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COMPLETIONS OF CATEGORIES AND SHAPE THEORY

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Abstract. Completions of categories with respect to an arbitrary class $\mathbb{D}$ of small categories are investigated. Generalizing the classical construction of pro-categories by Grothendieck and Verdier we give a sufficient condition on $\mathbb{D}$ for every category to admit a one step construction of a $\mathbb{D}$-(co)completion. General shape categories are embedded into these completions and a short proof of the Deleanu-Hilton equivalence theorem is given. Finally, by a generalized "Čech condition", shape categories are described via relatively adjoint functors.
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0. Introduction

There are mainly two ways to endow a given category K with (co)limits of a prescribed type: adjoin the needed (co)limits to K "stepwise" by an iterated process (cf. Isbell [15], Trnková [29]), or try to embed K into some larger (co)complete category $\hat{K}$ and consider the hull of K in $\hat{K}$ under the formation of the needed (co)limits (cf. Lambek [18], Trnková [30], Kelly [17]). If K is a category with small hom-classes, the suitable embedding is the Yoneda embedding

$Y_K : K \to \hat{K} = [K^{op}, Set]$, $A \mapsto K(-,A)$,

and the cocompletion of K (with respect to all small colimits) is given by the full subcategory of small functors in $\hat{K}$, i.e. the left Kan extensions along functors $H : \mathcal{D} \to K^{op}$ with small $\mathcal{D}$; this subcategory

(a) $\overline{K}$ is like K a legitimate category with small hom-classes,

(b) $\overline{K}$ is cocomplete,

(c) K is dense in $\overline{K}$, i.e. every $\overline{K}$-object is a canonical colimit of K-objects,

(d) the embedding $K \to \overline{K}$ is universal, i.e. every functor $T : K \to L$ such that colim TH exists for all small diagrams H in K is the restriction of a unique functor $S : K \to L$ preserving the canonical colimits mentioned in (c).

One cannot expect that the embedding $K \to \overline{K}$ preserves colimits: there is a

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* I am indebted to Armin Frei who introduced me to Categorical Shape Theory while I was on leave at the University of British Columbia at Vancouver in 1980.

counter-example by Trnková [29] concerning finite coproducts and one by Isbell [16] concerning coequalizers. This phenomen does not occur if one allows categories to have proper hom-classes as Trnková [29] showed.

Completely analogous results were obtained before by Grothendieck and Verdier [13] who constructed a cocompletion of K with respect to all small filtered colimits.In this paper we take up again their construction, but with respect to an arbitrary class $\mathbb{D}$ of small categories. It is shown that everything works nicely (i.e. properties (a)-(d) are satisfied relatively to $\mathbb{D}$), if the family $\overline{\mathbb{D}}$ of all categories which appear as the codomains of final functors with domain in $\mathbb{D}$ are closed under the formation of colimits of type $\mathbb{D}$ in CAT (see Proposition 5 and Theorems 2 and 4). The main tool to get this result is to embed the $\mathbb{D}$-cocompletion $\mathrm{Ind}(\mathbb{D},K)$ into the comma category CAT/K instead of the functor category $\hat{K}$, since then $\mathrm{Ind}(\mathbb{D},K)$ turns out to be a reflective subcategory of $\overline{\mathbb{D}}/K$ which is the full subcategory of CAT/K containing all functors with domain in $\overline{\mathbb{D}}$ (see Theorem 3).

The advantage of this construction is that it gives a $\mathbb{D}$-cocompletion in one step: no ordinal induction is needed. Corresponding results were obtained recently for concrete categories (cf. [1,14,4]) which, however, do not cover the theory of completions of abstract categories.

Categorical shape theory goes back to Mardešić [23] who described shape morphisms of topological spaces implicitly by natural transformations. Explicitly this was first done by Porter [27] and, more generally, by LeVan [19] who defined the shape category with respect to an arbitrary category L and a full subcategory K ; classically one takes for L the homotopy category of topological spaces and for K the subcategory of polytopes. Deleanu and Hilton [5,6,7], Frei [9,10], and MacDonald [21] considered shape categories even with respect to arbitrary functors and gave also non-topological examples. They rediscovered Linton's construction of the full clone of operations of an arbitrary functor (cf. [20]). In this note we give, for a *shape adequate* functor $F: K \to L$ (this is "Condition C" in Frei [9]), a new characterization of the shape category $\mathrm{Sh}(F)$ and the canonical functor $S_F : L \to \mathrm{Sh}(F)$: it is universal with respect to the property that the composition $S_F F$ is codense (see Theorem 5).

Another categorical approach to shape theory is close to the classical dessription by ANR-systems (cf. Mardešić and Segal [24,25]) and uses inverse systems of CW-complexes (cf. Morita [26]); it amounts to define the shape category of (pointed) topological spaces as a certain full subcategory of a pro-category (cf. Dydak and Segal [8], Borsuk and Dydak [2]). The equivalence of both approaches was shown in full generality by Deleanu and Hilton [5]; here we give a very short proof of their theorem (see Theorem 6).

Finally we discuss the so-called *Čech condition* for F; it is equivalent to the existence of a "Čech functor" $C : L \to \text{Pro-}K$ which is left adjoint to the induced functor $\text{Pro-}K \to \text{Pro-}L$ relatively to the embedding $L \to \text{Pro-}L$. (cf. Theorem 7). The shape category is then isomorphic to the full image of C ; this gives precisely the description [8,2] in case of topological spaces and polytopes.

We use MacLane's terminology [22], but categories are always assumed to have small hom-sets. To avoid too many contravariant functors we investigate co-completions instead of completions and coshape categories instead of shape categories, except in Section 5; there is no difficulty to dualize the notions and results.

