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CATEGORICAL ALGEBRAIC PROPERTIES. A COMPENDIUM ON AMALGAMATION, CONGRUENCE EXTENSION, EPIMORPHISMS, RESIDUAL SMALLNESS, AND INJECTIVITY

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Introduction

The existence of amalgamated products, injectives, non-surjective epimorphisms or large subdirectly irreducible objects in particular categories are closely related and frequently discussed questions in algebra and topology. Their solution becomes in many cases easier from a more general, say universal algebraic or categorical, point of view. For instance, one can often make use of some general theorems which are not always well-known for the specialist of a concrete field. The aim of this paper is to give a survey on concrete and general results in this area. The first part gives the definitions, some fundamental general theorems and key references concerning them. In the second part we present a table which lists for many familiar categories whether they enjoy the amalgamation property, the intersection property of amalgamations (hence also the strong amalgamation property), the congruence extension property; whether they have surjective epimorphisms, enough absolute retracts, cogenerating sets, injective hulls; whether they are residually small. The article ends with a classified bibliography which is intended to include all known results concerning these topics by May 1983. By the word 'classified' we mean that for each item we indicate which of the properties discussed in our survey are treated there. Of course, the authors do realize that their intention of completeness of references may not have been achieved, and they apologize to those colleagues whose work may be missing here.

The present survey grew out of a preprint with a similar intention of the last named author: *Amalgamations in categories,* Seminarberichte, Fachbereich Mathematik, Fernuniversität Hagen, **5** (1979), 121—151, and its principles have been agreed upon during a visit of his in Budapest in December 1979, sponsored by the J. Bolyai Mathematical Society within the frameworks of an agreement with the Deutsche Mathematiker-Vereinigung. The authors are indebted to many colleagues for helpful suggestions, especially to K. Głazek, H.-J. Hoehnke, J. R. Isbell, T. Katriňák, F. E. J. Linton, L. N. Shevrin, L. A. Skornjakov, and above all, to G. M. Bergman, whose contribution would have justified him to become a co-author.

1. General results

§0. Introductory remarks. In this first part of our paper we sum up the most important universal algebraic or categorical results concerning our topic, and we also present some general methods and ideas of proofs which facilitate settling problems of these kinds in concrete classes. We do not deal with the model-theoretic (or algebraic-logical) aspects of these problems, but papers treating them are included in the bibliography. Especially amalgamation and related properties have a rich literature of this kind, giving e.g. syntactic characterizations for them, and showing that some of them are equivalent to definability or interpolation properties. Without claiming completeness even among key references on this topic, we refer to Andréka and Sain [81], Bacsich [74], [75a], Bacsich and Rowlands Hughes [74], Comer [69], Pigozzi [71], Preller [69].

The first five sections deal with classes of algebras. In §1 we define the notions which will be investigated in the sequel, and give the basic connections among them. §§2—5 treat residual smallness, congruence extension, amalgamation, and injectivity, respectively. §6 describes how the preceding considerations carry over to abstract categories.

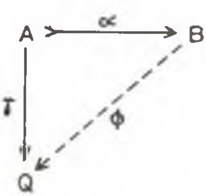
In order to give due credit to the people who invented the notions we are going to investigate, we shall begin by giving exact references of the first occurrences of these concepts (as far as we know). Amalgamation was first considered for groups by Schreier [27], the general form of the property had its first appearance in Fraïssé [54], whereas the strong amalgamation property was introduced by Jónsson [56]. The term 'intersection property of amalgamations' in the present meaning was first used by Ringel [72], but the name turned up in Dwinger [70] and meant what we call strong amalgamation property, whereas the property itself had been investigated before, first probably by Isbell [66a]. Semigroup theorists call the same notion 'special amalgamation property'; however, the latter name has a different meaning in Grätzer and Lakser [71]. The classical form of the property 'epimorphisms are surjective' goes back to Isbell [57]. The congruence extension property was introduced by Grätzer and Lakser [71], and the transferability property appeared first in Banaschewski [70] and then, under this name, in Bacsich [72c]. The existence of a cogenerating set was first considered by Grothendieck [57], that of enough absolute retracts by Węglorz [66], whereas the notion of residual smallness is due to Taylor [72]. Having enough injectives was first established by Baer [40] for abelian groups, the general notion appeared in Buchsbaum [55]. The existence of injective hulls was proved for modules by Eckmann and Schopf [53], the general form of this property goes back to Mitchell [65].

We use notations and terminology of Grätzer [79], but make no distinction between an algebra and its underlying set, and write the mappings on the left. The sign □ after a statement means that it can be proved by the reader with no difficulty.

§1. Generalities. For the following arguments we fix a class $\mathscr{K}$ of algebras with $\mathrm{HS}\mathscr{K} \subseteq \mathscr{K}$ and suppose that all algebras — unless otherwise stated — are in $\mathscr{K}$. We must emphasize that the closedness under HS is not necessary, but it makes life easier. However, the reasonings can be carried over to rather general arbitrary categories, see §6.

Before getting down to the investigations of our properties separately let us put their interrelations in their proper light. What follows is belonging to the folklore, the main references are: Banaschewski [70], Taylor [72], Grätzer, Lakser [71, 72a].

Let us start with some definitions. Our first concept appears in several places in mathematics. An algebra Q (not necessarily in $\mathscr{K}$) is said to be *injective* over $\mathscr{K}$ if whenever a diagram



is given with an injective homomorphism α then there exists a φ such that $\varphi\alpha=\gamma$. $\mathcal{K}$ is said to have *enough injectives* (EI) if each object can be embedded into an injective one (in and over $\mathcal{K}$).

In order to characterize this concept let us generalize it. We say that a subalgebra A of B ($A \leqq B$ in notation) is a *retract* of B if it is the image of an idempotent endomorphism of B. R is called *absolute retract* in $\mathcal{K}$ if it is a retract in each of its extensions (in $\mathcal{K}$).

Proposition 1.1. *The injectives in $\mathcal{K}$ are absolute retracts.* □

If we want to get EI via absolute retracts, we have to face two problems: 1) embedding each object into an absolute retract, and 2) finding conditions which ensure the injectivity of the latter.

A class $\mathcal{K}$ is said to have EAR *(enough absolute retracts)* if each of its objects can be embedded into an absolute retract in $\mathcal{K}$. This concept has as yet been investigated only for varieties, where it is equivalent to having only a set of (non-isomorphic) subdirectly irreducible algebras. Classes with this latter property are named *residually small* (RS). Results on RS varieties are summed up in §2 below.

In order to facilitate finding absolute retracts, let us make some general observations. For the first question call a subdirectly irreducible (for short: SI) algebra S *maximal subdirectly irreducible* in $\mathcal{K}$ if it cannot be properly embedded into any SI algebra in $\mathcal{K}$.

Proposition 1.2. *The maximal SI algebras are absolute retracts.*

This statement is clear by virtue of the following definition and claims. An extension $A \leqq B$ is called *essential* if each non-0 congruence of B restricts to a non-0 one of A.

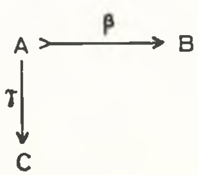
Proposition 1.3. (a) *Essential extensions of SI algebras are SI.* □

(b) *If $A \leqq B$ then among the congruences 0 on B with $0 \restriction A = 0$ there is a maximal one, 0_0, and the extension $A \leqq B/0_0$ is essential.* □

Proposition 1.4. *An algebra has a proper essential extension iff it is not an absolute retract.* □

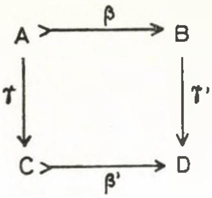
Now let us turn to the question: when are absolute retracts injective? We say that the *injections are transferable* in $\mathcal{K}$ or $\mathcal{K}$ has the *transferability property* (TP) if each diagram

(1)



can be completed to a commutative square as follows:

(2)

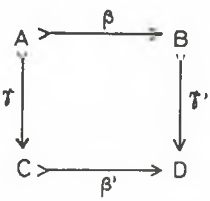


PROPOSITION 1.5. *EI implies TP. Conversely, if $\mathcal{K}$ has TP then the absolute retracts are injective.* □

TP is a join of two properties which are very important in themselves. Namely, if γ in (1) is supposed to be surjective, then we get the *congruence extension property* (CEP), which can be formulated in a more algebraic way: we say that an algebra A has CEP if all congruences on its subalgebras can be extended to A, and by the definition above, a class $\mathcal{K}$ has CEP if each of its objects has CEP.

For the other concept let us call a quintuple $(A; \beta, B; \gamma, C)$ an *amalgam* if $\beta: A \rightarrowtail B$ and $\gamma: A \rightarrowtail C$ are injective homomorphisms (the mappings are sometimes omitted; here A is not empty since it is an algebra). We say that this amalgam can be *completed* if β, γ admit a commutative square of the form

(3)



$\mathcal{K}$ is said to have the *amalgamation property* (AP) if each amalgam can be completed in $\mathcal{K}$. We mention that some difficulties can arise with respect to the empty subalgebra. For this problem see Lakser [73] and, concerning injectivity, Day [72].

PROPOSITION 1.6. *CEP and AP imply TP. Conversely, TP implies CEP and, if $\mathcal{K}$ has finite products, also AP.*

To explain the last assertion we mention that if γ in (1) is injective then TP does not give AP since γ' in (2) is not necessarily 1-1. So we must apply TP "on the other side" and then consider a product (cf. Proposition 4.1).

COROLLARY 1.7. *If $\mathcal{K}$ has finite products then EI is equivalent to AP and CEP and EAR.* □

AP has a frequently investigated stronger version. We say that an amalgam $(A; \beta, B; \gamma, C)$ can be *embedded* if it can be completed as in (3) so that

$$\operatorname{Im} \beta' \cap \operatorname{Im} \gamma' = \operatorname{Im} \gamma'\beta.$$

If each amalgam can be embedded then $\mathcal{K}$ has the *strong amalgamation property* (SAP).

To capture the extent to which SAP is stronger than AP we make an observation.

