Guillaume Brümmer and "Categorical Topology"
My scattered memories of the 1975-85 decade

Walter Tholen

York University, Toronto, Canada

Commemorating the contributions of G.C.L. Brümmer
at the occasion of the ninetieth anniversary of his birth

Cape Town
12-13 December 2024

What to expect

PART I: Topological Functors

- The origins: Grothendieck, Bourbaki, Čech
- The unifying role of Guillaume's thesis of 1971:
  A categorical study of initiality in uniform topology
- "Categorical Topology" in the early Seventies
- External characterizations
- Kan extensions and faithfulness
- A personal letter
- Topologicity as enriched total cocompleteness

PART II: Wait and see!

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Highlight: First self-duality theorems, initially over $\mathcal{X} = \text{Set}$, then more generally!

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Guillaume’s PhD thesis, Cape Town 1971, written under the supervision of Keith Hardie, completed while on a sabbatical leave at the University of Würzburg, Germany.

Features:

- Thorough literature research that includes almost all of the items mentioned so far
- Esthetically perfect presentation with clearly formulated goals – a work of love!
- The key definition of *initial completeness* relative to a functor stands the test of time
- “Clean” proofs of self-duality, inheritance of limits and colimits from the base category
- Focussed and well-motivated study of sections and retractions in $\mathrm{CAT}_{\mathrm{ff,a}}/\mathcal{X}$ and their applications in unform topology

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Reading from the dedication and the *Acknowledgements*

(Google translated from Afrikaans) *...and confirm the work of our hands on us, yes, the work of our hands, confirm it!*

Dedicated to the memory of my father, JOHAN NIKOLAAS BRÜMMER, 1899-1967.

[Dr. Hardie's] example has sharpened my ideal of research and teaching as an integral human activity that takes place in a community.

He taught me never to be satisfied that a result is definite enough, or that an approach is universal enough, but he also taught me to write up and get on.

He read and criticized various drafts of chapter 1 of this thesis; the work would have been incomparably better had I been able to follow many of the suggestions with which he so often harrassed me.

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- G.C.L. Brümmer: *Topological functors and structure functors*
- G.C.L. Brümmer, R.-E. Hoffmann: *An external characterization of topological functors*

THEOREM: A faithful and amnestic functor $P : \mathcal{A} \to \mathcal{X}$ is topological

$\iff$ every comm. diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "U(\mathrm{ff.,a.})"] \arrow[d, "K(\mathrm{ff.,a.})"', "\mathrm{full}"] & \mathcal{A} \arrow[d, "P"] \\ \mathcal{Y} \arrow[r, "V(\mathrm{ff.,a.})"] & \mathcal{X} \end{tikzcd}$$

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$\iff$ $(\mathcal{A}, P)$ is an injective object in $\mathrm{CAT}_{\mathrm{ff,a}}/\mathcal{X}$ with respect to full functors

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The Hušek-Brümmer-Hoffmann Theorem proves half of the following

COROLLARY (Adámek-Herrlich-Rosický-T, 2002):

$(\{\text{full functors}\}, \{\text{amn. topol. functors}\})$ is a weakly orthogonal factorization system of CAT.

In particular: A functor is amnestic & topological iff it is weakly orthogonal to full functors.

Briefly: Believing in injectivity, or in full functors, forces believing in topological functors!

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No uniqueness of diagonal functors, but ...

Brümmer-Hoffmann:
$$\begin{tikzcd}
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\end{tikzcd}$$
but there are finest and coarsest diagonals!

So, what is going on here: Is there a universal characterization in (the 2-category) CAT?