1. Categories of diagrams

Let K be a category, and let $\mathbf{D}$ be a class of small categories containing the terminal category $1 = \{\cdot\}$. A *diagram in* K *of type* $\mathbf{D}$ is a functor $H : \mathcal{D} \to K$ with $\mathcal{D} \in \mathbf{D}$. A *morphism* $(F,\varphi) : H \to J$ *of diagrams* in K is given by a functor $F : \mathcal{D} \to E$ (=dom J) and a natural transformation $\varphi : H \to JF$. For another morphism (G,ψ) one defines the composition with (F,φ) by $(G,\psi)(F,\varphi) = (G F, \psi F \cdot \varphi)$. This way we obtain the category $\mathbf{D}/\!/K$ of $\mathbf{D}$-diagrams over K. It contains the (not necessarily full) subcategory $\mathbf{D}/K$ containing all morphisms (F,φ) with $\varphi = 1$, i.e. $JF = H$. There is a canonical (but not necessarily faithful) functor

$$U_K : \mathbf{D}/\!/K \to Cat\ ,\ (F,\varphi) \mapsto F\ .$$

For every $\mathcal{D} \in \mathbf{D}$, the functor category $[\mathcal{D},K]$ can be embedded (not necessarily fully) into $\mathbf{D}/\!/K$ by the functor

$$I_{\mathcal{D},K} : [\mathcal{D},K] \to \mathbf{D}/\!/K\ ,\ \varphi \mapsto (\mathrm{Id}_{\mathcal{D}},\varphi)\ .$$

In case $\mathcal{D} = 1$ we obtain (up to an isomorphism $K \cong [1,K]$) the full embedding

$$\mathbb{A}_K : K \to \mathbf{D}/\!/K$$

which assigns to every object A the constant functor $\Delta A : 1 \to K$ with value A.

Our first proposition describes the behaviour of $\mathbb{A}_K$ with respect to limits and connected colimits and is easily proved:

PROPOSITION 1. $\mathbb{A}_K$ *preserves all limits and connected colimits.*□

Coproducts are <u>not</u> preserved in general: if $\mathbf{D}$, considered as a full subcategory of Cat, is closed under the formation of coproducts (i.e. disjoint unions), then - independently of the existence of coproducts in K - the category $\mathbf{D}/\!/K$ always has coproducts and they are formed like in Cat, i.e. U_K preserves them. With respect to limits and coequalizers one can prove:

THEOREM 1. (1) *For a small category* C, *let* $\mathbf{D}$ *be closed under* C*-limits in*

Cat . Then, if K *is C-complete, so is* $\mathbf{D}/\!/K$ *, and all the functors* U_K *and* $I_{\mathcal{D},K}$ *preserve C-limits.*

(2) *Let* $\mathbf{D}$ *be closed under coequalizers in Cat . Then, if* K *is cocomplete,* $\mathbf{D}/\!/K$ *has coequalizers, and all the functors* U_K *and* $I_{\mathcal{D},K}$ *preserve them.*

In (2), the cocompleteness of K is needed in order to construct certain left Kan extensions. For an explicit proof see Guitart - van den Bril [32], p. 376.□

REMARK. (Co)completeness of K is not a necessary condition for (co)completeness of $\mathbf{D}/\!/K$: consider $\mathbf{D} = \mathrm{Ob}\mathcal{C}at$ and $K = \{\cdot\,\cdot\} \cong 1 \sqcup 1$. Then a functor $H : \mathcal{D} \to K$ is completely determined by a pair $(\mathcal{D}_o, \mathcal{D}_1)$ with $\mathcal{D}_o \sqcup \mathcal{D}_1 \cong \mathcal{D}$. From this one easily sees that $\mathbf{D}/\!/K$ is equivalent to the complete and cocomplete category $\mathcal{C}at \times \mathcal{C}at$.

2. The Yoneda- and the Kan-representation of a functor

Let $F : K \to L$ be a functor. The *(right) Yoneda-representation* of F is the canonical functor

$$Y_F : L \to [K^{op}, \mathcal{S}et]$$

which assigns to every object $X \in L$ the functor $L(F-,X) : K^{op} \to \mathcal{S}et$. (The reader may disregard the fact that $[K^{op}, \mathcal{S}et]$ is a category in a higher universe. The same holds for the meta-category CAT/K of all categories over K which is defined like $\mathcal{C}at/K$.) The *(right) Kan-representation* of F is the functor

$$K_F : L \to CAT/K$$

which assigns to every object $X \in L$ the projection functor $P_X : F/X \to K$ where F/X is the comma category of F-comorphisms over X : objects are pairs (A,u) with $A \in \mathrm{Ob}K$ and $u : FA \to X$ in L ; a morphism $f : (A,u) \to (B,v)$ is given by a K-morphism $f : A \to B$ satisfying $v \cdot Ff = u$. Every $w : X \to Y$ in L induces canonically a functor $F/w : F/X \to F/Y$ which commutes with the projections, so we can take it as the value under K_F.

These two representations are related via the so called *element construction*. We define a functor

$$E_K : [K^{op}, \mathcal{S}et] \to \mathcal{C}at/K$$

as follows: for $T : K^{op} \to \mathcal{S}et$ let $El(T)$ be the (comma-)category whose objects are pairs (A,a) with $A \in \mathrm{Ob}K$ and $a \in TA$; a morphism $f : (A,a) \to (B,b)$ is given by a K-morphism $f : A \to B$ with $(Tf)(b) = a$; now E_K assigns to T the canonical projection functor $\pi_T : El(T) \to K$. For a natural transformation $\alpha : T \to S$, there is a functor $E_K\alpha : EL(T) \to EL(S)$, $(A,a) \mapsto (A,(\alpha A)(a))$, which commutes with the projections.

The following proposition is easily verified:

PROPOSITION 2. E_K *is a full embedding which commutes with Yoneda- and Kan- representations, i.e. the following triangle is commutative:*

$$\begin{tikzcd} & L \arrow[dl, "Y_F"'] \arrow[dr, "K_F"] & \\ {[K^{op},Set]} \arrow[rr, "E_K"'] & & CAT/K \end{tikzcd} \square$$

COROLLARY. *The following assertions are equivalent:*

(i) F *is dense (= colimit dense),*

(ii) Y_F *is full and faithful ,*

(iii) K_F *is full and faithful .*

Proof. (i) <==> (ii) holds by definition of density, and (ii) <==> (iii) follows from Proposition 2.□

PROPOSITION 3. Y_F *and* K_F *preserve all existing limits in* L .