PROPOSITION 1.8. *The following two conditions are equivalent.*
(a) *If an amalgam can be completed, it can be embedded as well.*
(b) *Each amalgam of the form* $(A; \beta, B; \beta, B)$ *can be embedded.* □

(Note that an amalgam of the form $(A; \beta, B; \beta, B)$ can always be completed by B.)

Condition (a) is called the *intersection property of amalgamation* (IPA) and (b) is sometimes named the *special amalgamation property.*

PROPOSITION 1.9. $\mathscr{K}$ *has SAP iff it has AP and IPA.* □

Our last property to deal with is the following. Consider $\mathscr{K}$ as a category, then one can define epimorphisms in the usual sense: they are the morphisms α with the property that for all β and γ, $\beta\alpha=\gamma\alpha$ implies $\beta=\gamma$. We say that *epimorphisms are surjective* in $\mathscr{K}$ (ES) if the epimorphisms are onto mappings.

In many cases the validity of ES can be decided positively:

PROPOSITION 1.10. *IPA implies ES.* □

We are going to give a refined version of this statement in §4 as well as an example of a variety showing that the converse of 1.10 fails. However, we can prove the following:

PROPOSITION 1.11. *In a variety having AP the properties ES and IPA are equivalent.*

We shall prove this proposition at the end of §4, but let us remark here that the statement holds in more general categories, too (see §6).

We close with a practical remark.

PROPOSITION 1.12. *Suppose that in* $\mathscr{K}$ *each algebra can be embedded into a simple one, and that* $\mathscr{K}$ *is closed under finite products. Then*
(a) $\mathscr{K}$ *does not have CEP,*
(b) $\mathscr{K}$ *has only trivial absolute retracts.* □

Note that a weaker condition (instead of the existence of finite products) would also do. If $\mathscr{K}$ has the stronger property that each — in some way naturally defined — "partial" algebra (e.g. an amalgam) can be embedded into a simple algebra in $\mathscr{K}$ then SAP holds as well.

This property occurs several times. The embedding is usually carried out by constructing a larger partial algebra which kills the congruences of the original one, and then one repeats the procedure countably many times. This method works e.g. for non-unary similarity types and quasigroups.

§2. Residual smallness. This is a frequently discussed question in universal algebra, or, to be more precise, in the theory of varieties. The following is a basic result here.

THEOREM 2.1 (Taylor [72]). *The following are equivalent for a variety* V:
(i) V *has only a set of (isomorphism types of) SI algebras.*
(ii) $V=ISP(\mathscr{K})$ *for some subset* $\mathscr{K}\subseteq V$.

(iii) *Each algebra in V has only a set of (pairwise nonequivalent) essential extensions.*

(iv) *Each algebra in V can be embedded into an absolute retract in V.*

Sets $\mathcal{K}$ as in (ii) are called *cogenerating sets* for V; property ECS means the Existence of a Cogenerating Set.

This theorem is not at all easy to prove, in fact there are two difficult parts in it. The proof of (iv)⇒(i) goes through the theory of equationally compact algebras. The reader may get acquainted with it by reading the paper of Banaschewski and Nelson [72]. Note that (i)⇒(iv) is not trivial: the way which seems promising, namely, to construct the required absolute retracts as products of maximal SI algebras, cannot be followed because in spite of the trivial fact that products of injectives are injective, the product of two absolute retracts need not be an absolute retract even in a CD (congruence distributive) variety (Taylor [73]).

For proving (i)⇒(iii) we make use of another concept which occurs in several places. A formula $\varphi(x, y, z, u)$ in the first order language of V is called *congruence formula if*

(I) φ is positive,

(II) $\forall y, z(\exists x\ \varphi(x, x, y, z) \Rightarrow y = z)$.

It is clear that $A \models \varphi(a, b, c, d)$ implies that $c \equiv d\ \theta(a, b)$, where $\theta(a, b)$ denotes the smallest congruence collapsing a and b. Conversely, Mal'cev's lemma implies that if $c \equiv d\ \theta(a, b)$ in A, then $A \models \varphi(a, b, c, d)$ for some congruence formula $\varphi(x, y, z, u)$ of the form

$$\exists z_0, \dots, z_n \Big(z_0 = z \wedge z_n = u \wedge \bigwedge_{i=0}^{n-1} \exists x_0, \dots, x_m (z_i = \tau_i \wedge z_{i+1} = \tau_i(\sigma)) \Big)$$

where each τ_i is a term with variables among $x, y, z, u, x_0, \dots, x_m$ and σ is the substitution switching x and y. Formulas of this form are called *Mal'cev schemes* and they are said to be *restricted* if the τ_i depend only on x, y, z and u.

The congruence formulas describe how the congruences spread, and though they cannot be handled in most of the cases, sometimes they prove very useful. This is so in our case as well, as we have:

THEOREM 2.2 (Taylor [72]). *A variety V is RS iff for each congruence formula φ in the language of V there exists a finite number n such that*

$$V \models \forall y, z[\exists x_1, \dots, x_n(\bigwedge_{1 \le i < j \le n} \varphi(x_i, x_j, y, z)) \Rightarrow y = z].$$

In fact this statement provides a deep insight into the behaviour of RS varieties. It is worth mentioning that its proof is based on a Ramsey-type theorem of combinacorial set theory due to Erdős and Radó. This way of proof yields even some numerital results on the size of subdirectly irreducible algebras.

THEOREM 2.3 (Taylor [72]). *Let $\varkappa = \aleph_0 +$ (the number of operations of V). Then if V is RS then each SI algebra in V has power $\le 2^{\varkappa}$ and their number is $\le 2^{2^{\varkappa}}$. It is also true that in V each essential extension of any $A \in V$ has at most $2^{|A| + \varkappa}$ elements.*

Concerning the number and the size of subdirectly irreducible algebras in a variety, further information can be found in McKenzie—Shelah [74], Baldwin—Berman [75], Baldwin [80].

There is also another problem to investigate. In most cases we are given a set (class) $\mathcal{K}$ of algebras and we have to say something about the SI algebras in the variety V generated by $\mathcal{K}$. If V is CD then they are in $HSU_p(\mathcal{K})$ (Jónsson [67]), but generally the problem is very hard. Anyhow, in many familiar varieties we find arbitrarily large SI algebras as soon as an infinite one can be constructed. This explains the importance of the following fundamental observation.

THEOREM 2.4 (Quackenbush [71]). *If a locally finite variety contains an infinite SI algebra then the size of its finite SI algebras is not bounded.*

The converse problem had a great influence on investigations concerning RS varieties and is still unsolved.

PROBLEM (Quackenbush [71]). Suppose that V is generated by a finite algebra and that V has infinitely many finite SI algebras. Is it true that V has an infinite SI algebra?

McKenzie [81, 82] described all the varieties of rings and semigroups which are RS, in the semigroup case up to groups. These results show that if the finite algebra in the above problem is a ring or a semigroup then the answer is yes. The same holds for all finite algebras generating a congruence modular (CM) variety because of the following deep result.

THEOREM 2.5 (Freese—McKenzie [81]). *The following are equivalent for a finite algebra A generating a CM variety V:*

(i) *V is RS.*

(ii) *Each SI in V has power $\leq (l+1)!m$ where $m=|A|$ and $l=m^{m^{m+1}}$.*

(iii) *Each subalgebra of A satisfies the commutator identity $[x, x]\wedge y \leq [x, y]$.*

Here $[x, y]$ denotes the commutator of congruences x and y. For groups, this notion coincides with mutual commutator subgroup, whereas in rings we have $[I, J]=IJ+JI$ (replacing congruences by normal subgroups and ideals, respectively). Commutators can be defined in any CM variety (for a very readable account see H. P. Gumm, An easy way to the commutator in modular varieties, *Arch. Math. (Basel)* **34** (1980), 220—228) and they can be handled essentially as in groups and rings. Considering commutators generally yields much information in the investigation of CM varieties, e.g. it is condition (iii) which makes Theorem 2.5 so effective.

Among others the well-known fact that a finite group generates an RS variety iff its Sylow subgroups are abelian, is a consequence of Theorem 2.5.

There is still another type of algebras for which the answer to the Quackenbush problem is known to be "yes". We say that a variety V has *definable principal congruences* (DPC) if there exists a formula $\varphi(x, y, z, u)$ in the first order language of V such that for each $a, b, c, d \in A \in V$ we have

$$c \equiv d\ \theta(a, b) \quad \text{iff} \quad A \models \varphi(a, b, c, d).$$

It can be easily seen that in this case φ is equivalent to a finite disjunction of Mal'cev schemes (which is a congruence formula). We have

THEOREM 2.6 (Baldwin—Berman [75]). *Let V be an RS variety with DPC. Then there exists a natural number N such that each SI algebra in V has $\leqq N$ elements.*

Note that locally finite CEP varieties have DPC (Baldwin—Berman [75]). Theorem 2.6 follows clearly from Theorem 2.2.

On the other hand two interesting 'almost counterexamples' to the Quackenbush problem can be found in Baldwin—Berman [75], and Baldwin [80].

Similar investigations of simple algebras instead of SI ones have been carried out in Magari, R., Una dimostrazione del fatto che ogni varietà ammette algebre semplici, *Ann. Univ. Ferrara, Sez. VII* **14** (1969), 1—4, Lampe, W. A. — Taylor, W., Simple algebras in varieties (preprint), McKenzie—Shelah [74], Freese—McKenzie [81]. In fact, in many cases the negation of RS is proved by finding arbitrarily large simple algebras.

§3. Congruence extension. It is not easy to obtain information about the CEP in general. Many results exist, however, dealing with CD (and recently CM) varieties. The CEP varieties of groups, rings, semigroups, and monoids, respectively, have also been described (see Biró—Kiss—Pálfy [82]).

Most works are based on the following observation.

PROPOSITION 3.1 (Day [71], Grätzer—Lakser [72b]). *Suppose $B\leqq A$ and each principal congruence on each C with $B\leqq C\leqq A$ can be extended to A. Then each congruence on B can be extended to A.*

The CEP is hereditary for subalgebras but not for homomorphic images (Fried [78]). It is preserved by direct limits but neither inverse limits (Biró—Kiss—Pálfy [82]) nor direct products. However, the situation changes in particular classes.