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Universal characterization of topological functors in CAT

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "M", "\Uparrow\mu"'] \arrow[dd, "K"'] & & \mathcal{A} \arrow[dd, "P"] \\
& & \\
\mathcal{Y} \arrow[rr, "U"', "\Uparrow\eta"] \arrow[uurr, dashed, "F"] & {} \arrow[uu, Rightarrow, "\psi"] & \mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
F & PFK \arrow[dr, Rightarrow, dashed, "\eta K"'] \arrow[drr, Rightarrow, dashed, "P\mu"] & & \\
& & UK \arrow[r, Rightarrow, "\psi"] & PM \\
G \arrow[uu, dotted, "\delta"] & PGK \arrow[uu, Rightarrow, "P\delta K"] \arrow[ur, Rightarrow, "\vartheta K"] \arrow[urr, Rightarrow, "P\nu"'] & &
\end{tikzcd}$$

$\forall\psi : UK \to PM\ \exists\eta : PF \to U,\ \mu : FK \to M : P\mu = \psi\cdot\eta K\ \&\ \forall\vartheta : PG \to U, \nu : GK \to M \ldots$

THEOREM (W. T. & M.B. Wischnewsky, 1979): Equivalent are for a functor $P : \mathcal{A} \to \mathcal{X}$:

(i) P is topological;

(ii) The above universal property holds, with $\eta = 1_U$ (i.e. $PF = U$);

(iii) The above universal property holds for $\psi = 1_{UK}$ (i.e. $UK = PM$) and K fully faithful, and with $\eta = 1_U$ and μ an isomorphism. [Note: $\mu = 1_M$ achievable iff P is amnestic].

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Universal characterization of topological functors in CAT

$$\begin{tikzcd}[column sep=huge, row sep=huge] \mathcal{B} \arrow[r, "M"] \arrow[d, "K"'] \arrow[d, phantom, shift left=8ex, "\psi"] & \mathcal{A} \arrow[d, "P"] \\ \mathcal{Y} \arrow[r, "U"'] \arrow[ur, dashed, "F"] \arrow[u, phantom, shift right=8ex, "\Uparrow\eta\quad\Uparrow\mu"] & \mathcal{X} \end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge] F \\ G \arrow[u, Rightarrow, dotted, "\delta"] \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=large] PFK \arrow[dr, Rightarrow, dashed, "\eta K"'] \arrow[drr, Rightarrow, dashed, "P\mu"] & & \\ & UK \arrow[r, Rightarrow, "\psi"] & PM \\ PGK \arrow[uu, Rightarrow, "P\delta K"] \arrow[ur, Rightarrow, "\vartheta K"] \arrow[urr, Rightarrow, "P\nu"'] & & \end{tikzcd}$$

$\forall\psi : UK \to PM\ \exists\eta : PF \to U,\ \mu : FK \to M : P\mu = \psi \cdot \eta K\ \&\ \forall\vartheta : PG \to U, \nu : GK \to M ...$

THEOREM (W. T. & M.B. Wischnewsky, 1979): Equivalent are for a functor $P : \mathcal{A} \to \mathcal{X}$:

(i) P is topological;

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$$
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\end{tikzcd}
\qquad
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F & PFK \arrow[drr, Rightarrow, dashed, "P\mu"] \arrow[dr, Rightarrow, dashed, "\eta K"'] & & \\
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What happens for $\mathcal{X} = \mathbf{1}$?

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "M"] \arrow[rr, phantom, shift right=3, "\Uparrow\mu"] \arrow[d, "K"'] & & \mathcal{A} \\
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\end{tikzcd}
\qquad\qquad
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\end{tikzcd}$$

COROLLARY: Equivalent are for a category $\mathcal{A}$:

(i) $\mathcal{A}$ is a large complete preordered class;

(ii) For all M, K as above, the right Kan extension $(F, \mu) = \mathrm{Ran}_K M$ of M along K exists;

(iii) For all M, K as above, K fully faithful, $(F, \mu) = \mathrm{Ran}_K M$ exists with μ an isomorphism.
[Note: $\mu = 1_M$ is achievable iff the preorder of $\mathcal{A}$ is antisymmetric.]

$$
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The Kan-type characterization of topological functors in CAT$//_{\Leftarrow}\mathcal{X}$

$$\begin{tikzcd}
(\mathcal{B}, PM) \arrow[rr, "{(M,1_{PM})}"] \arrow[d, "{(K,\psi)}"'] & \arrow[d, phantom, "\Uparrow\mu"] & (\mathcal{A}, P) \\
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ψ

THEOREM: Equivalent are for a category $(\mathcal{A}, P)$ over $\mathcal{X}$:

(i) $(\mathcal{A}, P)$ is initially complete;

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\mathcal{Y} \arrow[urr, dashed, "F"] \arrow[rr, "U"'] \arrow[rr, phantom, shift left=3, "\Uparrow\eta"] \arrow[u, phantom, shift left=6, "\Uparrow\psi"] & & \mathcal{X}
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The Kan-type characterization of topological functors in CAT$//_{\Leftarrow}\mathcal{X}$

$$\begin{tikzcd}
(\mathcal{B}, PM) \arrow[rr, "{(M,1_{PM})}"] \arrow[rr, phantom, shift right=4, "\Uparrow\mu"] \arrow[d, "{(K,\psi)}"'] & & (\mathcal{A}, P) \\
(\mathcal{Y}, U) \arrow[urr, dashed, "{(F,\eta)}"'] & &
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\qquad
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$\mathrm{CAT}/\!/_{\Leftarrow}\mathcal{X}$?

- Objects: $(\mathcal{A}, P)$ where $P : \mathcal{A} \longrightarrow \mathcal{X}$
- Morphisms: $(F, \varphi) : (\mathcal{A}, P) \longrightarrow (\mathcal{B}, Q)$ where

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{\Longleftarrow}{\varphi}"] & \mathcal{B} \arrow[dl, "Q"] \\ & \mathcal{X} & \end{tikzcd}$$

- 2-cells: $\alpha : (F, \varphi) \Longrightarrow (F', \varphi')$ where $F \xRightarrow{\ \alpha\ } F'$

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How did “faithful” disappear?

From Guillaume’s survey article on *Topological categories* (Top. Appl. 18, 1984):

The faithfulness [of a topological functor $P : \mathcal{A} \to \mathcal{X}$*] was first proved by Hoffmann [op.cit.] under the restriction that* $\mathcal{A}$ *has small hom-sets [...]. The restriction to small hom-sets was removed by Börger and Tholen [...] using a combinatorial result with four remarkable corollaries:*
(i) Cantor’s theorem that $2^{|X|} > |X|$*;*
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Guillaume's personal letter of 23.10.78, translated from German

Dear Mechthild, dear Walter,

The three beautiful days with you did a lot of good and added much to the already good memories. My cordial thanks for that and for all the efforts and preparations that you put on yourselves because of me.

Since then I thought of you all the time, because I meant to write. And why do I write only now? Because of lots of bad manners? Yes, also, but really because already on my flight back I swore to myself not to write before the writing of my Berlin paper would be done; and that I unfortunately finished only yesterday.

Why am I so sloppy with urgent work? Because I always imagine it to be super super urgent and thereby put great pressure on myself, which in this case was already very big, because of the many other duties I found here. Already in Berlin, Bremen, and especially in Hagen, I thought to have to finish the paper in very few days, and that I would be able to do so; and in this way I made myself especially unable to work normally and quickly. Otherwise the paper would have taken only two or three weeks.

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Well, after this damage estimate, also the positive: I learned a lot during this time, and the paper offers in almost perfect clarity precisely what I wanted to say. Because the due date is only November 15th, the paper will surely be accepted. Surely, the Bremen paper, for Horst, has to be finished very soon, and by the way, the Bremen paper was not included in the oath concerning you. Enough of that.

That Saturday when you took me to Düsseldorf, I had a pleasant bus trip. We stopped in Trier for 2 1/2 hours. I had a nice walk to the Porta Nigra and discovered a happy market place, where a beautiful young lady selected with great care a good peach for me which, washed in the fountain, I ate on the market place; whereupon a very lovely lady with even greater care sold me a newspaper, from which she tenderly and with lots of chatter removed the advertisement section, since I had complained about the uselessness of that ("In there you could find a new position, or a new house, or, only the Lord knows, whatever you are in need of", she said, and I: "See, I am already very happy with my position, with my wife who is really quite a hit, and with my children, and with our house; I am content with everything I have (only not with myself); hence an incredibly happy man!");

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whereupon a very knowledgable waitress brought me, still there on the market place in the sun, a perfectly dry Moselle wine, with which I devoured with pleasure the sandwich you had given me. – The further travel was good; in the Athens transit hall, at midnight, I then also found what Mientje had insisted on as a gift: "a white queen, completely white, with white lips, white eyes, white cheeks–all completely white". The queen was a 25cm tall salt statue, very enchanting, and it cost only seven German marks. And Mientje, who never forgets, inquired immediately after my arrival about the white queen. The next day she was a bit careless: statue's head off, still sits on my work bench in front of me, until I have time to glue the head on and to create a niche for the statue in the wall of Mientje's room.