The proof is straightforward.□

If F is full and faithful, one can prove a stronger result for Y_F . The proof is not more difficult than in the case $F = Id$ when one has the usual Yoneda embedding of a category:

PROPOSITION 4. *If* F *is full and faithful,* Y_F *is (colimit-)dense.* □

REMARK. If F is not full (and faithful), Y_F need not be dense. This may be seen from the functor $F : K \to 1$; here Y_F maps the only object in 1 to the functor $\Delta 1 : K^{op} \to Set$, $A \mapsto 1$. On the other hand, the functors $G : 1 \to K$ show that K_G need not be dense even if G is full and faithful (as all functors $1 \to K$ are). Namely, $K_G : K \to CAT \cong CAT/1$ assigns to each $X \in ObK$ the hom class $K(X_G, X)$ considered as a discrete category; here X_G denotes the only value of G .

3. The $\mathbb{D}$-cocompletion of a category

Like in Section 1, let $\mathbb{D} \subseteq ObCat$ with $1 \in \mathbb{D}$ be given. In order to find a suitable extension of K which has sufficiently many colimits of type $\mathbb{D}$ and which belongs to the same universe, we first consider the (meta-)category $\hat{K} = [K^{op}, Set]$ into which K is embedded by the Yoneda embedding Y_K . We then form like in Section 1 the (meta-)category $\mathbb{D} /\!/ \hat{K}$. The functor Y_K can be canonically extended to a functor

$\mathbb{D} /\!/ Y_K : \mathbb{D} /\!/ K \to \mathbb{D} /\!/ \hat{K}$ with $(\mathbb{D} /\!/ Y_K) \Delta_K = \Delta_{\hat{K}} Y_K$.

Since $\hat{K}$ is cocomplete, $\Delta_{\hat{K}}$ embeds $\hat{K}$ as a full reflexive subcategory into $\mathbb{D} /\!/ \hat{K}$. The reflector $\mathrm{colim}_{\hat{K}} : \mathbb{D} /\!/ \hat{K} \to \hat{K}$ assigns to every $H : \mathcal{D} \to \hat{K}$ its colimit. It can be chosen in such a way that $\mathrm{colim}_{\hat{K}} \Delta_{\hat{K}} = \mathrm{Id}_{\hat{K}}$ holds. We now consider the functor

$$L_{\mathcal{K}} = (\mathbb{D}/\!/\mathcal{K} \xrightarrow{\mathbb{D}/\!/Y_{\mathcal{K}}} \mathbb{D}/\!/\widehat{\mathcal{K}} \xrightarrow{\operatorname{colim}_{\widehat{\mathcal{K}}}} \widehat{\mathcal{K}})$$

and form its full image factorization

$$L_{\mathcal{K}} = (\mathbb{D}/\!/\mathcal{K} \xrightarrow{C_{\mathcal{K}}} \operatorname{Ind}(\mathbb{D},\mathcal{K}) \xrightarrow{J_{\mathcal{K}}} \widehat{\mathcal{K}}).$$

So the category $\operatorname{Ind}(\mathbb{D},\mathcal{K})$ of *inductive systems over* $\mathcal{K}$ *of type* $\mathbb{D}$ has the same objects as $\mathbb{D}/\!/\mathcal{K}$, and its hom classes are given by

$$\begin{aligned}
\operatorname{Ind}(\mathbb{D},\mathcal{K})(H,J) &= \widehat{\mathcal{K}}(L_{\mathcal{K}}H, L_{\mathcal{K}}J) \\
&= \widehat{\mathcal{K}}(\operatorname*{colim}_{D} Y_{\mathcal{K}}HD,\ L_{\mathcal{K}}J) \\
&\cong \lim_{D} \widehat{\mathcal{K}}(Y_{\mathcal{K}}HD,\ L_{\mathcal{K}}J) \\
&\cong \lim_{D} (L_{\mathcal{K}}J)(HD) \\
&= \lim_{D} \operatorname*{colim}_{E} \mathcal{K}(HD,\ JE)\ .
\end{aligned}$$

$$\begin{tikzcd}
 & \mathbb{D}/\!/\mathcal{K} \arrow[d, "C_{\mathcal{K}}"] \arrow[ddr, "L_{\mathcal{K}}"] & \\
 & \operatorname{Ind}(\mathbb{D},\mathcal{K}) \arrow[dr, "J_{\mathcal{K}}"] & \\
\mathcal{K} \arrow[uur, "\Delta_{\mathcal{K}}"] \arrow[ur, "M_{\mathcal{K}}"] \arrow[rr, "Y_{\mathcal{K}}"'] & & \widehat{\mathcal{K}}
\end{tikzcd}$$

This shows that $\operatorname{Ind}(\mathbb{D},\mathcal{K})$ has small hom classes, so it is a legitimate category. The functor $C_{\mathcal{K}}$ is the identity on objects whereas $J_{\mathcal{K}}$ is the identity on morphisms. We now consider the functor $M_{\mathcal{K}} = C_{\mathcal{K}}\Delta_{\mathcal{K}}$. Since $J_{\mathcal{K}}$ is full and faithful we get immediately from the corresponding statement about $Y_{\mathcal{K}} = J_{\mathcal{K}}M_{\mathcal{K}}$ (cf. Proposition 4):

PROPOSITION 5. $M_{\mathcal{K}} : \mathcal{K} \to \operatorname{Ind}(\mathbb{D},\mathcal{K})$ *is a full and (colimit-)dense embedding and preserves all limits.* □

In order to study the category $\operatorname{Ind}(\mathbb{D},\mathcal{K})$ in more detail it is useful to consider the Kan representation of $M_{\mathcal{K}}$, i.e.