PROPOSITION 3.2 (a) (Kiss [81]). *In a congruence permutable (CP) variety homomorphisms preserve CEP.* □

(b) *In CD varieties finite products preserve CEP.* □

(c) (Kiss [a]). *In a CM variety a finite product is CEP provided that the square of each factor is CEP.*

We can use Proposition 3.1 to characterize CEP by means of Mal'cev schemes.

PROPOSITION 3.3. *A variety V has CEP iff for each Mal'cev scheme φ there exists a restricted one ψ such that $V\models\varphi\Rightarrow\psi$.* □

Now we have two aims: to find properties of CEP varieties and to provide sufficient conditions for a variety to have CEP. It may be surprising that there are natural 'non-artificial' examples for varieties having a single restricted Mal'cev scheme for all congruences — call them URCS varieties after Fried—Grätzer—Quackenbush [80b].

PROPOSITION 3.4 (Fried—Pixley [79]). *Discriminator and dual discriminator varieties are URCS, and so have CEP by Proposition 3.3.*

Thus Werner [78] gives us a wide range of examples of CEP varieties. Unfortunately one cannot get but CD examples in this way (Fried—Kiss [a]). It seems to be very hard to find conditions under which a general variety has CEP. Much work has been done in this direction in CD varieties (Quackenbush [74a], Davey [77], Kollár

[80]). All these results can be generalized to CM varieties and are summed up in the following results. We work within a fixed CM variety V.

PROPOSITION 3.5 (Kiss [a]). *$A \times A$ has CEP iff*
(a) *A has CEP,*
(b) *A satisfies the commutator identity*

$$[x, y] = x \wedge y \wedge [1, 1],$$

(c) *for each $B \leqq A$ and congruences θ, ψ on A,*

$$[\theta, \psi] \restriction B = [\theta \restriction B, \psi \restriction B].$$

THEOREM 3.6 (Kiss [a]). *V has CEP iff $U_p Si(V)$ has CEP and the square of each SI has CEP. ($Si(\mathscr{K})$ stands for the class of SI members in $\mathscr{K}$.)*

THEOREM 3.7 (Kiss [a]). *Suppose that the free algebra on four generators in V is finite. Then V has CEP iff the square of each SI has CEP.*

THEOREM 3.8 (Kiss [a]). *If V_1 and V_2 are two subvarieties of V with CEP then so is their join.*

These results give one the feeling that CEP for $\mathrm{Si}(V)$ is not sufficient to imply CEP for all of a variety V. Indeed, Day [73] has given a counterexample; though his example is not CM, the methods of commutator theory enable one to produce modular examples, too.

§4. Amalgamation and surjectivity. These areas are among the neglected fields of universal algebra. There seems to exist no general theory or result which would provide deeper information.* There are only some easy technical observations which apply in certain cases.

On the other hand there are strong theorems on concrete structures (especially lattices and semigroups).

Both the algebraic and model theoretic aspects of AP are summed up in the very readable dissertation of Zeitler [76]. Here we restrict ourselves to listing two of the most fundamental facts.

PROPOSITION 4.1 (Grätzer—Lakser [71]). *The amalgam $(A; \beta, B; \gamma, C)$ over a variety V can be completed in V iff for each $b_1 \neq b_2 \in B$ there exist a $D \in V$ and homomorphisms $\gamma_1 : B \to D$, $\beta_1 : C \to D$ such that $\gamma_1 \beta = \beta_1 \gamma$ and $\gamma_1(b_1) \neq \gamma_1(b_2)$ and the same holds for C.* □

(Indeed, the amalgam can be completed by the product of these D-s.)

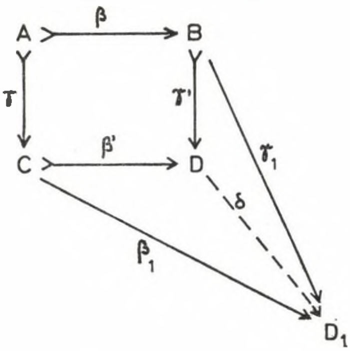
PROPOSITION 4.2 (Quackenbush [74b]). *Let V be a CEP variety containing only simple SI algebras. The amalgam $(A; \beta, B; \gamma, C)$ can be completed in V iff for any maximal congruence θ of B there exists a maximal congruence ψ of C with the same restriction Φ to A such that $(A/\Phi; B/\theta; C/\psi)$ can be completed by a simple algebra of V, and the same condition for B and C interchanged.* □

* Added in proof. For recent progress see C. H. Bergman [81].

COROLLARY 4.3 (Quackenbush [74b]). *A variety generated by a quasi-primal algebra A has the AP iff each proper inner automorphism of A extends to an automorphism (that is, A is demi-semi-primal).*

(This corollary is easy to prove by using basic facts about quasiprimal algebras (see e.g. Werner [78]).)

AP can be related to the solvability of algebraic equations (Hule [76], [78], [79]), and to free products (Grätzer—Lakser [71]) as well. Moreover, one can define in an arbitrary $\mathcal{K}$ the *free product* of algebras B, C *with amalgamated subalgebra* A to be an algebra $D\in\mathcal{K}$ together with embeddings $\gamma': B \rightarrowtail D$, $\beta': C \rightarrowtail D$ which coincide on A, with the property that for each D_1 and homomorphisms $\gamma_1: B\to D_1$, $\beta_1: C\to D_1$ coinciding on A there exists a unique homomorphism $\delta: D\to D_1$ with $\delta\beta'=\beta_1$ and $\delta\gamma'=\gamma_1$.



PROPOSITION 4.4. *In varieties with AP the free products with amalgamated subalgebra exist.* □

For a general idea of settling some specific structures note that the passage from semigroups to semigroup rings and from rings to their multiplicative semigroups makes it possible to carry over several results concerning AP, SAP, IPA, and ES from semigroups to rings and vice versa. One also often investigates in concrete cases which algebras A have the property that all amalgams $(A; \beta, B; \gamma, C)$ can be completed (embedded).

Finally we mention that many papers deal with the problem of amalgamating several (concrete) structures with all intersections prescribed. For a general investigation see Lanckau [69, 70] and Iskander [65].

There is only one purely algebraic paper which deals with the surjectivity problem in the most general setting. Let us recall its main result.

First of all it is clearly sufficient to investigate whether embeddings of proper subalgebras can be epimorphisms. If $A\leq B$ then define the *dominion* $\mathrm{Dom}_B(A)$ of A in B to be the set of all elements b of B with the property that for each pair of homomorphisms $\alpha, \beta: B\to C$ into some $C\in\mathcal{K}$ $\alpha\restriction A=\beta\restriction A$ implies $\alpha(b)=\beta(b)$. Clearly, $\mathrm{Dom}_B(A)=B$ iff the embedding of B is an epimorphism; such subalgebras are called *dense*. An algebra A is called *saturated* if for each $B\gneqq A$ we have $\mathrm{Dom}_B(A)\lneqq B$ and it is *absolutely closed* if $\mathrm{Dom}_B(A)=A$ for each $B\geqq A$.

PROPOSITION 4.5. *A variety V has IPA iff each $A \in V$ is absolutely closed. V has ES iff each $A \in V$ is saturated.* □

One has the following characterization of the dominion in classes admitting coproducts (e.g. varieties; we do not assume that the coordinate mappings are injective).

ZIG-ZAG THEOREM 4.6 (Isbell [66a]). *Suppose $\mathcal{K}$ admits coproducts and $A \leq B \in \mathcal{K}$. Let $B * B$ be the coproduct of two copies of B with coordinate mappings $\varrho_1, \varrho_2 : B \to B * B$. The following are equivalent for a $d \in B$:*

(i) $d \in \mathrm{Dom}_B(A)$.

(ii) *There exists a finite sequence $w_0 = \varrho_1(d), \ldots, w_n = \varrho_2(d)$ in $B * B$ such that for each $0 \leq i < n$ the element (w_i, w_{i+1}) lies in the subalgebra of $(B * B) \times (B * B)$ generated by all elements of three forms (x, x); $(\varrho_1(a), \varrho_2(a))$; $(\varrho_2(a), \varrho_1(a))$ $(a \in A)$.*

(iii) *$(\varrho_1(d), \varrho_2(d))$ is in the congruence of $B * B$ generated by the pairs $(\varrho_1(a), \varrho_2(a))$ $(a \in A)$.* □

COROLLARY 4.7 (Isbell [66a]). *Dom_B is a closure operator on the subalgebras of each $B \in \mathcal{K}$. No object of $\mathcal{K}$ is the domain of a proper class of inequivalent epimorphisms (since the cardinalities of the dominions are bounded). If $\mathcal{K}$ admits infinite coproducts as well, then each object can be embedded into an absolutely closed algebra.*

Now we prove Proposition 1.11. Suppose V is a variety having AP and ES, and $A \leq B \in V$. It is enough to show, by Proposition 4.5, that for each $c \in B \setminus A$ there exist homomorphisms α, β into some $C \in V$ coinciding on A but not at c. Consider the subalgebra B' of B generated by c and A. Because of ES there exist homomorphisms α' and β' from B' to some $C' \in V$ coinciding on A but not on B' and hence not at c. Replacing C' by $B' \times C'$ we may suppose that α' and β' are injective. Now completing the amalgam $(B'; \mathrm{id}, B; \alpha', C')$ we can "extend" α' to B. Extending β' similarly we are ready with the proof.

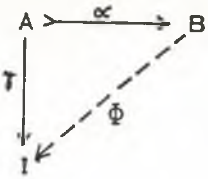
Finally, we give an example of a variety V which has ES but not IPA. Let $n \geq 4$ and consider the variety V_n of all semigroups with zero satisfying $S^n = 0$. We claim that V_n does the job. First we prove ES. In fact, if $A \lneqq B \in V_n$ then it is easy to see that $A \cup B^2 \lneqq B$. Therefore the natural mapping $B \to B/A \cup B^2$ coincides with the 0 mapping on A but not on B.

