A note on local compactness

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Abstract

We propose a categorical definition of locally-compact Hausdorff object which gives the right notion both, for topological spaces and for locales. Stability properties follow from easy categorical arguments. The map version of the notion leads to an investigation of restrictions of perfect maps to open subspaces.

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1 Introduction

Both, for topological spaces and for locales, locally compact Hausdorff spaces are characterized as the spaces which are openly embeddable into compact Hausdorff spaces. While in $\mathcal{T}op$ this is an obvious consequence of Alexandroff's one-point-compactification, in $\mathcal{L}oc$ one uses results of Vermeulen [10] to establish this result. In this note we show that, taking this characterization as the defining property for (Hausdorff) local compactness, one establishes practically all standard stability properties of local compactness – with the big exception of Whitehead's Theorem – with very brief and purely categorical arguments. Hence, $\mathcal{T}op$ and $\mathcal{L}oc$ may actually be replaced by an arbitrary category $\mathcal{X}$ which comes equipped with a proper factorization system and a closure operator, as in [4], [5], [2]. The only sticky point at this level of generality is the pullback behaviour of c-open maps (w.r.t. the closure operator c), which is not as smooth as in $\mathcal{T}op$ or $\mathcal{L}oc$, but which has been described in general in [6]. We briefly recall this and all other needed tools at the beginning of Section 2, where we introduce local c-compactness and derive first properties. Section 3 presents further properties which are derivable in the presence of the Stone-Čech compactification (w.r.t. c). All this is done in the context of topological spaces, while Section 4 explains how to pass from spaces to objects in a category. This passage pays off as one can now make the passage from spaces (or objects) to maps (or morphisms), by considering the notion of local compactness in the slices of the given category. Hence, in Section 5 we exploit the properties derived previously in the case of locally c-compact maps, which are simply restrictions of c-perfect maps to c-open subobjects.

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2 Locally c-compact spaces

2.1 Preliminaries. In what follows, “space” means “topological space” and “map” means “continuous mapping”. A *closure operator* c on the category $\mathcal{T}\!op$ of spaces and maps assigns to all $M \subseteq X \in \mathcal{T}\!op$ a set $c_X(M) \subseteq X$, such that c_X is extensive and monotone and the c-continuity condition is satisfied: $f(c_X(M)) \subseteq c_Y(f(M))$ for all maps $f : X \to Y$ and $M \subseteq X$; equivalently, $c_X(f^{-1}(N)) \subseteq f^{-1}(c_Y(N))$ for all f and $N \subseteq Y$ (cf. [4], [5]). Obviously, $M \subseteq X$ is called *c-closed* (*c-dense*) in X if $c_X(M) = M$ ($c_X(M) = X$, respectively). c is *idempotent* (*weakly hereditary*) if $c_X(M)$ is c-closed in X (if M is c-dense in $c_X(M)$, respectively), for all $M \subseteq X$. (Subsets are always provided with the subspace topology.) c is *hereditary* if $c_Y(M) = Y \cap c_X(M)$ for all $M \subseteq Y \subseteq X \in \mathcal{T}\!op$. The following classes of maps will be of interest:

$$
\begin{array}{llll}
f \in \mathcal{M} & :\Leftrightarrow\ f \textit{ embedding} & \Leftrightarrow & f \text{ induces a homeomorphism } X \xrightarrow{\sim} f(X) \subseteq Y, \\
f \in \mathcal{E} & :\Leftrightarrow\ f \textit{ surjective} & \Leftrightarrow & f(X) = Y, \\
f \in \mathrm{Ds}(c) & :\Leftrightarrow\ f \ c\textit{-dense} & \Leftrightarrow & f(X)\ c\text{-dense in } Y, \\
f \in \mathrm{Cl}(c) & :\Leftrightarrow\ f \ c\textit{-closed} & \Leftrightarrow & f(c_X(M)) = c_Y(f(M)) \text{ for all } M \subseteq X, \\
f \in \mathrm{Op}(c) & :\Leftrightarrow\ f \ c\textit{-open} & \Leftrightarrow & f^{-1}(c_Y(N)) = c_X(f^{-1}(N)) \text{ for all } N \subseteq Y, \\
f \in \mathrm{Ini}(c) & :\Leftrightarrow\ f \ c\textit{-initial} & \Leftrightarrow & c_X(M) = f^{-1}(c_Y(f(M))) \text{ for all } M \subseteq X, \\
f \in \mathrm{Fin}(c) & :\Leftrightarrow\ f \ c\textit{-final} & \Leftrightarrow & c_Y(N) = f(c_X(f^{-1}(N))) \text{ for all } N \subseteq Y.
\end{array}
$$