$$W_{\mathcal{K}} = K_{M_{\mathcal{K}}} : \operatorname{Ind}(\mathbb{D},\mathcal{K}) \to \mathbf{CAT}/\mathcal{K}\ .$$

From the Corollary of Proposition 2 we know that $W_{\mathcal{K}}$ is full and faithful. We want to show that, for the element functor $E_{\mathcal{K}} : \mathcal{K} \to \mathbf{CAT}/\mathcal{K}$, we have $E_{\mathcal{K}}J_{\mathcal{K}} \cong W_{\mathcal{K}}$. For this let $H : \mathcal{D} \to \mathcal{K}$ be in $\operatorname{Ind}(\mathbb{D},\mathcal{K})$. The objects of the category $\operatorname{El}(J_{\mathcal{K}}H)$ are pairs (A,a) with $A \in \operatorname{Ob}\mathcal{K}$ and

$$u \in (J_{\mathcal{K}}H)A = \operatorname{colim} \mathcal{K}(A,H-) \cong \operatorname{Ind}(\mathbb{D},\mathcal{K})(\Delta_{\mathcal{K}}A,H) = \operatorname{Ind}(\mathbb{D},\mathcal{K})(M_{\mathcal{K}}A,H)\ ;$$

therefore they correspond bijectively to the objects of $M_{\mathcal{K}}/H$. Now one easily proves that this is the object part of a bijective functor $I_H : \operatorname{El}(J_{\mathcal{K}}H) \to M_{\mathcal{K}}/H$ which commutes with the projections of the comma categories. Finally, since I_H is natural in H, we have a natural equivalence

$$\begin{tikzcd}
\operatorname{Ind}(\mathbb{D},\mathcal{K}) \arrow[rr, "W_{\mathcal{K}}"] \arrow[dr, "J_{\mathcal{K}}"'] & \arrow[d, phantom, "\overset{\cong}{\Longrightarrow}"] & \mathbf{CAT}/\mathcal{K} \\
 & \widehat{\mathcal{K}} \arrow[ur, "E_{\mathcal{K}}"'] &
\end{tikzcd}$$

LEMMA 1. *For every* $H : \mathcal{D} \to \mathcal{K}$ *with* $\mathcal{D} \in \mathbf{D}$ *there is a final functor*

$$R_H : \mathcal{D} \to M_{\mathcal{K}}/H \quad \text{with} \quad P_H R_H = H$$

satisfying the following universal property: for every $J \in \mathrm{Ob}\ \mathrm{Ind}(\mathbf{D}, \mathcal{K})$ *and every functor* $G : \mathcal{D} \to M_{\mathcal{K}}/J$ *with* $P_J G = H$ *there is a unique* $\mathrm{Ind}(\mathbf{D}, \mathcal{K})$*-morphism* $\gamma : H \to J$ *with* $(M_{\mathcal{K}}/\gamma) R_H = G$ *, i.e. in* $\mathsf{CAT}/\mathcal{K}$ *we have the following diagram:*

$$\begin{tikzcd} H \arrow[r, "R_H"] \arrow[dr, "G"'] & W_{\mathcal{K}}H \arrow[d, "W_{\mathcal{K}}\gamma"] \\ & W_{\mathcal{K}}J \end{tikzcd} \qquad \begin{tikzcd} H \arrow[d, dashed, "\gamma"] \\ J \end{tikzcd}$$

Proof. By the above isomorphism, we identify $M_{\mathcal{K}}/H$ with $\mathrm{El}(J_{\mathcal{K}}H)$. We use the usual colimit construction in $\mathcal{S}et$: objects in $\mathrm{El}(J_{\mathcal{K}}H)$ are given by pairs $(A,[f,D])$ with $A \in \mathrm{Ob}\ \mathcal{K}$, $D \in \mathrm{Ob}\ \mathcal{D}$, and $f \in \mathcal{K}(A,HD)$; here $[f,D]$ denotes the equivalence class of (f,D) in

$$(J_{\mathcal{K}}H)A = \mathrm{colim}\ \mathcal{K}(A,H-) = (\textstyle\coprod_{D'} \mathcal{K}(A,HD'))/\!\sim ,$$

where $\sim$ is smallest equivalence relation with $(f,D) \sim (Hd \cdot f, D')$ for all $d : D \to D'$ in $\mathcal{D}$. A morphism $h : (A,[f,D]) \to (B,[g,E])$ in $\mathrm{El}(J_{\mathcal{K}}H)$ is a $\mathcal{K}$-morphism $h : A \to B$ with $[f,D] = [gh,E]$. On objects, R_H is now defined by $R_H D = (HD,[1_{HD},D])$; on morphisms it is determined by $P_H R_H = H$. To prove that R_H is final is left to reader.

The universal property can be proved as follows: Since $M_{\mathcal{K}}$ is dense we have a canonical colimit $\Lambda_H : M_{\mathcal{K}} P_H \to \Delta H$, and since R_H is final also $\Lambda_H R_H : M_{\mathcal{K}} H \to \Delta H$ is a colimit. Therefore, if J and G are given, one gets a unique $\mathrm{Ind}(\mathbf{D}, \mathcal{K})$-morphism $\gamma : H \to J$ with $\Delta\gamma \Lambda_H R_H = \Lambda_J G$. This equation can be easily seen to be equivalent with $(W_{\mathcal{K}}\gamma) R_H = G$. □

COROLLARY. *For every object* H *in* $\mathrm{Ind}(\mathbf{D}, \mathcal{K})$ *one has* $H \cong \mathrm{colim}\ M_{\mathcal{K}}H$.□

THEOREM 2. *Let* $T : \mathcal{K} \to \mathcal{L}$ *be any functor such that, for every* $H : \mathcal{D} \to \mathcal{K}$ *with* $\mathcal{D} \in \mathbf{D}$, $\mathrm{colim}\ TH$ *exists in* $\mathcal{L}$ *. Then there is a functor* $S : \mathrm{Ind}(\mathbf{D}, \mathcal{K}) \to \mathcal{L}$ *with* $SM_{\mathcal{K}} = T$ *and* $S(\mathrm{colim}\ M_{\mathcal{K}}H) \cong \mathrm{colim}\ TH$ *for all* H *, and* S *is, up to natural equivalence, uniquely determined.*