Now we show that IPA is not satisfied. We use Proposition 4.5. Consider the free semigroup S on the generators x, y, z, and let B be the subsemigroup generated by xz, zy, z. Set $\bar{S} = S/S^4$ and $\bar{B} = B \cup S^4/S^4$. Then $\bar{S} \in V_n$ and for the images $\bar{x}, \bar{y}, \bar{z}$ of x, y, z we have $\overline{xz}, \overline{zy}, \bar{z} \in \bar{B}$ but $\overline{xzy} \notin \bar{B}$ (since $xzy \notin B \cup S^4$). But $\overline{xzy} \in \mathrm{Dom}_{\bar{S}}(\bar{B})$ is shown by the Zig-Zag Theorem.

We mention that the same computations show that the variety of rings satisfying $R^n = 0$ also has ES but not IPA if $n \geq 4$.

§5. Injectivity. What more can we say about this concept? Whenever we have to decide whether a class has EI, we can just test the conditions having been settled in the preceding paragraphs. Nevertheless, it is possible that if a generating subset of a variety is given with some property on injectivity, then this "goes up" despite the fact that this is not necessarily true e.g. for amalgamation. Also, we may be interested in the structure of injective algebras in order to see whether or not there are enough of them.

Universal algebraic theorems of this kind have been proved only for varieties being very close to CD ones so far. Before presenting them we have to introduce an "intermediate" concept. We say that I is a *weak injective* (over an HS-closed $\mathscr{K}$) if each "injectivity diagram" with epic γ



can be completed. We have

PROPOSITION 5.1 (Grätzer—Lakser [72a]). *Each injective is a weak injective and each weak injective is an absolute retract. Conversely, if $\mathscr{K}$ has CEP then absolute retracts are weak injectives.* □

We have already mentioned in §2 that products of injectives are injective but this is not true for absolute retracts. Let us call a subalgebra $A \leqq \prod_{i \in I} A_i$ a *subdirect retract* of the family $\{A_i \colon i \in I\}$ if it is a retract of $\prod_{i \in I} A_i$ and all its projections are onto the A_i-s.

PROPOSITION 5.2 (Grätzer—Lakser [72a]). *A retract of an injective is an injective. A subdirect retract of weak injectives is a weak injective.* □

There have been very nice initiatives to describe injectives as Boolean extensions in Day [72], Quackenbush [74a] and Davey [76], and of course there were similar results in concrete classes. All these results can be put under a common roof as was shown by Davey and Werner [79]. In order to formulate their main theorem we remind the reader of some definitions.

For a finite algebra A and a Boolean algebra B the *bounded Boolean power*, $A[B]^*$, is defined as the algebra of continuous functions from the Boolean space of prime ideals of B into the discretely topologized algebra A.

A first order formula $\alpha(x, y)$ which is a $\exists\forall$ conjunct of equations is said to be a *simplicity formula* for a class $\mathscr{K}$ if for each $a, b \in C \in \mathscr{K}$, $C \models \alpha(a, b)$ iff $\theta(a, b)$ is trivial (that is, it is the least or the greatest congruence on C). Finally we say that $\mathscr{K}$ has *factorizable congruences* if for all n and all $A_0, \ldots, A_n \in \mathscr{K}$ the natural map from $\mathrm{Con}\,(A_0) \times \ldots \times \mathrm{Con}\,(A_n)$ to $\mathrm{Con}\,(A_0 \times \ldots \times A_n)$ is onto.

THEOREM 5.3 (Davey—Werner [79]). *Let V be a variety, let $\mathscr{K}$ be a finite set of finite algebras from V, and assume that:*

(a) $\mathrm{Si}(V) \subseteq \mathrm{IS}(\mathscr{K})$,

(b) *there exists a simplicity formula for $\mathscr{K}$,*

(c) *$\mathscr{K}$ has factorizable congruences.*

Then I is a (weak) injective in V iff I is isomorphic to $A_0[B_0]^* \times \ldots \times A_n[B_n]^*$, where for all $j \leqq n$ $A_j \in H(\mathscr{K}) \cap \mathrm{Si}(V)$, A_j is a (weak) injective over V, B_j is a complete Boolean algebra, and the algebras A_j are pairwise nonisomorphic.

In Davey—Werner [79] there is also a complete discussion concerning the applications of this theorem for proving known and new concrete and general results.

$A[B]^*$ is always a subdirect retract of copies of A (Davey [77]), hence the "if" part is clear. If V is CD then (c) is satisfied.

A frequent choice of $\mathcal{K}$ is the set of maximal SI algebras of V. If $\mathcal{K}$ consists of simple algebras then (b) holds, and we have the same in some particular classes of lattice-ordered algebras.

The last question is: how to find (weak) injectives over V in $H(\mathcal{K})\cap \mathrm{Si}(V)$? This leads us to the field of "going up" theorems. We present the most general ones.

THEOREM 5.4 (Davey [77]). *A SI member of a CD variety V is a weak injective over V iff it is a weak injective over $U_p\mathrm{Si}(V)$.*

THEOREM 5.5 (Kollár [80]). *Let V be a CD variety generated by finitely many finite algebras and set $\mathcal{K}=\mathrm{HSSi}(V)$. Then V has enough injectives iff*

(i) *each maximal SI is injective over $\mathcal{K}$,*

(ii) *every retract of any maximal SI is the direct product of SI algebras which are injective over $\mathcal{K}$.*

Note that (ii) cannot be omitted (Kollár [80]).

We mention a result settling injective hulls. Fixing an HS-closed $\mathcal{K}$ we say that $Q\geqq A$ is an *injective hull* of A if Q is injective and this extension is essential. We have:

THEOREM 5.6 (Banaschewski [70], Bacsich [72b]). *The injective hull of A is unique up to isomorphism over A. If $\mathcal{K}$ has EI then each $A\in\mathcal{K}$ has an injective hull. In this case each maximal essential extension and each minimal injective extension is the injective hull.* □

The theory of *equationally compact* algebras, that is, algebras in which whenever each finite subsystem of a system of equations can be solved, then the whole system admits a solution, is related to injectivity as well. Indeed, if we consider a special class of monomorphisms (the so-called pure embeddings) then the resulting injectivity concept is exactly equational compactness. For an exposition of this topic see Appendix 6 (written by G. H. Wenzel) in Grätzer [79].

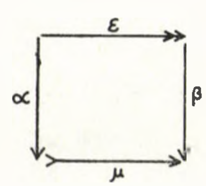
We mention that all members of a variety V are injective iff they are all equationally compact (Mamedov [78]).

§6. Categorical generalizations. In what follows we show how to carry over the preceding considerations from classes of algebras to abstract categories. We fix a category $\mathcal{K}$ with small Hom-sets and two classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ containing all $\mathcal{K}$-isomorphisms and being closed under them, such that the following holds:

(I) Every $\alpha\in\mathcal{K}$ allows a factorization $\alpha=\mu\varepsilon$ with $\mu\in\mathcal{M}$ and $\varepsilon\in\mathcal{E}$.

(II) Every $\varepsilon\in\mathcal{E}$ is a $\mathcal{K}$-epimorphism and every $\mu\in\mathcal{M}$ is a $\mathcal{K}$-monomorphism.

(III) For every commutative diagram



with $\varepsilon\in\mathcal{E}$ and $\mu\in\mathcal{M}$ there is a (necessarily unique) diagonal morphism δ with $\delta\varepsilon=\alpha$ (and $\mu\delta=\beta$).

(IV) For every object A, there is only a set of nonisomorphic $\mathcal{M}$-morphisms with codomain A and only a set of nonisomorphic $\mathcal{E}$-morphisms with domain A.

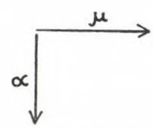
From (I)—(III) one gets the following properties: $\mathcal{E}$ and $\mathcal{M}$ are closed under composition; $\mathcal{E}\cap\mathcal{M}$ is the class of all isomorphisms; $\mathcal{E}$ and $\mathcal{M}$ are uniquely determined by each other; $\mathcal{E}$ is right cancellable (i.e., $\varepsilon\alpha\in\mathcal{E}$ only if $\varepsilon\in\mathcal{E}$) and, dually, $\mathcal{M}$ is left cancellable; $\mathcal{E}$ contains all extremal epimorphisms and, dually, $\mathcal{M}$ contains all extremal monomorphisms. (An epimorphism ε is *extremal* if it does not factorize over a proper monomorphism, i.e. $\varepsilon=\mu\beta$ with a monomorphism μ only if μ is an isomorphism; extremal monomorphism is dual.) □

For a class $\mathcal{K}$ of algebras the natural choice for $\mathcal{E}$, $\mathcal{M}$ is $\mathcal{E}=$ {surjective homomorphisms} and $\mathcal{M}=$ {injective homomorphisms}. If $\mathcal{K}$ is closed under H or S (and isomorphisms), conditions (I)—(IV) hold. Categorically spoken, $\mathcal{E}$ is just the class of extremal epimorphisms and $\mathcal{M}$ is the class of all monomorphisms of $\mathcal{K}$. Note that, up to categorical equivalence, here the $\mathcal{E}$-morphisms with fixed domain A describe the congruences on A.

Replacing injective homomorphisms by $\mathcal{M}$-morphisms and congruences by $\mathcal{E}$-morphisms we are now able to generalize the notions introduced before. Properties (I)—(IV) are not needed in full for the next propositions but, for simplicity, it is convenient to have them present throughout this section. We begin with the properties transferability (TP), congruence extension (CEP), and amalgamation (AP), which now depend on the choice of $\mathcal{E}$ and $\mathcal{M}$.

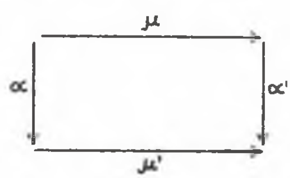
$\mathcal{K}$ satisfies (TP) ((CEP); (AP) resp.), if each span

(4)



with $\mu\in\mathcal{M}$ ($\mu\in\mathcal{M}$ and $\alpha\in\mathcal{E}$; $\mu\in\mathcal{M}$ and $\alpha\in\mathcal{M}$ resp.) can be completed to a commutative diagram

(5)



with $\mu'\in\mathcal{M}$ ($\mu'\in\mathcal{M}$; $\mu'\in\mathcal{M}$ and $\alpha'\in\mathcal{M}$ resp.). Proposition 1.6 *remains true* in this general setting, i.e. for $\mathcal{K}$ having finite products one has

$$(\mathrm{TP})\Leftrightarrow(\mathrm{CEP})\wedge(\mathrm{AP}).\quad\Box$$

If $\mathcal{K}$ has pushouts, it can be assumed without loss of generality that (5) is a pushout. So (AP) means the existence of free products with amalgamated

$\mathscr{M}$-subobject (cf. Proposition 4.4). Cf. Banaschewski [70], Dwinger [70], and Bacsich [72c].