For c the usual closure, the last five notions assume the expected meaning; for interrelationships between them, see [6], [3]. In particular, we record the following useful observations made in [2], [6]: denoting by $m : M \hookrightarrow X$ the inclusion map of $M \subseteq X$ we have

(1) m c-closed $\Leftrightarrow$ M c-closed in X and m c-initial;

(2) m c-closed $\Leftrightarrow$ M c-closed in X, provided that c is weakly hereditary;

(3) c hereditary $\Leftrightarrow$ m is c-initial for all $M \subseteq X \in \mathcal{T}\!op$.

We also use the following important result of [6].

2.2 Pullback Ascent and Descent. Consider the pullback diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ \ g\ \ } & V \\
{\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\
X & \xrightarrow{\ \ f\ \ } & Y
\end{array}
\qquad (1)
$$

(Hence, $U \cong X \times_Y V = \{(x, z) \mid f(x) = v(z)\} \subseteq X \times V$.) Then

(1) *g is c-closed (c-open, c-initial, c-final) if f has the respective property, provided that u is c-initial;*

(2) *f is c-closed (c-open, c-initial, c-final) if g has the respective property, provided that v is c-final.*

Note that the provisions in (1), (2) are essential (see [6]). Finally we mention that, trivially, $g \in \mathcal{M}$ ($g \in \mathcal{E}$) if $f \in \mathcal{M}$ ($f \in \mathcal{E}$, respectively); one says that $\mathcal{M}$ and $\mathcal{E}$ are (pullback-)*stable*, but we note that in general none of the other classes is.

2.3 c-Hausdorff and c-compact spaces. A space X is *c-Hausdorff* if the diagonal $\Delta_X = \{(x,x) \,|\, x \in X\}$ is c-closed in $X \times X$; equivalently (see [2]), if the map $\delta_X : X \to X \times X$, $x \mapsto (x,x)$, is c-closed. X is *c-compact* if the projection $X \times Y \to Y$ is c-closed for every space Y. In categorical generality, these notions seem to have appeared first in [8] but got treated subsequently by many authors. Denoting by Haus(c) and Comp(c) the subcategories arising, here we record the following properties (see [2]):

(1) $X, Y \in \mathrm{Haus}(c)$ $(\mathrm{Comp}(c)) \Rightarrow X \times Y \in \mathrm{Haus}(c)$ $(\mathrm{Comp}(c)$, resp.);

(2) $M \subseteq X \in \mathrm{Haus}(c) \;\Rightarrow\; M \in \mathrm{Haus}(c)$;

(3) $M \subseteq X$ c-closed, $X \in \mathrm{Comp}(c)$, c weakly hereditary $\Rightarrow M \in \mathrm{Comp}(c)$;

(4) $X \in \mathrm{Comp}(c)$, $Y \in \mathrm{Haus}(c) \;\Rightarrow\; f : X \to Y$ c-closed.

2.4 Definition. A space X is called *locally c-compact* if there is a c-Hausdorff c-compact space K and a c-open embedding $f : X \hookrightarrow K$. Hence, local c-compactness entails c-Hausdorffness. It is clear that for c the usual closure and for X Hausdorff, this definition gives the usual notion. For the equivalence of this notion with the one proposed in [3] in the presence of the c-Stone-Čech compactification, see 3.2 below. We show a number of stability properties for locally c-compact spaces.