Proof. On objects, S is defined by $SH = \mathrm{colim}\ TH$. Then one first considers the morphisms $\alpha : M_{\mathcal{K}}A \to H$ with $A \in \mathrm{Ob}\mathcal{K}$ and puts $S\alpha = \lambda_H(A,\alpha)$; here $\lambda_H : TP_H \to \Delta SH$ is a colimit which exists since colim TH exists and since R_H with $P_H R_H = H$ is final. Now, if we consider more generally a morphism $\beta : H \to J$ in $\mathrm{Ind}(\mathbf{D}, \mathcal{K})$, S is already defined on all morphisms $\beta\alpha = \beta\Lambda_H(A,\alpha) : M_{\mathcal{K}}A \to J$ (Λ_H is like in the proof of Lemma 1). This way we get a cocone $TP_H \to \Delta SJ$ which, by the colimit property in $\mathcal{L}$, defines the morphism

$$S\beta : SH \to SJ \quad \text{with} \quad \Delta S\beta \cdot \lambda_H = S(\Delta\beta \cdot \Lambda_H) .$$

S is well defined and does everything one wants. □

Completeness and cocompleteness properties of $\mathrm{Ind}(\mathbf{D},\mathcal{K})$ can now be derived from those of $\mathcal{CAT}/\mathcal{K}$. For this we first modify the class $\mathbf{D}$ as is suggested by Lemma 1:

$$\overline{\mathbf{D}} = \{\mathcal{D} \in \mathrm{Ob}\ \mathcal{CAT} \mid \exists\ R : \mathcal{E} \to \mathcal{D} \text{ final with } \mathcal{E} \in \mathbf{D}\}.$$

Since final functors do not change colimits we obviously have that $\mathrm{Ind}(\mathbf{D},\mathcal{K})$ and $\mathrm{Ind}(\overline{\mathbf{D}},\mathcal{K})$ are equivalent even though $\overline{\mathbf{D}}$ may contain non-small categories. Now the universal property of Lemma 1 reads as follows:

THEOREM 3. $\mathrm{Ind}(\mathbf{D},\mathcal{K})$ *can be embedded into* $\overline{\mathbf{D}}/\mathcal{K}$ *as a full reflective subcategory.* □

This is the main result of the paper as full reflective subcategories inherit (co)completeness. Therefore, if $\overline{\mathbf{D}}$ is closed under certain colimits in $\mathcal{CAT}$, then $\overline{\mathbf{D}}/\mathcal{K}$ is closed under these colimits in $\mathcal{CAT}/\mathcal{K}$ and so is $\mathrm{Ind}(\mathbf{D},\mathcal{K})$. For a well chosen class $\mathbf{D}$ one should expect at least that $\overline{\mathbf{D}}$ is closed under the colimit types given by $\mathbf{D}$. We therefore define:

DEFINITION. $\mathbf{D} \subseteq \mathrm{Ob}\mathcal{CAT}$ is called *admissible*, iff $\overline{\mathbf{D}}$ is closed under $\mathcal{D}$-colimits in $\mathcal{CAT}$ for all $\mathcal{D} \in \mathbf{D}$.

THEOREM 4. *For an admissible class* $\mathbf{D}$, *the category* $\mathrm{Ind}(\mathbf{D},\mathcal{K})$ *is* $\mathbf{D}$*-cocomplete, i.e.* $\mathcal{D}$*-cocomplete for all* $\mathcal{D} \in \mathbf{D}$. □

With respect to shape theory we are mainly interested in the class $\mathbf{D}_r$ of all r-directed sets (considered as small categories) where r is a regular cardinal. Analogously to the Grothendieck-Verdier proof in case $r = \aleph_0$ (cf. [13], p. 65) one can show that $\overline{\mathbf{D}}_r$ consists of all r-filtered categories $\mathcal{D}$ such that there exists a subset $K \subseteq \mathrm{Ob}\mathcal{D}$ with the following property: for every $D \in \mathrm{Ob}\mathcal{D}$ there is a $f : D \to E$ in $\mathcal{D}$ with $E \in K$. Since r-directed colimits in $\mathcal{CAT}$ are formed like in $\mathcal{S}et$ one can easily see that $\overline{\mathbf{D}}_r$ is closed under r-directed colimits in $\mathcal{CAT}$. Thus we have that $\mathbf{D}_r$ is an admissible class whence Theorem 4 applies to $\mathbf{D}_r$. In case $r = \aleph_0$ the category $\mathrm{Ind}(\mathbf{D}_r,\mathcal{K})$ is denoted by Ind-$\mathcal{K}$. From Theorems 2,3,4 we get:

COROLLARY. *For every category* $\mathcal{K}$, *the functor* $M_{\mathcal{K}}$ *is a full and dense embedding of* $\mathcal{K}$ *into the category* Ind-$\mathcal{K}$ *which has all small filtered colimits. Every functor* T *from* $\mathcal{K}$ *into a category* $\mathcal{L}$ *with small filtered colimits factors, up to a natural equivalence, uniquely over* $M_{\mathcal{K}}$ *by a functor* S *which preserves filtered colimits.* □

The category of *projective systems over* $\mathcal{K}$ *of type* $\mathbf{D}$ is defined as

$$\mathrm{Pro}(\mathbf{D},\mathcal{K}) = (\mathrm{Ind}(\mathbf{D}^{op},\mathcal{K}^{op}))^{op}$$

with $\mathbf{D}^{op} = \{\mathcal{D}^{op} \mid \mathcal{D} \in \mathbf{D}\}$. The canonical embedding

$$N_{\mathcal{K}} : \mathcal{K} \to \mathrm{Pro}(\mathbf{D},\mathcal{K})$$

is defined by $N_K = (M_{K^{op}})^{op}$. For $\mathbf{D}$ being the class of all (downwards) directed sets one obtains the so called *pro-category of* K as $\text{Pro-}K = \text{Pro}(\mathbf{D}, K)$.

