns to a V-category (X, a) the W-category (X, b), with the least W-category structure with $b \geq \varphi a$; likewise when (X, a) is an $(\mathbb{U}, \mathsf{W})$-category.]

$\mathbb{T} = (\mathsf{T}, \mathsf{m} : \mathsf{TT} \to T, \mathsf{e} : \mathrm{Id} \to \mathsf{T})$ **Set**-monad, $\hat{\mathsf{T}} : \mathsf{V}$-**Rel** $\longrightarrow$ V-**Rel** lax ext.

$(\mathbb{T}, \mathsf{V})$-Cat : $\quad (X, a : \mathsf{T}X \to\!\!\!\!\!\!+\, X)$: $\quad \mathsf{k} \leq a(\mathsf{e}_X(x), x)$

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Lax Eilenberg-Moore via (Kleisli)$^{\mathrm{op}}$: $\mathsf{e}_X^{\circ} \leq a$

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$(\mathbb{T}, \mathsf{V})$-categories: “Topology = Lax Algebra”

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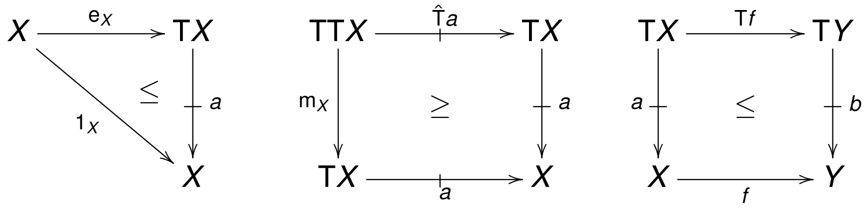

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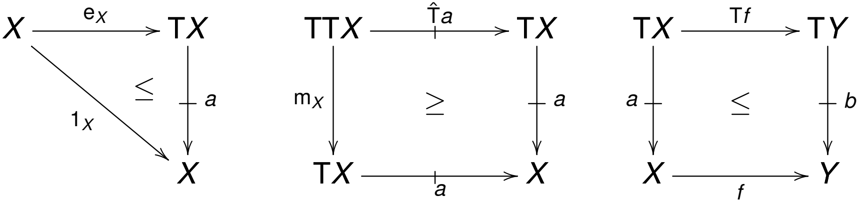

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Hausdorff's vision, III: "What about my nbhd systems?"

$(X, \mathcal{U} : X \longrightarrow \mathsf{F}X)$ $\quad \{\dot{-}\} \leq \mathcal{U} \qquad \forall A \in \mathcal{U}(x) : x \in A$

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[Gähler *et al.* 1992]

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Topological spaces are monoids in the (ordered) Kleisli category of the filter monad $\mathbb{F} = (\mathsf{F}, \Sigma, \{\dot{-}\})$ on **Set**:
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More generally:
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A glance at properties: Hausdorffness, compactness...

Trivially for a relation r:

$r : X \nrightarrow Y$ is a map $\iff$ $r \cdot r^\circ \leq 1_Y$ $r(x)$ uniquely defined

$1_X \leq r^\circ \cdot r$ $r(x)$ defined everywhere

[Kamnitzer 1974]

$a\colon TX \nrightarrow X$ Hausdorff: $a \cdot a^\circ \leq 1_X$ "$\mathfrak{x}$ has at most one pt of conv"

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If V is such that V-relations satisfying both inequalities are maps:

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[Manes 1969] **CompHaus** is monadic over **Set**.

A glance at properties: Hausdorffness, compactness...

Trivially for a relation r:

$r : X \to\!\!\!\!\!\!+\!\!\to Y$ is a map $\iff$ $r \cdot r^\circ \leq 1_Y$ $r(x)$ uniquely defined
$1_X \leq r^\circ \cdot r$ $r(x)$ defined everywhere

[Kamnitzer 1974]

$a\colon \mathsf{T}X \to\!\!\!\!\!\!+\!\!\to X$ Hausdorff: $a \cdot a^\circ \leq 1_X$ "$\mathfrak{x}$ has at most one pt of conv"
compact: $1_{\mathsf{T}X} \leq a^\circ \cdot a$ "$\mathfrak{x}$ has at least one pt of conv"

If V is such that V-relations satisfying both inequalities are maps:

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... also being equationally defined in a lax environment

$(\mathbb{T}, \mathsf{V})$-**Cat** :

$f : (X, a) \longrightarrow (Y, b)$ characterized by: $f \cdot a \leq b \cdot f$
f $(\mathbb{T}, \mathsf{V})$-proper defined by: $f \cdot a = b \cdot f$

Under mild conditions on $\mathbb{T}, \mathsf{V}$:
X Hausdorff $\iff$ $X \longrightarrow X \times X$ proper
X compact $\iff$ $X \longrightarrow 1$ proper

Other important equationally-defined map and object concepts:
open, local homeomorphism, regular, exponentiable, ...

[Hofmann-Seal-T (eds) 2014]: *Monoidal Topology*

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Finally: Letting Hausdorff's target set N "float"

[Kopperman 1988]
Continuity space $(X, d\colon X \times X \longrightarrow A, (A, +), P \subseteq A)$
"All topologies come from generalized metrics"

[Henriksen-Kopperman 1991]
Structure space $(X, d\colon X \times X \longrightarrow A, A = 2^R)$
Transition to keeping "N" fixed: [Flagg 1997], [Flagg-Kopperman 1997]
V-continuity space: essentially V-**Cat**!

A more recent "floater": [Dikranjan-Zava 2017]
Coarse space (via balleans): $(X, P, B\colon X \times P \longrightarrow 2^X)$
Rewrite B as $X \times X \longrightarrow 2^P$: see structure space above!

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Selected milestones leading to $(\mathbb{T}, \mathsf{V})$-**Cat**, and beyond

- Enriched category theory:
  [Eilenberg-Kelly 1966], [Lawvere 1973], [Kelly 1982], ...
- Monad theory and its application to topology:
  [Eilenberg-Moore 1965], [Kleisli 1965], [Manes 1969], [Barr 1970],.
- Combining the two:
  [Clementino-Hofmann 2003], [Clementino-T 2003],
  [Clementino-Hofmann T 2003], [Seal 2005], [Hofmann 2007],
  [Hofmann-Seal-T (eds) 2014], [T 2016], [Hofmann-Nora 2017], ...
- Resources from categorical topology:
  [Brümmer 1971], [Penon 1972], [Herrlich 1974], [Manes 1974],
  [Kamnitzer 1974], [Möbus 1981], [Wyler 1995], [Lowen 1997],
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Your questions, comments?

THANK YOU !