2.5 Proposition. *For X locally c-compact, a subspace M is locally c-compact, provided that one of the following conditions is satisfied:*

(a) *the embedding $m : M \hookrightarrow X$ is c-open;*

(b) *M is c-closed in X, and c is idempotent and weakly hereditary;*

(c) *$M = A \cap B$ with $B \subseteq X$ c-closed and $A \subseteq X$ such that $a : A \hookrightarrow X$ is c-open, and c is idempotent and weakly hereditary.*

Proof. (a) is trivial since $\mathrm{Op}(c) \cap \mathcal{M}$ is closed under composition.

(b) Consider $f : X \hookrightarrow K$ in $\mathrm{Op}(c) \cap \mathcal{M}$ with $K \in \mathrm{CompHaus}(c) = \mathrm{Comp}(c) \cap \mathrm{Haus}(c)$ and form $N := c_K(f(M))$. Since c-closedness of M and c-openness of f give

$$M = c_X(M) = c_X(f^{-1}(f(M))) = f^{-1}(c_K(f(M))),$$

one has the following pullback diagram:

$$\begin{array}{ccc} M & \xrightarrow{\;f'\;} & N \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle n} \\ X & \xrightarrow[\;f\;]{} & K \end{array} \tag{2}$$

With 2.2(1) and 2.3(2),(3), $f' \in \mathrm{Op}(c) \cap \mathcal{M}$ and $N \in \mathrm{CompHaus}(c)$, hence (a) applies.

(c) By hypothesis, there is a pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\;a'\;} & B \\ {\scriptstyle b'}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ A & \xrightarrow[\;a\;]{} & X \end{array} \tag{3}$$

in which b and therefore also b' is c-closed, and in which A is locally c-compact, by (a). Hence, the assertion follows with (b). □

2.6 Proposition.

(1) *$X \times Y$ is locally c-compact if X and Y are, and if* $\mathrm{Op}(c) \cap \mathcal{M}$ *is stable.*

(2) *X is locally c-compact if $X \times Y$ is, if Y has a c-closed point, and if c is idempotent and weakly hereditary.*

Proof. (1) Consider $f : X \hookrightarrow K$, $g : Y \hookrightarrow L$ in $\mathrm{Op}(c) \cap \mathcal{M}$ with $K, L \in \mathrm{CompHaus}(c)$. Since

$$\begin{array}{ccc} X \times Y & \xrightarrow{\;f \times 1_Y\;} & K \times Y \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\;f\;]{} & K \end{array} \qquad \begin{array}{ccc} K \times Y & \xrightarrow{\;1_K \times g\;} & K \times L \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow[\;g\;]{} & L \end{array} \tag{4}$$

are pullback diagrams, $f \times g = (1_K \times g)(f \times 1_Y) \in \mathrm{Op}(c) \cap \mathcal{M}$, by hypothesis, with $K \times L \in \mathrm{CompHaus}(c)$, by 2.3(1).

(2) The c-closed point $y \in Y$ gives a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\;< 1_X, y\cdot !_X >\;} & X \times Y \\ {\scriptstyle !_X}\big\downarrow & & \big\downarrow \\ 1 & \xrightarrow[\;y\;]{} & Y \end{array} \tag{5}$$

with $< 1_X, y\cdot !_X >$ c-closed. Hence, 2.5(b) applies. □

2.7 Remark.

(1) It is easy to show that for Y c-Hausdorff, every point in Y is c-closed. Hence, in 2.6(2) it suffices to ask for the existence of a point in Y: for $X \neq \emptyset$, there is an embedding $Y \hookrightarrow X \times Y$ with $X \times Y$ c-Hausdorff, so that Y is c-Hausdorff, and for $X = \emptyset$ the assertion of 2.6(2) is trivial. See also 4.4 below.

(2) In general, the class $\mathrm{Op}(c) \cap \mathcal{M}$ fails to be stable under pullback, even when c is idempotent and weakly hereditary: see Example 5.7.2 in [6].