By the way, all is good here, all children except Stefaans are healthy, Stefaans is slowly becoming stronger, and Niko will soon be able to put aside the crutches. The politics is becoming, if I may say, interesting. And all of us are sending you our cordial greetings.

Yours,
Guillaume

Fast forward to 2014: Garner's "enriched characterization"

R. Garner: *Topological functors as total categories*, Theory Appl. Categories 29, 2014.

Idea: For any faithful functor $P : \mathcal{A} \to \mathcal{X}$, one has $\mathcal{A}(A, B) \hookrightarrow \mathcal{X}(PA, PB)$. That is: we consider being a *morphism* $A \to B$ in $\mathcal{A}$ as a *property* of the *maps* $PA \to PB$ in $\mathcal{X}$.

In order to capture *all* potential properties of the maps of $\mathcal{X}$, he forms the category $\mathcal{Q}_{\mathcal{X}}$:

- the objects of $\mathcal{Q}_{\mathcal{X}}$ are those of $\mathcal{X}$
- $\mathcal{Q}_{\mathcal{X}}(X, Y) = \{\mathbf{f} \mid \mathbf{f} \subseteq \mathcal{X}(X, Y)\} = \mathrm{P}(\mathcal{X}(X, Y))$ ($\mathcal{X}$ should be locally small!)
- $\mathbf{g} \cdot \mathbf{f} = \{g \cdot f \mid f \in \mathbf{f}, g \in \mathbf{g}\}, \quad \mathbf{1}_X = \{1_X\}$

$\mathcal{Q}_{\mathcal{X}}$ *is a category enriched in* Sup (= cat. of complete lattices with sup-preserving maps), and *every concrete category* $(\mathcal{A}, P)$ *over* $\mathcal{X}$ *is described as a category enriched in* $\mathcal{Q}_{\mathcal{X}}$, by

a class $\mathrm{ob}(\mathcal{A})$, a map $P : \mathrm{ob}\mathcal{A} \to \mathrm{ob}\mathcal{X}$, a family $\mathcal{A}(A, B) \in \mathcal{Q}_{\mathcal{X}}(PA, PB)$ $(A, B \in \mathrm{ob}\mathcal{A})$, s. th.

$\mathbf{1}_{PA} \subseteq \mathcal{A}(A, A)$, $\mathcal{A}(B, C) \cdot \mathcal{A}(A, B) \subseteq \mathcal{A}(A, C)$ for all $A, B, C \in \mathrm{ob}(\mathcal{A})$. This leads Garner to:

$$\mathrm{CAT}/_{\mathrm{ff}}\mathcal{X} \cong \mathcal{Q}_{\mathcal{X}}\text{-}\mathrm{CAT}$$

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Fast forward to 2014: Garner's "enriched characterization"

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Idea: For any faithful functor $P : \mathcal{A} \to \mathcal{X}$, one has $\mathcal{A}(A, B) \hookrightarrow \mathcal{X}(PA, PB)$. That is:
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Garner's Theorem

Garner described final liftings in $\mathcal{A}$ of (large) P-structured sinks as (large) weighted colimits in the $\mathcal{Q}_{\mathcal{X}}$-enriched category $\mathcal{A}$ and thereby obtained:

THEOREM: Equivalent are for a concrete category $(\mathcal{A}, P)$ over $\mathcal{X}$:

(i) $(\mathcal{A}, P)$ is finally complete ($\iff$ initially complete);

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For quantaloid-enriched categories and a detailed proof of Garner's Theorem, see also:

- I. Stubbe: Theory Appl. Categories 14, 2005, and 16, 2006.
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Recall for $P : \mathcal{A} \to \mathcal{X}$:

P topological $\iff$ P fibration + P cofibration + P-fibres are large-complete lattices

In the language of quantale-enriched categories, the condition on the P-fibres amounts to: the $\mathcal{Q}_{\mathcal{X}}$-enriched category $\mathcal{A}$ is *order-complete*.