4. The coshape category of a functor

For a general functor $F : K \to L$ the *coshape category* $Sh^*(F)$ of F is defined to be the full image of the Yoneda functor Y_F (cf. Section 2), i.e. objects of $Sh^*(F)$ are the objects of L, and the hom-sets are given by the formula

$$Sh^*(F)(X,Y) = K(Y_FX, Y_FY) = \text{Nat}(L(F-,X), L(F-,Y)) \cong (CAT/K)(K_FX, K_FY)$$

(Proposition 2). We have the canonical factorization

$$\begin{tikzcd} & Sh^*(F) \arrow[dr, "I_F"] & \\ L \arrow[ur, "S_F^*"] \arrow[rr, "Y_F"'] & & \hat{K} \end{tikzcd} .$$

S_F^* acts on morphisms like Y_F does, and I_F is full and faithful. Therefore, from the Corollary of Proposition 2 and from the Propositions 2 and 3 we obtain immediately:

PROPOSITION 6 . S_F^* *preserves all limits. If* F *is full and faithful, both,* S_F^* *and* I_F *are dense.* S_F^* *is a bijective functor, iff* F *is dense* .□

Surprisingly, the composition $F^o = S_F^*F : K \to Sh^*(F)$ might be dense even for non-faithful F. The most general sufficient condition for F^o to be dense was pointed out by Frei [9]:

DEFINITION. F is called *coshape-adequate*, if for all $A \in \text{Ob}K$ and $X \in \text{Ob}L$, the functor Y_F induces a bijection $L(FA,X) \longrightarrow \text{Nat}(L(F-,FA), L(F-,X))$.

For every $X \in \text{Ob}L$, let $\lambda_X : FP_X \longrightarrow \Delta X$ be the canonical cocone with $\lambda_X(A,u) = u$, $(A,u) \in \text{Ob}(F/X)$. The following has been proved in [9] (except the equivalence of assertions (2) (ii) and (iii)):

PROPOSITION 7. (1) *If* F *is full or dense, then* F *is coshape adequate.*

(2) *The following assertions are equivalent:*

(i) F *is coshape adequate.*

(ii) *For all* $X \in \text{Ob}L'$, S_F^* *induces a bijective functor* $(S_F^*)_X : F/X \to S_F^*F/X$

(iii) *The functors* $(S_F^*)_X$, $X \in \text{Ob}L$, *are final.*

(iv) *For all* $A \in \text{Ob}K$, *the canonical cocone* $\lambda_{FA} : FP_{FA} \to \Delta FA$ *is a colimit in* L.

(3) *If* F *is coshape adequate, then, with* $F^o = S_F^*F$, $S_{F^o} : Sh^*(F) \to Sh^*(F^o)$ *is a bijective functor, and* F^o *is dense; moreover, for all* $X \in \text{Ob}L$, *the canonical cocone* $S_F^*\lambda_X : F^oP_X \to \Delta X$ *is a colimit in* $Sh^*(F)$. □

The colimit presentation of the objects in $\mathrm{Sh}^*(F)$ is characteristic for the coshape construction: the following theorem describes S_F^* as the "universal densificator" of F.

THEOREM 5. *If* F *is coshape adequate and if* $T : \mathcal{L} \to \mathcal{M}$ *is any functor such that* TF *is dense and* T *induces a final functor* $T_X : F/X \to TF/TX$ *for all* $X \in \mathrm{Ob}\mathcal{L}$, *then there is a unique functor* $U : \mathrm{Sh}^*(F) \to \mathcal{M}$ *with* $US_F^* = T$.

Proof. Necessarily the cocone $S_F^*\lambda_X$ is transformed into $T\lambda_X : TFP_X \to \Delta TX$ by U. Since TF is dense and T_X is final this cocone is a colimit in $\mathcal{M}$. Therefore, for every $\beta : X \to Y$ in $\mathrm{Sh}^*(F)$, $U\beta$ must be the unique $\mathcal{M}$-morphism rendering the diagram

$$\begin{tikzcd} TFP_X \arrow[r, "T\lambda_X"] \arrow[dr, "T\tilde{\beta}"'] & \Delta TX \arrow[d, "\Delta U\beta"] \\ & \Delta TY \end{tikzcd}$$

commutative

where $\tilde{\beta}$ is defined by $\tilde{\beta}(A,\alpha) = \beta\alpha$ for all $(A,\alpha) \in S_F^*F/X \cong F/X$ (this construction is similar to that one in the proof of Theorem 2). □

REMARK. The condition that induced functors T_X are final is satisfied, if T is full and faithful on the hom sets $\mathcal{L}(FA,X)$, $A \in \mathrm{Ob}\mathcal{K}$. For these functors there is also a *co*universal characterization of the shape category contrasting with Theorem 5 (cf. Mardešić [23], Theorem 3.1 and Frei [9], Theorem 4.6). If all functors T_X are final, then T is full on the hom-sets $\mathcal{L}(FA,X)$, but not necessarily faithful.

We will now present coshape categories as categories of inductive systems. For this we assume that we can choose a class $\mathbb{D}$ of small categories such that all comma categories F/X, $X \in \mathrm{Ob}\mathcal{L}$, belong to $\overline{\mathbb{D}}$ as defined before Theorem 3. We also assume that only at most one of these comma categories is empty. One now considers the full embedding

$$\mathrm{Ind}(\mathbb{D},\mathcal{K}) \cong \mathrm{Ind}(\overline{\mathbb{D}},\mathcal{K}) \xrightarrow{W_{\mathcal{K}}} \overline{\mathbb{D}}/\mathcal{K}, \quad H \mapsto (M_{\mathcal{K}}/H \to \mathcal{K}) .$$

LEMMA 2. *For all* $X \in \mathrm{Ob}\mathcal{L}$, *the projection functor* $P_X : F/X \to \mathcal{K}$ *is isomorphic to* $W_{\mathcal{K}}P_X$ *in* $\overline{\mathbb{D}}/\mathcal{K}$.