3 In the presence of Stone-Čech

3.1 A space X is *c-Tychonoff* if there is an embedding $f : X \hookrightarrow K$ with $K \in \mathrm{CompHaus}(c)$. In this section *we assume that* $\mathrm{CompHaus}(c)$ *is (c-dense)-reflective in the resulting subcategory* $\mathrm{Tych}(c)$ *of* $\mathcal{T}op$; hence, for every $X \in \mathrm{Tych}(c)$ there is a c-dense map $\beta_X : X \to \beta X$ with $\beta X \in \mathrm{CompHaus}(c)$ through which every other map $X \to Y$ with $Y \in \mathrm{CompHaus}(c)$ factors uniquely. Clearly, for $X \in \mathrm{Tych}(c)$ one has $\beta_X \in \mathcal{M}$ (cf. [2]).

3.2 Theorem. *Assume c to be hereditary, or that* $\mathrm{Op}(c) \cap \mathcal{M}$ *is stable. Then X is locally c-compact if and only if X is c-Tychonoff and β_X is c-open.*

Proof. For the non-trivial direction, we assume to have $f : X \hookrightarrow K$ in $\mathrm{Op}(c) \cap \mathcal{M}$ with $K \in \mathrm{CompHaus}(c)$ and first show that the commutative diagram

$$\begin{array}{ccc} X & \overset{\beta_X}{\hookrightarrow} & \beta X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle t} \\ X & \overset{f}{\hookrightarrow} & K \end{array} \qquad (6)$$

(in which t is uniquely determined) is a pullback. In fact, if one forms the pullback diagram

$$\begin{array}{ccc} P = X \times_K \beta X & \overset{f'}{\hookrightarrow} & \beta X \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ X & \overset{f}{\hookrightarrow} & K \end{array} \qquad (7)$$

then the map $d : X \to P$ with $t' \cdot d = 1_X$ and $f' \cdot d = \beta_X$ is a c-dense embedding. This follows immediately from the c-density of β_X once one shows that $f' \in \mathcal{M}$ is c-initial; but this is trivially true when c is hereditary, and also when $\mathrm{Op}(c) \cap \mathcal{M}$ is stable, since $\mathrm{Op}(c) \cap \mathcal{M} \subseteq \mathrm{Ini}(c)$. Since $P \in \mathrm{Haus}(c)$ (by 2.3(1),(2)), the c-dense embedding d is an epimorphism in $\mathrm{Haus}(c)$ and must therefore be a homeomorphism. Consequently, (6) is a pullback diagram. Since 1_X is trivially c-initial, an application of 2.2(1) yields c-openness of β_X. □

3.3 In [2], a map $f : X \to Y$ was called *c-compact* if for every pullback diagram (1), the map g is c-closed. For c hereditary and $X, Y \in \mathrm{Tych}(c)$, it was shown that these are exactly the maps for which

$$\begin{array}{ccc} X & \xhookrightarrow{\beta_X} & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta f} \\ Y & \xhookrightarrow{\beta_Y} & \beta Y \end{array} \qquad (8)$$

is a pullback diagram, i.e. that the c-version of the Henriksen-Isbell characterization holds true; they are also called *c-perfect.*

3.4 Proposition. *Let c be hereditary* or $\mathrm{Op}(c) \cap \mathcal{M}$ *stable, and let the map $f : X \to Y$ be surjective and βf be c-open. Then Y is locally c-compact if Y is c-Tychonoff and X locally c-compact.*

Proof. It is elementary to show that with

$$\beta f \cdot \beta_X = \beta_Y \cdot f$$

also β_Y is c-open when f is surjective. □

3.5 Theorem. ([3]) *Let c be hereditary and $f : X \to Y$ in* $\mathrm{Tych}(c)$ *be c-perfect. Then*

(1) *local c-compactness of X implies the same for Y if f is surjective,*

(2) *local c-compactness of Y implies the same for X if f is c-initial,* or *if* $\mathrm{Op}(c) \cap \mathcal{M}$ *is stable.*

Proof. (1) Since $\beta_Y \cdot f = \beta f \cdot \beta_X$ is c-dense when f is surjective, also βf is c-dense. But by 2.3(4) βf is also c-closed, hence surjective, even c-final. Now, with 2.2(2) one concludes that c-openness of β_X implies the same for β_Y. (2) follows immediately from the fact that (8) is a pullback diagram and from 2.2(1).