One has the strict implications

P fibration + $\mathcal{A}$ order-complete $\Longrightarrow$ $\mathcal{A}$ *conically cocomplete*
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$\mathcal{A}$ tensored $\Longrightarrow$ P cofibration

and their dualizations! Therefore, from Stubbe's Theorem (1) one obtains (2) and (3):

THEOREM (1) [Stubbe] $\mathcal{A}$ (co)total $\iff$ $\mathcal{A}$ tensored + $\mathcal{A}$ conically cocomplete $\iff$
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PART II: Solid functors, and a close relative of theirs

- Relaxing initial and final lifts
- Duality and characterizations
- Topalg functors
- Taut lifting of adjoint functors
- Totality and the Special Adjoint Functor Theorem
- Final tribute

Guillaume Brümmer and "Categorical Topology"
My scattered memories of the 1975-85 decade

Walter Tholen

York University, Toronto, Canada

Commemorating the contributions of G.C.L. Brümmer
at the occasion of the ninetieth anniversary of his birth

Cape Town

12-13 December 2024

What you may have missed yesterday

PART I: Topological Functors

- The origins: Grothendieck, Bourbaki, Čech
- The unifying role of Guillaume's thesis of 1971:
  A categorical study of initiality in uniform topology
- "Categorical Topology" in the early Seventies
- External characterizations
- Kan extensions and faithfulness
- A personal letter
- Topologicity as enriched total cocompleteness

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Mini-quiz (Answers are due on the Chat within 30 minutes!)

The Yoneda embedding

$$\mathbf{y}_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}], \quad A \longmapsto \mathcal{C}(-, A)$$

of a locally small (but not necessarily small) category $\mathcal{C}$ has a left adjoint, for

- $\mathcal{C} = \mathrm{Set}$ YES □ NO □
- $\mathcal{C} = \mathrm{Grp}$ YES □ NO □
- $\mathcal{C} = \mathrm{Top}$ YES □ NO □

Bonus question:

Do you know proofs for your answers?

Recall yesterday's picture showing "P-initial/final lifting"

$$
\begin{tikzcd}
\mathcal{A} \arrow[d, "P"'] \\
\mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, dashed, "h"] \arrow[rr, bend right, "g_i"'] & A \arrow[r, "f_i"] & B_i \\
PC \arrow[r, "v"] \arrow[rr, bend right, "Pg_i"'] & X \arrow[r, "u_i"] & PB_i
\end{tikzcd}
\qquad (i \in I \text{ large}) \qquad
\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[rr, bend right, "g_i"'] & B \arrow[r, dashed, "h"] & C \\
PA_i \arrow[r, "u_i"] \arrow[rr, bend right, "Pg_i"'] & Y \arrow[r, "v"] & PC
\end{tikzcd}
$$

Let's relax the requirement for the existence of P-final liftings, from "on the nose" to "lax":

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A_i \arrow[rr, "f_i"] \arrow[rrr, bend right, "g_i"'] & & B \arrow[r, dashed, "h"] & PC \\
PA_i \arrow[r, "u_i"] \arrow[rrr, bend right=40, "Pg_i"'] & Y \arrow[r, "q"] \arrow[rr, bend right, "v"'] & PB \arrow[r, "Ph"] & PC
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & PB \arrow[d, dashed, "Ph"] \\
PA_i \arrow[urr, bend left, "Pf_i"] \arrow[r, "u_i"] \arrow[drr, bend right, "Pg_i"'] & Y \arrow[ur, "q"] \arrow[r, "v"'] & PC
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Solid functors

Definition:

$$P : \mathcal{A} \to \mathcal{X} \textit{ solid (a.k.a. semi-topological, or lax-topological)}$$

$\iff$ every P-structured sink has a *lax* P-final lifting

$\iff \forall\ (A_i)_{i\in I}$ in $\mathcal{A}$ (any class I) : $((A_i)_{i\in I} \downarrow \mathcal{A}) \longrightarrow ((PA_i)_{i\in I} \downarrow \mathcal{X})$ has a left adjoint

Some history:

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Examples and properties

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- Haus $\to$ Set; in fact: every reflective restriction of any topological functor is solid!
- Grp $\to$ Set; in fact: every monadic functor over Set is solid!
- Every locally presentable category admits a solid functor to a small power of Set.
- A composite of solid functors is solid.