The Strong Amalgamation Property (SAP) means that, for every span (4) with $\mu \in \mathscr{M}$ and $\alpha \in \mathscr{M}$, there is a pullback diagram (5) with $\mu' \in \mathscr{M}$ and $\alpha' \in \mathscr{M}$. If $\mathscr{K}$ has pushouts, (5) can be assumed to be also a pushout. The Intersection Property of Amalgamations (IPA) means that, for every span (4) with $\mu \in \mathscr{M}$ and $\alpha \in \mathscr{M}$ for which there is a commutative diagram (5) with $\mu' \in \mathscr{M}$ and $\alpha' \in \mathscr{M}$, there is even a pullback diagram (5) with $\mu' \in \mathscr{M}$ and $\alpha' \in \mathscr{M}$. Proposition 1.9 *remains true*, i.e.

$$(\mathrm{SAP}) \Leftrightarrow (\mathrm{AP}) \wedge (\mathrm{IPA}). \quad \square$$

Also, Proposition 1.8 *remains true*. Moreover, one has the following characterization of (IPA) (cf. Kelly [69], Ringel [72], Tholen [82a]):

Proposition 6.1. *A category $\mathscr{K}$ with pushouts satisfies (IPA) (with respect to $\mathscr{M}$), iff $\mathscr{M}$ consists of all regular monomorphisms.*

(A monomorphism $\mu\colon A \to B$ is regular, if every $\alpha\colon C \to B$ satisfying the equation $\xi\alpha = \eta\alpha$ whenever $\xi\mu = \eta\mu$ holds, factorizes as $\alpha = \mu\beta$; the class of regular monomorphisms coincides with that of equalizers in such a category.)

Since regular monomorphisms are in particular extremal, and since all extremal monomorphisms belong to $\mathscr{M}$, (IPA) implies that $\mathscr{M}$ is just the class of all extremal monomorphisms. Then $\mathscr{E}$ has to be the class of all epimorphisms. We denote the latter property by (ES), since, for a class of algebras (as at the beginning) with the natural factorization system, this means that epimorphisms are surjective. But note that in general (ES) depends, like (AP), (EI), etc., on the choice of $(\mathscr{E}, \mathscr{M})$.

Even without assuming the existence of pushouts one has

Corollary 6.2. *(IPA)$\Rightarrow$(ES).* $\square$

Note that Proposition 6.1 and Corollary 6.2 reformulate Proposition 4.5. (An object A in $\mathscr{K}$ is *saturated, absolutely closed* resp. if every monomorphism with domain A is extremal, regular resp.). Note furthermore that the converse implication in 6.2 does not hold, even for varieties (§4). Categories satisfying (ES) but not (IPA) contain extremal monomorphisms which are not regular. Now we get the following sharpening of Proposition 1.9:

Proposition 6.3 (Ringel [72]). *For $\mathscr{K}$ having pushouts of monomorphisms and equalizers one has*

$$(\mathit{SAP}) \Leftrightarrow (\mathit{AP}) \wedge (\mathit{ES}). \quad \square$$

Let us now consider the conditions (EAR), (EI), and (EIH). An object Q in $\mathscr{K}$ is an ($\mathscr{M}$-) *absolute retract*, if any $\mu \in \mathscr{M}$ with domain Q is a split monomorphism (i.e., has a left inverse). Q is ($\mathscr{M}$-) *injective*, if for all $\mu\colon A \to B$ in $\mathscr{M}$ and $\alpha\colon A \to Q$ in $\mathscr{K}$ there is a $\beta\colon B \to Q$ with $\beta\mu = \alpha$. Every injective object is an absolute retract, and the converse proposition holds under (TP) (cf. Proposition 1.5). A morphism $\mu\colon A \to B$ in $\mathscr{M}$ is called ($\mathscr{M}$-) *essential*, if for any $\gamma\colon B \to C$ one has $\gamma\mu \in \mathscr{M}$ only if $\gamma \in \mathscr{M}$. This is the same as to say that for every nonisomorphic $\varepsilon\colon B \to C$ in $\mathscr{E}$ one has $\varepsilon\mu \notin \mathscr{M}$. An essential morphism into an injective object is called an *injective hull* of its domain and is, up to isomorphisms, uniquely determined. $\mathscr{K}$ is said to satisfy (EAR) ((EI); (EIH) resp.) if for every object A there is an $\mathscr{M}$-morphism

$\mu: A \to Q$ with Q being an absolute retract (Q being injective; Q being injective and μ being essential resp.). Trivially one has

$$(\text{EIH}) \Rightarrow (\text{EI}) \Leftrightarrow (\text{EAR}) \wedge (\text{TP}).$$

(EI)$\Rightarrow$(EIH) holds if the category is, in a sense, finitary. However, this is false in the infinite case, as shown by the category of compact spaces. (Finitariness can be expressed by different conditions and is, implicitly, contained in conditions (V) and (VI) below.)

In order to analyze property (EAR) in more detail we put a further condition on our factorization system $(\mathcal{E}, \mathcal{M})$:

(V) For every well-ordered chain $(\alpha_{ij}: A_i \to A_j)_{0 \leqq i \leqq j < m}$ with $\alpha_{ii}=1$, $\alpha_{jk}\alpha_{ij}=\alpha_{ik}$ for $i \leqq j \leqq k$, and all α_{0i} in $\mathcal{M}$, one has an "upper bound" $(\alpha_i: A_i \to A)_{0 \leqq i < m}$ with $\alpha_j \alpha_{ij} = \alpha_i$ for $i \leqq j$ and α_0 in $\mathcal{M}$.

Using (V) one proves the categorical generalization of Proposition 1.3 (b) which is condition (E3) in Banaschewski [70]: For every $\mu \in \mathcal{M}$ there is an $\varepsilon \in \mathcal{E}$ such that $\varepsilon\mu$ is an essential $\mathcal{M}$-morphism (cf. Tholen [81]). Furthermore, in the presence of the harmless conditions (I)—(IV), condition (V) is strong enough to imply the following important result:

Proposition 6.4 (Banaschewski [71]). *If, for every object A in $\mathcal{K}$, there is only a set of nonisomorphic essential $\mathcal{M}$-morphisms with domain A, then $\mathcal{K}$ has property (EAR).*

One can even show that every object admits an essential $\mathcal{M}$-morphism into an absolute retract. Therefore, having (TP) one is able to construct injective hulls by 6.4.

The reverse implication in Proposition 6.4 does not hold in general; it does, if $\mathcal{K}$ fulfils the following weakening of condition (AP):

(ap) For every span (4) with $\mu \in \mathcal{M}$ and $\alpha \in \mathcal{M}$ one has a commutative diagram (5) with $\mu'\alpha \in \mathcal{M}$ (not necessarily $\mu' \in \mathcal{M}$).

(ap) follows from (EI) even if $\mathcal{K}$ does not have (finite) products. Therefore, in the presence of conditions (I)—(V), from Proposition 6.4 one gets the following generalization of Theorem 5.6:

Corollary 6.5. *(EIH)$\Leftrightarrow$(EI).*

Finally we want to consider the properties (ECS) and (RS) for abstract categories. A (small) set $\mathcal{C}$ of objects of $\mathcal{K}$ is called an ($\mathcal{M}$-) *cogenerating set* of $\mathcal{K}$, if all direct products of objects in $\mathcal{C}$ exist in $\mathcal{K}$ and if every object A in $\mathcal{K}$ admits some $\mathcal{M}$-morphism

$$A \to \prod_{i \in I} C_i$$

into a product of objects in $\mathcal{C}$; the latter is the same as to say that the canonical morphism

$$A \to \prod_{C \in \mathcal{C}} C^{\mathcal{K}(A,C)}$$

belongs to $\mathcal{M}$. If $\mathcal{M}$ is the class of all monomorphisms and if $\mathcal{K}$ has products, then $\mathcal{C}$ is a cogenerating set iff for every pair $\alpha, \beta\colon A \to B$ of different $\mathcal{K}$-morphisms one can find a morphism $\gamma\colon B \to C$ with $\gamma\alpha \neq \gamma\beta$ and $C \in \mathcal{C}$. The importance of the existence of a cogenerating set (ECS) in $\mathcal{K}$ was pointed out a long time ago: by the Special Adjoint Functor Theorem (cf. Freyd [64]), a functor from a category with (ECS) and certain limits has a left adjoint iff it preserves these limits.

The following theorem compares (ECS) with (EAR); it shows that Theorem 2.1 can be almost completely proved for abstract categories with (ap):

THEOREM 6.6. *Let $\mathcal{K}$ fulfil the property* (ap) *(see above). One then has the implication*

$$(ECS) \Rightarrow (EAR),$$

and (EAR) is equivalent to each of the following conditions:

(i) *Every object admits only a set of nonisomorphic $\mathcal{M}$-essential extensions.*

(ii) *For every object A there is an $\mathcal{M}$-morphism $\mu\colon A \to Q$ such that every $\mathcal{M}$-morphism $\nu\colon A \to B$ admits a morphism β with $\beta\nu = \mu$.*

The idea how to prove (ECS)$\Rightarrow$(ii) is in Barr [75]. For (EAR)$\Leftrightarrow$(i) see Proposition 6.4, and for (i)$\Leftrightarrow$(ii) cf. Tholen [81]. Note that (ii) implies (ap).