Proof (sketch). $W_{\mathcal{K}}P_X$ is the projection functor $M_{\mathcal{K}}/P_X \to \mathcal{K}$ (with $M_{\mathcal{K}}$ like in Section 3). Objects in $M_{\mathcal{K}}/P_X \cong \mathrm{El}(J_{\mathcal{K}}P_X)$ are pairs (A,u) with $A \in \mathrm{Ob}\mathcal{K}$ and $u \in \mathrm{colim}\, \mathcal{K}(A,P_X-)$. This colimit is explicitly given by $\mathcal{L}(FA,X)$ (since $\mathcal{L}(FA,-)$ is the left Kan extension of $\mathcal{K}(A,-)$ along F). This way we get a one-to-one correspondence between the objects of $M_{\mathcal{K}}/P_X$ and of F/X which is trivially extended to morphisms. □

Let $\mathrm{Ind}_F(\mathcal{K})$ be the full subcategory of $\mathrm{Ind}(\overline{\mathbb{D}},\mathcal{K})$ generated by all projections $P_X : F/X \to \mathcal{K}$, $X \in \mathrm{Ob}\mathcal{L}$.

THEOREM 6. $\mathrm{Ind}_F(K)$ *is isomorphic to* $\mathrm{Sh}^*(F)$.

Proof. On objects the bijection is given by $X \mapsto P_X$ (here we use that for only one X the comma category F/X may be empty) . Since W_K is full and faithful we have the isomorphisms

$$\begin{aligned}\mathrm{Ind}_F(K)(P_X,P_Y) &\cong \overline{\mathbf{D}}/K \quad (W_K P_X, W_K P_Y) \\ &\cong \overline{\mathbf{D}}/K \quad (P_X,P_Y) \qquad \text{(Lemma 2)} \\ &\cong \mathrm{CAT}/K \quad (K_F X, K_F Y) \\ &\cong \mathrm{Sh}^*(F) \quad (X,Y) \ ,\end{aligned}$$

and this establishes a bijective functor $\mathrm{Sh}^*(F) \to \mathrm{Ind}_F(K)$. □

5. The Čech condition and relative adjoints

The *shape category* $\mathrm{Sh}(F)$ of F is defined by $\mathrm{Sh}(F) = (\mathrm{Sh}^*(F^{op}))^{op}$ (with $F^{op} : K^{op} \to L^{op}$) . There is a canonical functor

$$S_F = (S^*_{F^{op}})^{op} : L \to \mathrm{Sh}(F)$$

which fulfills all the assertions one obtains by formally dualizing the previous results. Explicitly one has $\mathrm{Ob}\ \mathrm{Sh}(F) = \mathrm{Ob}L$ and

$$\begin{aligned}\mathrm{Sh}(F)(X,Y) &= \mathrm{Nat}(L(Y,F-),L(X,F-)) \\ &\cong \mathrm{Pro}(\overline{\mathbf{D}}^*,K)(Q_X,Q_Y)) \qquad \text{(Theorem 6)} \\ &\cong \lim_{Y\backslash F}\ \operatorname*{colim}_{X\backslash F}\ K(Q_X-,Q_Y-) \ ;\end{aligned}$$

here $Q_X : X\backslash F \to K$ is the projection functor of the comma category $X\backslash F$ of F-morphisms with domain X , and $\mathbf{D}$ is chosen in a way such that

$$\overline{\mathbf{D}}^* = \{\mathcal{D} \in \mathrm{ObCAT} \mid \exists R : E \to \mathcal{D} \text{ cofinal with } E \in \mathbf{D}\}$$

contains all comma categories $X\backslash F$.

The *Čech condition* which is defined below implies that these comma categories belong to $\overline{\mathbf{D}}^*$, if $\mathbf{D}$ is a given class of small, weakly cofiltered categories with $1 \in \mathbf{D}$. We remind the reader that a category $\mathcal{D}$ is *weakly cofiltered,* iff

(1) for all $D_o, D_1 \in \mathrm{Ob}\mathcal{D}$ there are $\mathcal{D}$-morphisms $d_i : D \to D_i$, $i = 0,1$,

(2) for all $\mathcal{D}$-morphisms $c_i : D_i \to C$, $i = 0,1$, there are $\mathcal{D}$-morphisms $d_i : D \to D_i$, $i = 0,1$, with $c_o d_o = c_1 d_1$.

DEFINITION. $F : K \to L$ satisfies the *Čech condition* (with respect to $\mathbf{D}$), iff for every $X \in \mathrm{Ob}L$ there exists a category $\mathcal{D}_X \in \mathbf{D}$, a functor $C_X : \mathcal{D}_X \to K$ and a cone $\pi_X : \Delta X \to FC_X$ such that the following holds:

(I) for all $A \in \mathrm{Ob}K$ and every $u : X \to FA$ in L there are $D \in \mathrm{Ob}\mathcal{D}_X$ and $f : C_X D \to A$ in K with $Ff \cdot \pi_X D = u$,

(II) for all $D \in \mathrm{Ob}\mathcal{D}_X$, $A \in \mathrm{Ob}K$ and $f,g : C_X D \to A$ with $Ff \cdot \pi_X D = Fg \cdot \pi_X D$ one has $\mathcal{D}_X$-morphisms $d,e : E \to D$ with $f \cdot C_X d = g \cdot C_X e$.