□

3.6 Remarks.

(1) In the context of 3.4, c-openness of βf implies c-openness of f.

(2) If, in 3.5(2), one assumes f to be an embedding (as it was done in [3]), one just obtains a weakened version of 2.5(b).

3.7 Theorem. *Let c be idempotent and hereditary. Then every c-dense embedding $f : X \to Y$ of a locally c-compact space X into a c-Tychonoff space is c-open.*

Proof. We first consider the case $Y \in \mathrm{CompHaus}(c)$. As in 3.2, one has the pullback diagram (6). Since c is hereditary, with $f = t \cdot \beta_X$ also t is c-dense, but also c-closed, by 2.3(4). Hence t is surjective, even c-final, so that 2.2(2) gives c-openness of f.

In the general case one considers the composite

$$X \overset{f}{\hookrightarrow} Y \overset{\beta_Y}{\hookrightarrow} \beta Y$$

which is c-dense when c is idempotent. Hence, as already shown, $\beta_Y \cdot f$ is c-open, which implies c-openness of f since $\beta_Y \in \mathcal{M}$. □

4 In a category

4.1 We adopt the setting of [2], [6] and let $\mathcal{X}$ be a finitely-complete category with a proper $(\mathcal{E}, \mathcal{M})$ factorization system and a closure operator c w.r.t. $\mathcal{M}$. With the understanding that now

- "space" means "object in $\mathcal{X}$"

- "map" means "morphism in $\mathcal{X}$"

- "subspace of X" means "equivalence class of morphisms in $\mathcal{M}$ with codomain X"

- preimages and images of subspaces are given by pullback and $(\mathcal{E}, \mathcal{M})$-factorization,

it is immediately clear that all statements (except Remark 2.7) carry over from $\mathcal{T}op$ to $\mathcal{X}$, as follows.

4.2 Theorem. *If $\mathcal{E}$ is stable under pullback, under the translations given in 4.1, all statements of Sections 2 and 3 are valid in $\mathcal{X}$.*

4.3 Remark. Since there are important examples in which $\mathcal{E}$ fails to be stable under pullback (see 4.5 below), it is worth analyzing to which extent this hypothesis is being used in the results of Sections 2 and 3. In its full generality, it is used in the Pullback Ascent and Descent Theorem 2.2, via the Beck-Chevalley Property (see [2], [6]). However, subsequently we apply this result only in very special situations, which require no or very limited use of pullback stability of $\mathcal{E}$. Specifically, the only places where additional hypotheses on $\mathcal{E}$ are needed are

- 3.5(1) where $\mathcal{E}$ needs to be stable under pullback along those morphisms in $\mathcal{M}$ that are c-dense or of the form $c(m)$ (i.e., c-closed, if c is idempotent),

- 3.5(2) for the case "f c-initial" only, where $\mathcal{E}$ needs to be stable along morphisms in $\mathcal{M}$,

- 3.7 where $\mathcal{E}$ needs to be stable along c-closed $\mathcal{M}$-morphisms.

4.4 Example. In the category $\mathcal{G}ph$ of (thin) directed graphs (where objects are sets with a binary relation written as $x \to y$, and where morphisms are maps preserving the relation) with its (surjective, embedding)-factorization structure, consider the up-closure $c = \uparrow$ with

$$\uparrow_X (M) = \{x \in X \mid x \in M \text{ or } \exists y \in M \text{ with } y \to x\},$$

which is hereditary but not idempotent. One then has:

$$X \in \mathrm{Haus}(\uparrow) \quad \Leftrightarrow \quad \forall x, y, z \in X \ \ (x \to y \ \ \& \ \ x \to z \ \Rightarrow \ y = z);$$
$$X \in \mathrm{Comp}(\uparrow) \quad \Leftrightarrow \quad \forall x \in X \ \ \exists y \in X \ : \ x \to y;$$

hence, CompHaus($\uparrow$) consists exactly of those graphs X whose relation is the graph of a mapping $X \to X$. For every $X \in \mathrm{Haus}(\uparrow)$ one may construct a reflexion $\beta_X : X \to \beta X$ into CompHaus($\uparrow$) by attaching to every $x \in X$ with $\uparrow_X (\{x\}) \setminus \{x\} = \emptyset$ a copy of $\mathbb{N}$ (with the successor relation). The natural embedding β_X is $\uparrow$-open, hence every $\uparrow$-Hausdorff graph is already locally $\uparrow$-compact.