COROLLARY 6.7. *(ECS)$\wedge$(AP)$\Rightarrow$(EAR).*

An object S in $\mathcal{K}$ is called ($\mathcal{M}$-) *subdirectly irreducible,* iff for every $\mathcal{M}$-morphism $\mu\colon S \to \prod_{i \in I} A_i$ into a direct product there is at least one index $i \in I$ with $\pi_i\mu \in \mathcal{M}$, π_i being a canonical projection. $\mathcal{K}$ is ($\mathcal{M}$-) *residually small,* iff $\mathcal{K}$ contains only a set of nonisomorphic subdirectly irreducible objects. One has (without any condition on $\mathcal{K}$):

LEMMA 6.8. *(ECS)$\Rightarrow$(RS).* □

To get further results on the relationships between (ECS), (RS), and (EAR) we shall restrict ourselves throughout the rest of this section to the case

$\mathcal{M}$=all monomorphisms

and impose a sixth condition (VI) which sharpens condition (V) if $\mathcal{K}$ has colimits of well-ordered chains.

(VI) $\mathcal{K}$ has products and a generating set $\mathcal{G}$ (this is dual to (ECS)) such that, for every $G \in \mathcal{G}$, for every pair of different morphisms $\xi, \eta\colon G \to A_0$, and for every well-ordered chain $(\alpha_{ij}\colon A_i \to A_j)_{0 \leq i \leq j < m}$ with $\alpha_{ii} = 1$, $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ for $i \leq j \leq k$, and $\alpha_{0i}\xi \neq \alpha_{0i}\eta$ for all i, there is an upper bound $(\alpha_i\colon A_i \to A)_{0 \leq i < m}$ with $\alpha_j\alpha_{ij} = \alpha_i$ for $i \leq j$ and $\alpha_0\xi \neq \alpha_0\eta$.

Conditions (I)—(VI) *are still satisfied by every quasi-variety of (finitary) universal algebras* and, more generally, by every $\aleph_0$-presentable category in the sense of Gabriel and Ulmer [71].

In every category satisfying conditions (I)—(VI) one has Birkhoff's Subdirect Representation Theorem:

PROPOSITION 6.9 (Tholen [81], [82b]). *Every object A admits a monomorphism* $\mu: A \to \prod_{i \in I} S_i$ *into a product of subdirectly irreducible objects such that all morphisms* $\pi_i \mu$ *are extremal epimorphisms* (π_i *are the projections*).

COROLLARY 6.10. $(ECS) \Leftrightarrow (RS)$.

The connection between (RS) and (EAR) is given by

LEMMA 6.11 (Tholen [81]). *Condition* (i) *of Theorem 6.6 implies (RS).*

We therefore get:

THEOREM 6.12. *If* $\mathscr{K}$ *fulfils (ap), then (ECS), (RS), and (EAR) are pairwise equivalent.*

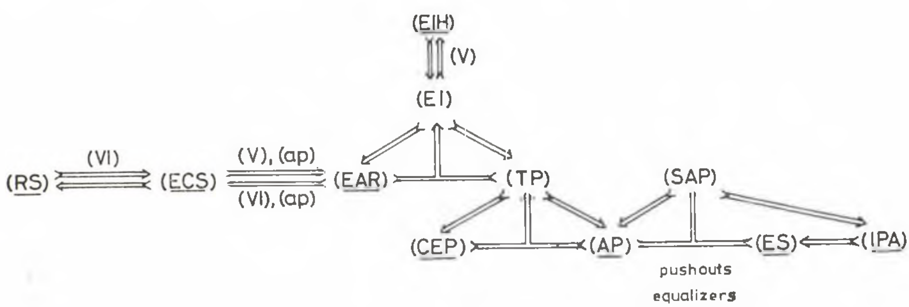
COROLLARY 6.13. $(ECS) \wedge (TP) \Leftrightarrow (RS) \wedge (TP) \Leftrightarrow (EIH)$.

2. Table of results

In what follows we list a number of categories of algebras and other structures, and indicate whether or not they possess the properties discussed above. In most cases we name only the objects of the categories, morphisms being the obvious ones (e.g. homomorphisms in categories of algebras); however, morphisms are specified in cases which might be ambiguous. For the readers' convenience we first repeat the abbreviations we use and the most important logical connections:

AP	Amalgamation Property	TP	Transferability Property
IPA	Intersection Property of Amalgamations	ECS	Existence of a Cogenerating Set
SAP	Strong Amalgamation Property	RS	Residual Smallness
ES	Epimorphisms are Surjective	EAR	Enough Absolute Retracts
CEP	Congruence Extension Property	EI	Enough Injectives
		EIH	Existence of Injective Hulls

For a category with direct products satisfying the (standard) assumptions (I)—(IV) (see §6) one has the implications below. Some of them require the additional assumptions (V) or (VI) which are still satisfied by every quasi-variety of universal algebras. Two implications require the additional assumption (ap) which is a weakening of (AP) (see the remark after Proposition 6.5); but (ap) is not needed in case of a variety. Another implication requires the existence of pushouts and equalizers which is also given in every quasi-variety.



The table contains only the underlined properties since the others can be obtained from them, even if the additional conditions (V), (VI), etc. are not satisfied.

We remind the reader that *all these properties depend on the choice of the factorization system* ($\mathscr{E}, \mathscr{M}$). If not otherwise stated (except for some cases where the choice is clear from the foregoing categories in the table) we choose $\mathscr{M}$=all monomorphisms and, consequently, $\mathscr{E}$=extremal epimorphisms. In most cases $\mathscr{E}$ is contained in the class of morphisms having underlying surjective mappings; then, if (ES) holds, epimorphisms are really surjective. But in general, (ES) just means that $\mathscr{E}$ is the class of all epimorphisms. So it may happen that (ES) holds even though epimorphisms are not surjective (Hausdorff spaces with $\mathscr{E}$=dense maps, for example), or that (ES) does not hold even though epimorphisms are surjective (topological spaces with $\mathscr{E}$=quotient maps). Those entries which may cause misunderstandings of (ES) are marked by †.

In many entries of the table we give the first reference (up to our knowledge) in which it is determined whether or not the category in question has the given property. In some cases, if the answer is easy, we refer to the first paper containing non-trivial results about the respective property. If the sign ∘ appears instead of a reference, this means we could not find any explicit reference, have checked the property ourselves, and felt that some of the readers might need a hint at the proof. These hints are put together and follow the table. Finally, if neither a reference nor the sign ∘ appears, this means that the given result is either well-known or easy to verify (maybe using some general theorems figuring in the previous part). For some varieties of algebras there is also a description of all its subvarieties possessing the given property; this is indicated by an * above the answer in the given entry, and an appropriate reference is also included. In some cases we do not know whether a category listed below enjoys some of the properties in question. In such a case either the entry is left blank, or a question mark is put there. The latter means that we think the problem is difficult.

The categories of lattices, modular lattices, and distributive lattices, respectively, are named with the supplement 'bounded or not'. In fact, it is easy to verify that the answers must be the same in the two cases. (In categories consisting of bounded lattices, morphisms are 0-1-preserving homomorphisms.)

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
sets	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
M-sets (M a monoid) = any variety of unary algebras	Yes$_\circ$	Yes	Yes	Yes	Yes	Yes	Yes	Yes$_\circ$ Berthiaume [67]
any similarity type of non-unary algebras	Yes	Yes	Yes	No	No	No	No	No
any functor category [$\mathscr{D}$, $\mathscr{S}et$] ($\mathscr{D}$ small)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Kacov [76]
any Grothendieck topos	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Ebrahimi [82]
any elementary topos	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
abelian groups	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Baer [40]
R-modules (R a unital ring) = any subvariety of such	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Eckmann—Schopf [53]
any additive functor category Add [$\mathscr{C}$, $\mathscr{A}b$] ($\mathscr{C}$ small, additive)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Weidenfeld [70]
any Grothendieck category (= abelian Ab5-categ.) with generator	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Grothendieck [57]
semigroups	No Kimura [57]	No	No	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No* Biró—Kiss—Pálfy [82]
finite semigroups	No Kimura [57]	No	No Howie—Isbell [67]	No	Yes	Yes	No$_\circ$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
commutative semigroups	No Kimura [57]	No	No* Higgins [83]	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No* Biró—Kiss—Pálfy [82]
regular semigroups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_0$	No
orthodox semigroups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_0$	No
inverse semigroups	Yes* Hall [75] Hall [78c]	Yes Howie—Isbell [67]	Yes	No	No$_0$	No	No$_0$	No
finite inverse semigroups	No Hall [75]	Yes Hall [78b]	Yes	No	Yes	Yes	No$_0$	No
commutative inverse semigroups	Yes* Imaoka [76a] Hall [78c]	Yes Imaoka [76a]	Yes	Yes$_0$	Yes	Yes	Yes	Yes Schein [76]
unions of groups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_0$	No
semilattices of groups (=Clifford semigroups)	No Hall [75]	Yes$_0$	Yes	No	No	No	No$_0$	No
bands	No* Hall [78a]	No Scheiblich [76]	?	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No
normal bands	Yes* Imaoka [76b] Biró—Kiss—Pálfy [82]	Yes Imaoka [76b] Scheiblich [76]	Yes	Yes* Biró—Kiss—Pálfy [82]	Yes	Yes	Yes	Yes* Biró—Kiss—Pálfy [82]
semilattices	Yes	Yes Horn—Kimura [71]	Yes	Yes	Yes	Yes	Yes	Yes Bruns—Lakser [70]
compact Lawson semilattices	No Hofmann—Mislove [76]	Yes	Yes Hofmann—Mislove [75]	No Stralka [77]				No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
left cancellative semigroups	No Howie [63]	No	No	No	No	No	No$_o$	No
commutative cancellative semigroups	Yes$_o$	No	No$_o$	Yes$_o$	Yes	Yes	Yes	Yes$_o$
monoids (with 1-preserving homomorphisms)	No$_o$	No	No	No$^*_o$	No	No	No	No$^*_o$
commutative monoids	No$_o$	No	No	No	No	No$_o$	No	No$^*_o$ Banaschewski [70]
commutative cancellative monoids	Yes$_o$	No	No$_o$	Yes$_o$	Yes	Yes	Yes	Yes Georgescu [71b]
small categories	No Trnková [65]	No	No Isbell [68b]	No	No	No	No	No
quasi-groups	Yes Ježek—Kepka [79]	Yes Ježek—Kepka [79]	Yes	No	No	No	No	No
commutative quasi-groups	Yes* Ježek—Kepka [77]	Yes* Ježek-Kepka [77]	Yes*	No	Yes	Yes Ježek—Kepka [77]	Yes	No
medial quasi-groups	Yes* Ježek—Kepka [77]	Yes* Ježek-Kepka [77]	Yes*	No	Yes	Yes Ježek—Kepka [77]	Yes	No
Steiner quasi-groups	Yes$_o$	Yes$_o$	Yes	No* Quacken-bush [76]				No Quackenbush [76]
loops	Yes$_o$	Yes$_o$	Yes	No	No	No	No	No
Steiner loops	Yes$_o$	Yes$_o$	Yes	No* Quacken-bush [76]				No Quackenbush [76]