Properties of all solid functors $P : \mathcal{A} \to \mathcal{X}$:

- P is faithful (Börger-T, 1978) and has a left adjoint (consider $I = \emptyset$).
- If $\mathcal{X}$ has colimits of shape $\mathcal{I}$, so does $\mathcal{A}$ (Hoffmann, 1972).
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Question: What about the "lifting" of limits?

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The Lax Duality Theorem (as in W. T, Math. Arb. 7, Bremen, 1976)

Existence of lax-final liftings of sinks *vs.* existence of rigid lax-initial lifting of sources:

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\begin{tikzcd}
 & & PB \arrow[dd, dashed, "Ph"] \\
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COROLLARY:

For $P : \mathcal{A} \to \mathcal{X}$ laxtop, if $\mathcal{X}$ has limits of shape $\mathcal{I}$, so does $\mathcal{A}$.

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Consequences: Two characterization theorems

THEOREM (W. T., Oberwolfach 1977, published* in T, JPAA 15, 1979):

Equivalent are for a functor $P : \mathcal{A} \to \mathcal{X}$:

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(A category $\mathcal{A}$ satisfying properties 2 & 3 is called $\mathcal{E}$-*cocomplete*.)

* (i) $\Longleftrightarrow$ (ii) may also be found in:

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One doesn't have to waste time on questions of priority.

But then he immediately turns to a more recent matter:

Our dear Rudolf sent me a copy of the [publisher's] proofs of his "Note on semi-topological functors"; it appears in Math. Z., "received August 19, 1977". I am stunned that in this paper no paper of yours or of Manfred is cited, although his main Theorem 2.1 reads "Every semi-topological functor $V : \mathcal{A} \to \mathcal{X}$ is a restriction of a topological functor to a full reflective subcategory of its domain."

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A couple of (unexpected?) corollaries (as noted in T, 1979)

(1) Any composite functor $\mathcal{B} \xrightarrow{I} \mathcal{A} \xrightarrow{J} \mathcal{K} \xrightarrow{Q} \mathcal{X}$ is solid, provided that

1. Q is topological;
2. J is the inclusion of a full epi-reflective subcategory;
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(2) Let $\mathcal{X}$ have orthogonal $(\mathcal{E}, \mathbb{M})$-factorizations of sources, with all sources in $\mathbb{M}$ monic. Then any composite functor $\mathcal{A} \xrightarrow{J} \mathcal{K} \xrightarrow{Q} \mathcal{X}^T \xrightarrow{U} \mathcal{X}$ is solid, provided that

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A close relative: topalg functors

- Y.H. Hong: *Studies on categories of universal topological algebras*, Thesis, McMaster University, Hamilton, 1974
- Y.H. Hong: *On initially structured categories*, J. Korean Math. Soc. 14, 1978

$P : \mathcal{A} \to \mathcal{X}$ *topalg (topologically algebraic)* $\iff$ every P-structured source factors into a P-epimorphism followed by the P-image of a P-initial source:

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What's the precise difference to solid?

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(Properties 2,3,4 $\Longleftrightarrow$ $\mathcal{A}$ admits orthogonal factorizations through $\mathcal{E}$ for all sources.)