In this example, the hypothesis of 2.6(2) that Y has a ($\uparrow$-closed) point is essential (i.e., it is not sufficient to require $Y \neq \emptyset$): let $X = \{0 \to 1\}$ with 0 having a loop, but 1 not so, and $Y = \{\cdot\}$ (the "naked point"); then Y and $X \times Y \cong \{\cdot \ \cdot\}$ are locally $\uparrow$-compact, but X is not.

4.5 Example. We consider the category $\mathcal{L}oc$ of locales (i.e., the dual of the category of frames, see [7]) with its usual closure c. Next we show that a Hausdorff locale is locally compact (in the usual sense, i.e it is a continuous lattice) if and only if it is openly embeddable into a compact Hausdorff locale.

Let us first observe that

- a locale is regular if it is compact Hausdorff (see [10], Prop. 2.3)

- the rather-below relation $\lessdot$ implies the way-below relation $\ll$ in a compact locale (see [10], Lemma 4.1).

These two properties together guarantee that a compact Hausdorff locale is locally c-compact. Furthermore, local c-compactness is obviously open-hereditary.

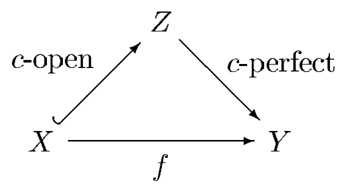
Conversely, one first checks that every locally compact locale X is covered by the interiors of its compact sublocales (see [10], Prop. 4.7). Then one constructs the one-point compactification of X by Artin-glueing (see [1]) a point to its filter of compact sublocales.

Hence, the definition given in 2.4 is equivalent to the usual notion of local compactness in $\mathcal{L}oc$.

5 Locally c-perfect maps

5.1 Having a categorical setting as in 4.1 makes it easy to extend object notions to morphisms, via slicing. As it is done in [2], the factorization structure $(\mathcal{E}, \mathcal{M})$ and the closure operator c of $\mathcal{X}$ give corresponding structures in each comma category $\mathcal{X}/B$ (B an object of $\mathcal{X}$). A c-compact morphism $f : X \to Y$ (see 3.3) is simply a c-compact object in $\mathcal{X}/Y$; likewise, c-Hausdorffness is defined for morphisms, and c-compact c-Hausdorff morphisms are called c-perfect. For details we refer the Reader to [2].

5.2 Definition. A morphism $f : X \to Y$ is called *locally c-perfect* if it is locally c-compact as an object in $\mathcal{X}/Y$. These are exactly the restrictions of c-perfect morphisms to subobjects whose representing morphism is c-open:



(9)

Directly from the statements of Sections 2 and 3 (keeping in mind Theorem 4.2) one derives:

5.3 Corollary. *Let c be idempotent and weakly hereditary. Then, with f also $f \cdot m$ is locally c-perfect if the morphism $m \in \mathcal{M}$ is* (a) *c-open or* (b) *c-closed or* (c) *the intersection of a morphism of type* (a) *with one of type* (b).

5.4 Corollary. *Let* $\mathrm{Op}(c) \cap \mathcal{M}$ *be stable, and consider the pullback diagram* (1). *Then*

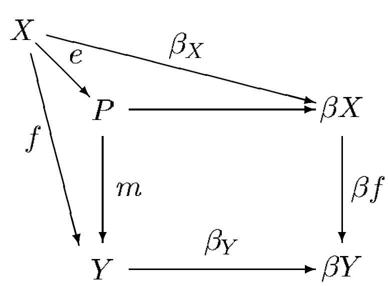
(1) *with f also g is locally c-perfect;*

(2) *with f and v also $f \cdot u = v \cdot g$ is locally c-perfect;*

(3) *with $f \cdot u = v \cdot g$ also f is locally c-perfect, provided that v is the retraction of a c-open or c-closed section and c is weakly hereditary.*