The main consequence of the Čech condition is that it allows a more "classical" description of the shape category $\mathrm{Sh}(F)$. The reason for this is:

LEMMA 3. *With the data as chosen above one obtains a cofinal functor* $\Pi_X : \mathcal{D}_X \to X\backslash F$ *satisfying the equations* $Q_X \Pi_X = C_X$ *and* $\Pi_X D = (\pi_X D, C_X D)$ *for all* $D \in \mathrm{Ob}\mathcal{D}_X$. □

By Lemma 3 we have that, in $\mathrm{Pro}(\overline{\mathbf{D}}^*, \mathcal{K})$, $C_X = Q_X \Pi_X \cong Q_X$. Therefore, for all $X, Y \in \mathrm{Ob}\mathcal{L}$, we get

$$\begin{aligned} \mathrm{Sh}(F)(X,Y) &\cong \mathrm{Pro}(\mathbf{D}, \mathcal{K})(C_X, C_Y) \\ &\cong \lim_{E \in \mathcal{D}_Y} \operatorname{colim}_{D \in \mathcal{D}_X} \mathcal{K}(C_X D, C_Y E) \ . \end{aligned}$$

So a shape morphism from X to Y may be described by a family $([f_E, D_E])_{E \in \mathrm{Ob}\mathcal{D}_Y}$ with $D_E \in \mathrm{Ob}\mathcal{D}_X$, $f_E \in \mathcal{K}(C_X D_E, C_Y E)$ such that, for all $e : E \to E'$, one has

$$[C_Y e \cdot f_E, D_E] = [f_{E'}, D_{E'}];$$

here the brackets denote the equivalence classes of the relation

$$(f,D) \sim (f',D') \iff \exists d : D'' \to D \ , \ d' : D'' \to D' \ : \ f \cdot C_X d = f' \cdot C_X d'$$

($\sim$ is an equivalence relation since $\mathcal{D}_X$ is weakly cofiltered) . If $([g_J, E_J])_{J \in \mathcal{D}_Z}$ describes a morphism in $\mathrm{Sh}(Y,Z)$, then the composition is described by

$$([g_J, E_J])_{J \in \mathrm{Ob}\mathcal{D}_Z} ([f_E, D_E])_{E \in \mathrm{Ob}\mathcal{D}_Y} = ([g_J f_{E_J}, D_{E_J}])_{J \in \mathcal{D}_Z}$$

compared to the elegant setting using natural transformations this is a rather clumsy description of shape morphisms, but it is used to prove the following theorem. To formulate it we need the canonical functor $\tilde{F} : \mathrm{Pro}(\mathbf{D}, \mathcal{K}) \to \mathrm{Pro}(\mathbf{D}, \mathcal{L})$ with $N_{\mathcal{L}} \tilde{F} = F N_{\mathcal{K}}$. ($N_{\mathcal{K}}$ is introduced at the end of Section 3).

THEOREM 7. *The following assertions are equivalent:*

(i) F *satisfies the Čech condition.*

(ii) *For every* $X \in \mathrm{Ob}\mathcal{L}$, *there are* $C_X \in \mathrm{Ob}\ \mathrm{Pro}(\mathbf{D}, \mathcal{K})$ *and a* $\mathrm{Pro}(\mathbf{D}, \mathcal{L})$*-morphism* $\pi_X : N_{\mathcal{L}} X \to \tilde{F} C_X$ *such that the following holds: every* $\mathrm{Pro}(\mathbf{D}, \mathcal{L})$*-morphism* $\varphi : N_{\mathcal{L}} X \to \tilde{F} H$ *with* $H \in \mathrm{Ob}\ \mathrm{Pro}(\mathbf{D}, \mathcal{K})$ *factorizes over* π_X *by a unique* $\mathrm{Pro}(\mathbf{D}, \mathcal{K})$*-morphism* $\psi : C_X \to H$:

$$\begin{tikzcd} N_{\mathcal{L}} X \arrow[r, "\pi_X"] \arrow[dr, "\varphi"'] & \tilde{F} C_X \arrow[d, "\tilde{F}\psi"] \\ & \tilde{F} H \end{tikzcd} \qquad \begin{tikzcd} C_X \arrow[d, dashed, "\psi"] \\ H \end{tikzcd}$$

(iii) *There is a functor* $C : \mathcal{L} \to \mathrm{Pro}(\mathbf{D}, \mathcal{K})$ *which is left adjoint to* $\tilde{F}$ *relatively to* $N_{\mathcal{L}}$ *(cf.* Ulmer [31]).

Proof (sketch). One first proves that (i) is equivalent to a restriction of (ii), namely where $H = N_{\mathcal{K}} A$ for an object $A \in \mathrm{Ob}\mathcal{K}$. Using the above description of shape morphisms one easily sees that condition (I) of the Čech condition is equivalent to the existence part of the factorization property whereas (II) expresses its uniqueness. Then it is not difficult to derive property (ii) in its general form from the "pointwise version" with $H = N_K A$.

The equivalence of (ii) and (iii) is well known and describes just the fact that "universal solutions" are functorial. □

COROLLARY. *If* F *satisfies the Čech condition, then* Sh(F) *is isomorphic to the full image of the functor* $C : L \to \mathrm{Pro}(\mathbf{D}, K)$ *which is left adjoint to* $\tilde{F}$ *relatively to* N_L .

REMARKS. (1) Taking in the Corollary L as the homotopy category of pointed topological spaces, K as the full subcategory of all pointed spaces homotopy equivalent to pointed polytopes and F as the inclusion one gets the shape category described by Dydak and Segal. For a pointed space (X,x_o) , $C(X,x_o)$ is its Čech system (cf. [8]).

(2) In the terminology of Pumplün [28] , property (iii) means that the pair $(N_L,\tilde{F})$ admits a universal solution. Mardešić calls F dense iff property (ii) is fulfilled, but this may be confused with the older categorical meaning: dense = colimit dense. Giuli [11] and Tozzi [12] have given a series of sufficient conditions for F to fulfill property (ii) using Adjoint Functor Theorem methods. Obviously the Čech condition (I) is already a kind of solution set condition.

(3) If F satisfies the Čech condition and if F is shape adequate (= dual of coshape adequate), i.e. S_F is full and faithful on the hom-sets $L(X,FA)$, $X \in \mathrm{Ob}L$ and $A \in \mathrm{Ob}K$, then, by the dual Proposition 7 (3) and by Lemma 3, we have a limit presentation $X \cong \lim F_o C_X$ with $F_o = S_F F$ in Sh(F) .

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