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THEOREM (Börger -T, 1979): Equivalent are for $P : \mathcal{A} \to \mathcal{X}$;

(i) P topalg;

(ii) P is an $\mathbb{M}$-functor, for some collection $\mathbb{M}$ of P-initial sources in $\mathcal{A}$ (T, Bremen 1976);

(iii) P has a left adjoint, and there is a class $\mathcal{E}$ of morphisms in $\mathcal{A}$ such that

1. all co-units of the adjunction lie in $\mathcal{E}$;
2. any pushout of an $\mathcal{E}$-morphism exists in $\mathcal{A}$ and lies in $\mathcal{E}$;
3. the co-intersection of any (large) family of $\mathcal{E}$-morphisms exists in $\mathcal{A}$ and lies in $\mathcal{E}$:
4. $\mathcal{E}$ is closed under composition.

(Properties 2,3,4 $\iff$ $\mathcal{A}$ admits orthogonal factorizations through $\mathcal{E}$ for all sources.)

How close a relative?

Clearly: $P : \mathcal{A} \to \mathcal{X}$ topalg $\Longrightarrow$ P solid

But $\not\Leftarrow$: R. Börger: *Semitopologisch $\neq$ topologisch algebraisch*, FU Hagen, 1977

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Characterization via completion

For $P : \mathcal{A} \to \mathcal{X}$ (amnestic) one has:

(1) P solid $\iff$ P has a reflective Dedekind-Mac Neille completion.

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Behavioural comparison in CAT (as a 1-category)

	contains all isos	closure under composition	pullback stable
topological	Yes	Yes	Yes
solid	Yes	Yes	No**
algtop	Yes	No*	No**

* Since there are laxtop functors that are not topalg.
** Since not even full reflective inclusions are stable under pullback.

COROLLARY (of the characterization theorem for laxtop functors):

{solid functors} is the compositional hull of {topalg functors} in CAT.

Taut lifting of left adjoints à la Wyler

$$\begin{tikzcd} \mathrm{TopGrp} \arrow[d, "\mathrm{top}"'] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, bend right] \arrow[d, "\mathrm{top}"] \\ \mathrm{Grp} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{A} \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\perp"] \arrow[d, "P"'] & \mathcal{B} \arrow[l, dashed, bend right, "F'"'] \arrow[d, "Q"] \\ \mathcal{X} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{Y} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Problem:

$GP = QG'$ (or $GP \cong QG'$), $F \dashv G$ $\quad ? \Longrightarrow ? \quad$ $\exists\, F' \dashv G'$ (ideally taut: $\kappa : FQ \Longrightarrow PF'$ iso)

Selected answers:

- (Wyler, 1971) P, Q top., G' maps P-initial(source)s to Q-initials $\Longrightarrow \exists\, F' \dashv G'$ taut
- (T, 1974) P, Q any, $\exists\, F' \dashv G'$ taut $\Longrightarrow$ G' maps P-initials to Q-initials
- (T, 1978) P $\mathbb{M}$-functor, Q any, $G'\mathbb{M} \subseteq \{Q\text{-initials}\} \Longrightarrow \exists\, F' \dashv G'$, taut if $\mathbb{M} = \{P\text{-init's}\}$
- (Street-T-Wischnewsky-Wolff, *Semi-topological functors III*, JPAA 16, 1980) P solid and G' mapping P-lax-initials to Q-lax-initials suffices.

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From Guillaume's letter dated 6.2.1978 (translated from German)

[...]
Please excuse me that I am answering only now. We were away four weeks for holidays, and afterwards I wanted to catch up a little with all that you got me to read. I am far from having finished with that, and I don't know what rescue there can be for me in this world, if there is one at all: even if I do nothing but reading, you still write faster than I can follow with the reading. So it is quite okay that I won't participate in [Manfred's and] your work with Wolff. Given the different speeds I didn't imagine it to work. It is very kind of you that you explained your discussions with Manfred further to me, and that you always tried to take me along on the steep climb to the high categorical mountains.
[...]

Quiz answers, I

R. Street, R. Walters: *Yoneda structures on 2-categories*, J. of Algebra 50, 1978:

- $\mathcal{C}$ *total* $:\iff$ $\mathbf{y}_{\mathcal{C}} : \mathcal{C} \longrightarrow \hat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ has a left adjoint.
- Every reflective subcategory of a presheaf category $\hat{\mathcal{D}}$ ($\mathcal{D}$ small) is total.
- In particular: reflective subcats of monadic cats over Set with rank are total.