Proof. (1) follows directly from the definition, and (2), (3) from 2.6, with the remark that 2.6(2) obviously holds also in the presence of a c-open point, rather than a c-closed point. □

5.5 Remark. For the remainder of the paper, *let c be hereditary, and let the hypothesis of* 3.1 *be satisfied.* With the characterization given in 3.3, it is easy to see that the reflexion of a morphism $f : X \to Y$ in Tych(c) (considered as an object of Tych(c)/Y) into a c-perfect morphism is given by the left commutative triangle of



(10)

with $P = Y \times_{\beta Y} \beta X$ (see [9]); in other words, the pullback projection m is the Stone-Čech compactification of f, with reflexion e. Now 3.2 gives:

5.6 Corollary. *A morphism f of c-Tychonoff objects X, Y is locally c-perfect if and only if its "antiperfect factor" e is open.*

5.7 Corollary. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in* Tych(c), *with f c-perfect. Then:*

(1) *g is locally c-perfect, if $g \cdot f$ has this property with $f \in \mathcal{E}$ and $\mathcal{E}$ stable under pullback along c-dense morphisms in $\mathcal{M}$ and c-closures of $\mathcal{M}$-morphisms (see* 4.3*).*

(2) *$g \cdot f$ is locally c-perfect, if g has this property and if f is c-initial.*

Proof. Follows from 3.5 applied to the morphism $f : (X, g\cdot f) \to (Y, g)$ in Tych$(c)/Z$. □

5.8 Corollary. *Let c be idempotent (in addition to being hereditary), and consider morphisms $f : X \hookrightarrow Y$ and $g : Y \to Z$ in* Tych(c). *If $f \in \mathcal{M}$ is c-dense, g c-perfect and $g\cdot f$ locally c-perfect, then f is c-open.*

5.9 In addition to these consequences of the facts presented in Section 3 we have the following improvement of 5.7(2):

Theorem. *If* Op$(c) \cap \mathcal{M}$ *is stable, then the class of locally c-perfect morphisms is closed under composition.*

Proof. It suffices to show that the composite $f = s \cdot g$ of a c-perfect morphism $g : X \to Z$ followed by a c-open morphism $s : Z \hookrightarrow Y$ in $\mathcal{M}$ is locally c-perfect, and for that we must show that its antiperfect factor e as in (10) is c-open. But the β-naturality diagram for f decomposes as

$$\begin{array}{ccc} X & \xrightarrow{\ \beta_X\ } & \beta X \\ {\scriptstyle g}\downarrow & \boxed{1} & \downarrow{\scriptstyle \beta g} \\ Z & \xrightarrow{\ \beta_Z\ } & \beta Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \beta s} \\ Y & \xrightarrow{\ \beta_Y\ } & \beta Y \end{array} \tag{11}$$

with the upper square $\boxed{1}$ a pullback, since g is c-perfect. With e' the antiperfect factor of s, diagram $\boxed{1}$ decomposes further, as follows:

$$\begin{array}{ccccc} X & \xrightarrow{\ e\ } & Y\times_{\beta Y}\beta X & \longrightarrow & \beta X \\ {\scriptstyle g}\downarrow & \boxed{2} & \downarrow{\scriptstyle 1_Y\times\beta g} & \boxed{3} & \downarrow{\scriptstyle \beta g} \\ Z & \xrightarrow{\ e'\ } & Y\times_{\beta Y}\beta Z & \longrightarrow & \beta Z \end{array}$$

Here $\boxed{2}$ is a pullback diagram since $\boxed{1}$ and $\boxed{3}$ are pullbacks. As the antiperfect factor of a c-open (and therefore locally c-perfect morphism), e' is c-open; consequently, also e is c-open. □

5.10 Remark. We restricted ourselves to considering only morphisms in Tych(c) from 5.5 on, only in order to have a convenient description of the antiperfect-perfect factorization of a morphism, as in [9]. However, it suffices to have just these factorizations, whose existence may be guaranteed in much more general situations; see [2].

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