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
groups	Yes Schreier [27]	Yes Schreier [27]	Yes	No* Biró—Kiss—Pálfy [82]	No	No	No	No
finite groups	Yes B. H. Neumann [54]	Yes B. H. Neumann [54]	Yes	No	Yes	Yes	No$_0$	No
solvable groups	No B. H. Neumann [60b]	Yes Hilton [72]	Yes	No	No	No$_0$	No$_0$	No
nilpotent groups	No Wiegold [59]	No$_0$	Yes Enjalbert [78]	No	No	No$_0$	No$_0$	No
torsion groups	No B. H. Neumann [60a]	Yes Hilton [72]	Yes	No	No	No$_0$	No$_0$	No
p-groups	No B. H. Neumann [60b]	Yes Hilton [72]	Yes	No	No	No$_0$	No$_0$	No
not necessarily associative rings	Yes Dididze [57]	Yes Dididze [57]	Yes	No	No	No	No	No
associative *R*-algebras (*R* a commutative unital ring)	No	No	No	No* E. W. Kiss [a]	No	No* McKenzie [82]	No	No
commutative associative *R*-algebras (*R* a commutative unital ring)	No	No	No	No* E. W. Kiss [a]	No	No* McKenzie [82]	No	No
Lie algebras (over a field)	Yes$_0$	Yes Reid [70]	Yes	No* E. W. Kiss [a]	No Pareigis—Sweedler [70]	No	No	No
regular rings	Yes$_0$	Yes Stenström [75]	Yes Gardner [75]	No	No	No	No$_0$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
commutative regular rings	Yes Cornish [77]	No	No$_o$	Yes Cornish [77]	No$_o$	No	No$_o$	No
integral domains	Yes Cornish [77]	No	No	No	No	No	No	No
Ore domains	Yes Felgner [a]	No	No$_o$	No	No	No	No	No
division rings	Yes Cohn [71]	Yes Cohn [71]	Yes	Yes	No	No	No	No
fields	Yes Jónsson [65]	No	No Burgess [65]	Yes	No	No	No	No
near-rings$_o$	No	No	No	No	No	No	No	No
preordered sets (with monotone mappings)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
partially ordered sets ($\mathcal{M}$=embeddings ($x\leq y\Leftrightarrow fx\leq fy$), $\mathcal{E}$=surj. morphisms)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Bacsich [72b]
lattices (bounded or not)	Yes Jónsson [60]	Yes Grätzer [78]	Yes	No	No	No	No	No
modular lattices (bounded or not)	No Grätzer—Jónsson—Lakser [73]	No	No Freese [79]	No	No	No	No	No
distributive lattices (bounded or not)	Yes Pierce [68]	No	No$_o$	Yes	Yes	Yes	Yes	Yes Banaschewski—Bruns [68]
complete lattices (with 0-1-preserving homomorphisms)	Yes$_o$	Yes$_o$	Yes	No	No$_o$	No$_o$	No$_o$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
algebraic lattices (with 0-1-preserving homomorphisms)	Yes$_o$	Yes$_o$	Yes	No	No$_o$	No$_o$	No$_o$	No
Stone algebras	Yes			Yes	Yes	Yes	Yes	Yes* Balbes—Grätzer [71] A. Day [72]
double Stone algebras	Yes*			Yes	Yes	Yes* Katriňák [74a]	Yes	Yes* Katriňák [77b]
distributive p-algebras	Yes* Grätzer—Lakser [71]			Yes Grätzer—Lakser [71]	No	No* Lee [70], Lakser [71]	No	No* A. Day [72] Grätzer—Lakser [72a]
distributive double p-algebras				Yes Katriňák [74b]	No	No Katriňák [80]	No	No
distributive Ockham algebras	Yes	No	No$_o$	Yes Berman [77]	Yes	Yes Berman [77]	Yes	Yes Goldberg [81]
de Morgan algebras	Yes			Yes	Yes	Yes	Yes	Yes Cignoli [75]
Kleene algebras	Yes			Yes	Yes	Yes	Yes	Yes Cignoli [75]
Heyting algebras	Yes A. Day [a]	Yes A. Day [a]	Yes	Yes A. Day [a]	No	No	No	No* A. Day [a]
Boolean algebras	Yes Dwinger—Yaqub [63]	Yes Dwinger—Yaqub [63]	Yes	Yes Sikorski [48]	Yes	Yes	Yes	Yes Halmos [61]
m-complete Boolean algebras	Yes Lagrange [74]	Yes Lagrange [74]	Yes	Yes	No	No Monk [67]	No	No Monk [67]

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
cylindric algebras (of fixed dimension α)	Yes if $\alpha=1$ No if $\alpha>1$ Comer [69] Pigozzi [71]	Yes if $\alpha=1$ No if $1<\alpha<\omega$	Yes if $\alpha=1$ Sain [a] No if $1<\alpha<\omega$ Andréka—Comer—Németi [a]	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
cylindric algebras of fixed characteristic $\neq 0$ (and of fixed dimension)	Yes Comer [68] Pigozzi [71]	Yes if $\alpha<\omega$	Yes if $\alpha<\omega$ Comer [a]	Yes	Yes	Yes Henkin—Monk—Tarski—Andréka—Németi [81]	Yes	Yes
locally finite cylindric algebras (of fixed infinite dimension)	Yes Daigneault [64b]	Yes Daigneault [64b]	Yes	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
representable cylindric algebras (of fixed dimension α)	Yes if $\alpha=1$ No if $\alpha>1$ Comer [69] Pigozzi [71]	Yes if $\alpha=1$ No if $1<\alpha<\omega$	Yes if $\alpha=1$ No if $1<\alpha<\omega$ Andréka—Comer—Németi [a]	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
cylindric-relativised set algebras (of fixed dimension)	Yes Németi [a]	Yes Németi [a]	Yes	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
weakly associative lattices	Yes Fried—Grätzer [76]	Yes Fried—Grätzer [76]	Yes	No* Fried [74b]	No	No	No	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
the variety generated by the weakly associative lattices with unique bound property	Yes Fried—Grätzer—Quackenbush [80a]	No	No$_o$	Yes Fried [74b]	No	No	No	No
topological spaces ($\mathscr{M}$=inj. cont. maps, $\mathscr{E}$=quotient maps)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
topological spaces ($\mathscr{M}$=embeddings, $\mathscr{E}$=surjective maps)	Yes	Yes	Yes	Yes	Yes	Yes		No Wyler [77]
T_o-spaces ($\mathscr{M}$=embeddings, $\mathscr{E}$=surjective maps)	Yes	No	No Baron [68]	Yes	Yes	Yes	Yes	No Banaschewski [77b]
T_1-spaces ($\mathscr{M}$=embeddings, $\mathscr{E}$=surjective maps)	Yes	Yes	Yes	Yes	No Mrówka [82]	Yes Vinárek [82]	No	No R.-E. Hoffmann [81]
Hausdorff spaces ($\mathscr{M}$=embeddings, $\mathscr{E}$=surjective maps)	No	No	No	No Kelly [69]	No			No
Hausdorff spaces ($\mathscr{M}$=closed embeddings, $\mathscr{E}$=dense maps)	Yes	Yes	Yes†	No Kelly [69]	No			No
completely regular Hausdorff spaces ($\mathscr{M}$=embeddings, $\mathscr{E}$=surjective maps)	Yes	No	No	Yes	Yes	Yes		
completely regular Hausdorff spaces ($\mathscr{M}$=closed embeddings, $\mathscr{E}$=dense maps)	Yes	Yes	Yes†	Yes	No Herrlich [67]		Yes	

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
compact Hausdorff spaces ($\mathscr{M}$ = closed embeddings = inj. maps., $\mathscr{E}$ = dense maps = surj. maps)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No Banaschewski [70]
compact 0-dim. Hausdorff spaces ($\mathscr{M}$ = closed embeddings = inj. maps, $\mathscr{E}$ = dense maps = surj. maps)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No$_\circ$
topological groups (not necessarily T_2)	Yes	No	No†	No	No	No	No	No
Hausdorff groups	No Tholen [82a]	No	No Nummela [78]	No	No			No
compact (Hausdorff) groups	No Bergman [d]	Yes Poguntke [73]	Yes Reid [70], Poguntke [70]	No	Yes$_\circ$	Yes$_\circ$	No$_\circ$	No
Hausdorff abelian groups	No Tholen [82a]	No	No					No
compact abelian groups	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No Banaschewski [70]
topological vector spaces (over a Hausdorff top. field)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
Hausdorff topological vector spaces	No	No	No					
locally convex spaces	Yes	No	No†	Yes	Yes	Yes	Yes	Yes

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
Hausdorff locally convex spaces	No	No	No		Yes	Yes		
metric spaces (with contractions) ($\mathscr{M}$ = isometric embeddings, $\mathscr{E}$ = surj. morphisms)	Yes	No	No	Yes	Yes	Yes	Yes	Yes Isbell [64a]
compact metric spaces	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Isbell [64a]
normed vector spaces (with linear contractions)	Yes	No	No	Yes	Yes	Yes	Yes	Yes Nachbin [50]
Banach spaces	Yes	No	No	Yes	Yes	Yes	Yes	Yes Phillips [40]
commutative C*-algebras	Yes	Yes	Yes Reid [70]	Yes	Yes	Yes	Yes	Yes Gleason [58]

Hints for proving some of the results in the table

M-sets: AP, EIH

Notice that algebras are assumed to be non-empty. The two answers in question are false if this assumption is cancelled and we have a variety of unary algebras with two constant operations which may or may not coincide (see Higgs [71] and Lakser [73]).