R.J. Wood: Talk at Oberwolfach,1979 (What about arbitrary monadic cats over Set?)

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For $P : \mathcal{A} \to \mathcal{X}$ solid, just consider

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with $L \dashv P$.

Since $\mathcal{X}(-, PA) \cong \mathcal{A}(L-, A)$ for $A \in \mathcal{A}$, one has $\mathbf{y}_{\mathcal{X}} P \cong Q\, \mathbf{y}_{\mathcal{A}}$, apply lifting of left adjoints:

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Why "colim" for the left adjoint of $\mathbf{y}_{\mathcal{X}}$? Quiz answers, II

Concepts for a locally small categories $\mathcal{A}$ defined by the existence of certain (colimits):

- $\mathcal{A}$ total $\iff \forall\, D : \mathcal{I} \to \mathcal{A}\ [\forall A \in \mathcal{A}\ (\mathrm{colim}\,\mathcal{A}(A, D-)$ ex. in Set$) \implies \mathrm{colim}\, D$ ex. in $\mathcal{A}]$
- (Isbell, 1968) $\mathcal{A}$ *compact* $\iff \forall\, H : \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}\ [\mathrm{colim}(\mathrm{el}\, H \to \mathcal{A})$ exists in $\mathcal{A}]$
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- (Börger-T-Wischnewsky-Wolff, JPAA 21, 1981)
  $\mathcal{A}$ *hypercomplete* $\iff \forall D : \mathcal{I} \to \mathcal{A}\ [\forall A \in \mathcal{A}\ (\mathrm{Nat}(\Delta A, D)$ small$) \implies \lim D$ ex. in $\mathcal{A}]$
- $\mathcal{A}$ Mono-complete $\iff \mathcal{A}$ has pullbacks of monos and arb. intersections of monos

THEOREM (R. Börger, W. T., Can. J. Math. 42, 1990) One has the (proper) implications

$$\begin{array}{ccccccc}
\text{total} & \Longrightarrow & \text{compact} & \Longrightarrow & \text{hypercomplete} & \Longrightarrow & \text{Mono-complete} \\
\Downarrow & & & & \Downarrow & & \\
\text{small-cocomplete} & & & & \text{small-complete} & &
\end{array}$$

(and no other implications), and all of these properties lift along solid functors.

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(and no other implications), and all of these properties lift along solid functors.

Why “colim” for the left adjoint of $\mathbf{y}_{\mathcal{X}}$? Quiz answers, II

Concepts for a locally small categories $\mathcal{A}$ defined by the existence of certain (colimits):

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One more theorem and some further reading on totality

THEOREM (Day, 1987; Börger-T, 1990: Enhanced Special Adjoint Functor Theorem)

(1) Every $\mathcal{E}$-cocomplete category with an $\mathcal{E}$-generator is total.

(2) There is a total category with a strong generator which does not admit arbitrarily wide pushouts of strong epimorphisms (and which, therefore, is not $\mathcal{E}$-cocomplete for any class $\mathcal{E}$ containing the strong epimorphisms, and which is not weakly cowellpowered).

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Homework Assignment (Take as much time as you wish!)

- Combine and extend the existing fragments of the theory of topological, topalg, and of solid functors in a 2-categorical or an enriched setting, and
- present the completed theory in a readable and attractive manner that adheres to Guillaume's standards!

From Guillaume's letters dated 4.12.78 and 20.7.82 (Translations)

... [The difficulties with obtaining payment] could of course also be caused by our bureaucracy which now, with latest scandals, prepares for repeated exponential growth in currency controls. With my previous travel it was already pretty bad. One shouldn't see politics everywhere but still, from your point of view: do you also think that there is a new epidemic of political hate speech, demagoguery and hypocrisy that is getting around the whole World?
All of us at home are doing very well now. A short while ago the University promoted me to Assoc. Prof., and Keith to Full Prof. For that I have to thank also Herr Pumplün and you. ...

... I am very glad that you think so much of Cape Town. As emphasized by me and the dean in the telephone conversation on the University's expense: don't forget to tell us if you get an offer somewhere else. ...

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Thank you for listening and keeping Guillaume's legacy alive!