Inverse semigroups, Unions of groups, Semilattices of groups, Commutative regular rings: EAR, ECS

Given a class $\mathscr{K}$ of algebras, if new operations can be introduced in these algebras so that their homomorphisms remain the same and $\mathscr{K}$ becomes a variety, then RS⇔EAR in $\mathscr{K}$ (in fact, between $\mathscr{K}$ and its enriched copy there is an isomorphism which commutes with the underlying set functor).

Commutative inverse semigroups: CEP

Make use of the fact that these semigroups are strong semilattices of abelian groups.

Semilattices of groups: IPA

These semigroups are strong semilattices of groups; first we amalgamate the corresponding components, then we extend these larger amalgams (by using the structure homomorphisms) to the 'skeleton' obtained by amalgamating the two semilattices.

Left cancellative semigroups: EAR

This class is a quasi-variety containing arbitrarily large subdirectly irreducible algebras, so it does not have EAR by Taylor [72].

Commutative cancellative semigroups

Embed the semigroups under consideration into their quotient groups.

Monoids, Commutative monoids, Commutative cancellative monoids

The results in question follow from those on the corresponding classes of semigroups by choosing an appropriate semigroup and adjoining an external unit to it.

Steiner quasi-groups, Loops, Steiner loops: SAP

Use the result that in these categories every partial algebra can be extended to a complete one (see Bruck [58]).

Finite groups: EAR

Notice that every finite group can be embedded in a suitable finite alternating group.

Solvable groups, Nilpotent groups, Torsion groups, p-groups: RS

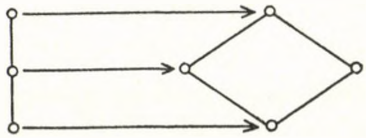
The central product of arbitrarily many copies of a fixed non-abelian group of order p^3 is subdirectly irreducible.

Near-rings

The ring constructions work.

Distributive lattices, distributive Ockham algebras, HSP*(weakly ass. lattices with UBP):* ES

Consider the embedding



Complete lattices, Algebraic lattices

Notice that every lattice can be embedded into a partition lattice, which is a simple algebraic lattice.

(HINTS BASED ON SUGGESTIONS OF G. M. BERGMAN)

Regular rings: EAR

The 2×2 matrix ring over R is an essential extension of R.

Regular, orthodox, inverse, finite inverse, finite semigroups: EAR

The same as above, using the 2×2 Rees matrix semigroup with the identity as sandwich matrix. We start by adjoining an external zero and an external identity to the given semigroup.

Solvable, nilpotent, torsion, and p-groups: EAR

For a group G and $n(=1, 2, \dots$ or $\infty)$ let $\bar{G}$ be the semidirect product of Z_n and G^n with generator x of Z_n acting on G^n by the shifting operator. Embed G as the diagonal; observe that x centralizes this subgroup but the commutator of x and $(e, a, \dots, a^{n-1})$ (or $(\dots, a^{-1}, e, a, \dots)$) is $(\dots, a, a, \dots)$ for each $a\in G$, $|a|\,|n$, thus G is not a retract in $\bar{G}$. Now for

torsion resp. p-groups: if $e\neq a\in G$ and $n=|a|$ then $\bar{G}$ is a torsion resp. p-group;

solvable groups: Let A be a non-trivial abelian normal subgroup of G, $n=\infty$, and consider the subgroup of $\bar{G}$ generated by Z_∞ and the elements $(\dots, ga^{-1}, g, ga, \dots)$ with $g\in G$, $a\in A$. This is solvable but G is not a retract of it;

nilpotent groups: the previous construction works with $A=Z(G)$.

Nilpotent groups: IPA

Let B be the group on $Q\times Q\times Q$ with rule of multiplication $(a, b, c)\times \times(a', b', c')=(a+a', b+b', c+c'+a'b)$ and $A<B$ be the subgroup consisting of the elements with entries in Z. Then $(0, 0, a)\in \mathrm{Dom}_B(A)$ for all $a\in Q$. In order to show this, write $(a, b, c)\in B$ as $p(a)q(b)r(c)$, where $p(a)=(a, 0, 0)$ etc., p, q, r are

homomorphisms of the additive group Q into B, and $q(a)p(b)=p(b)q(a)r(ab)$, further, make use of the fact that for each pair α, β of homomorphisms of Q to a nilpotent group, if $\alpha(Z)$ centralizes $\beta(Q)$ then so does $\alpha(Q)$.

Lie algebras: SAP

If we have an amalgam of A, B, C, $A \subseteq B$, $A \subseteq C$, then consider the universal enveloping algebras $k(A)$, $k(B)$, $k(C)$, and notice that $k(B)$ and $k(C)$ as modules are free over $k(A)$. Now the coproduct of $k(B)$ and $k(C)$ with amalgamation of $k(A)$ regarded as a Lie algebra can be seen to do the job.

Regular rings: AP

First observe that every regular ring is a subdirect product of regular rings containing prime fields, and that a product of regular rings is regular. This allows us to reduce to the case where our rings are algebras over a fixed *field* k. Second, note that every k-algebra is embeddable in a regular k-algebra, namely a full algebra of endomorphisms of a vector space. Hence it suffices to show that every amalgam of regular algebras over k can be completed with a not necessarily regular k-algebra; equivalently, that given regular k-algebras $A \subseteq B, A \subseteq C$, the algebras B and C embed in their coproduct over A. This can be deduced from a description of the coproduct due to P. M. Cohn [59]. Namely, it is shown that as left A-module, this coproduct is the direct limit of the chain $B \to C \otimes_A B \to B \otimes_A C \otimes_A B \to \dots$. Now since A is regular, tensoring with inclusions $A \subseteq B$ and $A \subseteq C$ gives 1-1 maps, so B embeds in the direct limit; likewise C does, as required.

Commutative regular rings: ES

If $A < B$ are fields of characteristic $p > 0$, and B is generated by an element x over A with $x^p \in A$, then $x \in \mathrm{Dom}_B(A)$, thus A is dense in B.

Ore domains: ES

Let B be the subring of $Q[x]$ generated by x and $A < B$ be generated by x^2 and x^3. Then we again have $x \in \mathrm{Dom}_B(A)$, thus $\mathrm{Dom}_B(A) = B$.

Compact 0-dimensional Hausdorff spaces: EIH

Since the essential embeddings here are clearly surjective, we have to show (by Stone duality) that not every Boolean algebra is projective. But the natural homomorphism of $P(\omega)$ to its factor modulo the ideal of finite subsets is easily seen not to be right invertible.

Compact groups: ECS, RS

The Peter—Weyl Theorem says that the Hilbert space of L^2 functions on a compact group is a Hilbert-space direct sum of finite-dimensional subspaces closed under the action of the group by translation. Since the action of the group on the full space separates group-elements, so do the actions on these finite-dimensional spaces, taken together. But a norm-preserving action on an n-dimensional Hilbert space is equivalent to a homomorphism into the unitary group $U(n)$, which is compact, so the unitary groups form a cogenerating set.

Compact groups: EAR

First note that every $U(n)$ can be embedded in a $PSU(n')$, $n' \geqq 4$. These are simple groups, hence a retract of a direct product of copies of these groups will be the direct product of a subfamily thereof. Now since $PSU(i) \subseteqq PSU(i+1)$, we see that every compact group G can be embedded, on the one hand, in a direct product of groups $PSU(i)$ where i takes on even values, and on the other hand, in a direct product where i takes on odd values. Hence if G is a retract of both of these products, it must itself be a direct product of both sorts; contradiction.

3. Bibliography

The following list of papers is intended to be (though it is certainly not) complete concerning items which might be useful in deciding whether a given category (of structures) enjoys one of the properties discussed in the preceding text and table. E.g., it contains papers determining rings over which injectivity of modules is equivalent with some weaker property. Furthermore, if not all objects of a category have a given property, it might be interesting to know which of them do; papers going in this line are also included here, e.g. investigations of semigroup amalgams which can be completed or embedded. In order to facilitate the work of a reader who wants to find information on a given topic, we also include classification codes of the enlisted papers. These codes are the following:

AP	Amalgamation Property	EIH	Existence of Injective Hulls
CEP	Congruence Extension Property	ES	Epimorphisms are Surjective
EAR	Enough Absolute Retracts	IPA	Intersection Property of Amalgamations
ECS	Existence of Cogenerating Sets	RS	Residual Smallness
EI	Enough Injectives	SAP	Strong Amalgamation Property

On the other hand, we restricted ourselves to consider only the validity of these properties and not to go further. E.g., the reader will find here neither the papers dealing with the structure of amalgamated free products of groups nor those investigating injective modules in order to solve ring theoretical problems, nor a comprehensive literature on equational compactness though, as we have seen in the expository part, the latter is in connection with injectivity.

ABRAMOVIČ, JU. A.
71 Injective hulls of normed lattices, *Dokl. Akad. Nauk SSSR* **197** (1971), 743—745 (in Russian). *MR* **44**#7257 EIH

AGAPITOV, K. V.
81 Epimorphisms of generalized uniserial rings and bicentralizers of torsion modules over bounded hereditary noetherian prime rings, *Abelian Groups and Modules*, ed. by L. A. Skornyakov, Tomsk University, Tomsk, 1981, 3—19 (in Russian). ES

AHSAN, J. and DEEBA, E. Y.
81 Injective BCK-algebras, *Math. Japon.* **26** (1981), 547—551. *MR* **82m**: 06011 EI

ALLOUCH, D.-A.
74 Monoïdes de fractions et enveloppe injective d'un monoïde, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 833—836. *MR* **51**#589 EIH

AMIR, D. and ARBEL, B.
73 On injections and surjections of continuous function spaces, *Israel J. Math.* **15** (1973), 301—310. *MR* **53**#1239 ES

ANDERSON, F. W.
78 Simple injective modules, *Math. Scand.* **43** (1978), 204—210. *MR* **80g**: 16027 EI

ANDERSON, M. and CONRAD, P.
81 Epicomplete *l*-groups, *Algebra Universalis* **12** (1981), 224—241. *MR* **82d**: 06017 ES

ANDRÉKA, H., COMER, S. D. and NÉMETI, I.
a Epimorphisms of finite dimensional cylindric algebras need not be surjective (manuscript). ES